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Localization for the Random XXZ Spin-J Chain and Embedded Eigenvalues in Defective Periodic Quantum Graphs

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Localization for the Random XXZ Spin-J Chain
and
Embedded Eigenvalues in Defective Periodic Quantum Graphs

DISSERTATION

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for the degree of

DOCTOR OF PHILOSOPHY

in Mathematics

by

Lee Fisher

Dissertation Committee:
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DEDICATION

To my wife, Tomoyo Aizawa

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I would like to thank...

Christoph Fischbacher, a collaborator and friend. His many insights and contributions helped me more than anything.

My father, who always encouraged me.

VITA

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- Joe P. Chen, Luke G. Rogers, Loren Anderson, Ulysses Andrews, Antoni Brzoska, Aubrey Coffey, Hannah Davis, Lee Fisher, Madeline Hansalik, Stephen Loew, and Alexander Teplyaev. Power dissipation in fractal AC circuits. *J. Phys. A*, 50(32):325205, 20, 2017.

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ABSTRACT OF THE DISSERTATION

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By

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Doctor of Philosophy in Mathematics

University of California, Irvine, 2023

Professor Abel Klein, Co-Chair
Professor Stephen Shipman, Co-Chair

This thesis is composed of two distinct projects. Although both projects are devoted to studying situations in which the eigenfunctions of certain Schrödinger operators exhibit interesting or exceptional behavior, the nature of the results as well as the tools used in each of the parts are drastically different. The first portion is dedicated to a particular spin-system model in which a type of many body localization is proved. We use the Multiscale Analysis to show that localization in higher spin systems occurs within a certain interval near the bottom of the spectrum. The second project demonstrates that a carefully constructed defect in specific kinds of periodic media, most notably multilayer quantum-graph graphene, can create an eigenvalue embedded in the continuous spectrum with a corresponding eigenfunction that has exceptional properties.

Chapter 1

Localization for the Higher Spin Random XXZ Chain

1.1 Introduction

The phenomenon of Anderson Localization for a single particle is well studied, see, e.g., [2, 3, 4, 5, 19, 24, 30, 31, 40, 44, 45, 46, 47, 50, 80]. The situation is different in the case where the number of particles is large, even growing in conjunction with the infinite volume limit. Due to a lack of spatial degrees of freedom, spin systems are a relatively simple setting to study many body localization phenomena. In particular, the random Heisenberg XXZ spin-1/2 model has been the subject of rigorous mathematical analysis. For this model localization in the droplet spectrum (the lowest spectral band above the vacuum energy) is proven in [14, 35] and consequences are studied in [36, 34]. Very recently, localization has been proven in any given energy interval at the bottom of the spectrum [32]. All these proofs of localization are based on the fractional moment method (FMM).

Here we prove localization for the random Heisenberg XXZ model for the general spin-J case.

Our proof is based on a multiscale analysis (MSA), extending the method developed in [33] for the spin-1/2 model.

There are differences between the spin-J case for $J \geq 1$ and the spin-1/2 case. A crucial difference between the spin-1/2 and the spin-J cases is that the interaction potential for the spin-J case is more complex. Unlike in the spin-1/2 case, the interaction potential does not simply count the number of clusters (connected components) of a spin configuration. However the notion of a droplet band still makes sense: if the particle number is sufficiently large ($N \geq 4J$), then states corresponding to energies in the droplet band are supported around single cluster configurations. This feature makes it amenable to techniques similar to ones developed in [33] to prove localization in the spin-1/2 case.

On the other hand, the situation is different for small particle numbers, which adds an additional technical difficulty. In this case, even for energies in the droplet band, states can be supported around multi-cluster configurations. Although the lower and higher particle number regimes have different features, it is possible to treat them separately. For the lower particle number case, the localization results follow from methods developed for many-particles Schrödinger operators in [23, 6, 51]. These publications treat the case of a fixed number of particles, and their methods apply to our lower particle number setting. The fixed particle number results can be used to cover the finitely many cases where $N < 4J$ – however they cannot be extended to prove localization that holds for all particle numbers N. The parameters have to be chosen large enough depending on N, where how large diverges with N.

We will proceed as follows: In Sections 1.2, 1.3, and 1.4 we cover prerequisite material for stating the main results. In Section 1.2 we introduce the XXZ Quantum spin-J model, as well as fundamental objects such as particle number, subintervals of $\mathbb{Z}$, and projection operators. In Section 1.3 we show that the XXZ-Hamiltonian is unitarily equivalent to a direct sum of Schrödinger operators on a particular graph; this equivalence enables us to apply some well known tools to study the XXZ model. In Section 1.4 we cover some essential operator bounds and their consequences for the spectral properties of the XXZ model, and describe the properties of energy minimizing configurations.

In Section 1.5 we outline the main results in the case where $N \geq 4J$. Our two main results are Theorem 1.5.2 and Theorem 1.5.4, which respectively prove eigenfunction and resolvent localization with exponentially high probabilty. These can be viewed as an extension of results from [36, 33] for the spin-1/2 model to the spin-J case. From Sections 1.6 to 1.9 we build up towards a proof of Theorem 1.5.4, which is presented in Section 1.10. In Section 1.11 we apply the already developed tools to give a proof of Theorem 1.5.2 as well.

In Sections 1.6 and 1.7 we detail how some common tools for the Multiscale Analysis can be adapted to the XXZ setting. In Section 1.6 we give a proof of the Combes-Thomas estimate that proceeds along an argument similar to the one found in [35]. We use the Combes-Thomas to prove Lemma 1.6.6, which is crucial for the Multiscale Analysis. In Section 1.7 we cover Wegner and Large Deviation estimates, these results allow us to easily handle the cases where N is large enough compared to the size of the subsystem.

We dedicate Sections 1.8 through 1.10 directly to the Multiscale Analysis. In Section 1.8 we prove the starting condition, that is, that there is a base case for which the hypothesis of Theorem 1.5.4 is satisfied. In Section 1.9 we cover buffered intervals, one more essential concept for the induction process of the MSA. Finally the proof of Theorem 1.5.4 is given in Section 1.10, using the MSA.

The case of low particle numbers ($N < 4J$) is treated in Section 1.12. Theorem 1.12.22 is the analog to Theorem 1.5.4 for low particle numbers.

1.2 The XXZ Quantum Spin-J model

Let $J \in \frac{1}{2}\mathbb{N} = \{1/2, 1, 3/2, \dots\}$ and let Λ be a finite subgraph of $\mathbb{Z}$. We let $\mathcal{H}_\Lambda$ be the Hilbert space $\bigotimes_{x\in\Lambda} \mathbb{C}^{2J+1}$. Now we introduce the matrix

$$S^3 = \mathrm{diag}(J, J-1, \dots, -J+1, -J) \tag{1.2.1}$$

Let e_i be a basis for $\mathbb{C}^{2J+1}$, where $i = \{0, \dots, 2J\}$. Then we have that $S^3 e_i = (J-i)e_i$. The spin raising and spin lowering operators are defined by their action on the basis vectors e_i.

$$S^+ e_i := \begin{cases} \sqrt{i(2J+1) - i^2}\, e_{i-1} & \text{if } i > 0 \\ 0 & \text{if } i = 0 \end{cases} \tag{1.2.2}$$

$$S^- e_i := \begin{cases} \sqrt{2J + i(2J-1) - i^2}\, e_{i+1} & \text{if } i < 2J \\ 0 & \text{if } i = 2J \end{cases} \tag{1.2.3}$$

Here, J is the highest spin and $-J$ is the lowest spin. In what follows, we will however interpret e_0 as the vacuum and e_{2J} as a slot being occupied by $2J$ particles. Other foundational objects are the matrices:

$$S^1 := \tfrac{1}{2}\left(S^+ + S^-\right), \tag{1.2.4}$$

$$S^2 := \tfrac{1}{2i}\left(S^+ - S^-\right). \tag{1.2.5}$$

Let $\Lambda' \subset \Lambda$. Then, if A is an operator acting on the tensor product $\mathcal{H}_{\Lambda'}$, we extend A to an operator on $\mathcal{H}_\Lambda$, $A_{\Lambda'}$, where $A_{\Lambda'}$ acts as the identity on the factors in the tensor product corresponding to the elements of $\Lambda \setminus \Lambda'$. Next, suppose that $\{x, x+1\} \subset \Lambda$ and introduce the two-site Hamiltonian

$$
\begin{aligned}
h_{x,x+1} &= J^2 - \frac{1}{\Delta}\left(S_x^1 S_{x+1}^1 + S_x^2 S_{x+1}^2\right) - S_x^3 S_{x+1}^3 \\
&= J^2 - \frac{1}{2\Delta}\left(S_x^+ S_{x+1}^- + S_x^- S_{x+1}^+\right) - S_x^3 S_{x+1}^3 .
\end{aligned}
\tag{1.2.6}
$$

Here, Δ is called the anisotropy parameter and in what follows, we will assume $\Delta > 2J$.

Another important object is the particle number operator

$$
\mathcal{N} = J - S^3 = \mathrm{diag}(0, 1, \ldots, 2J). \tag{1.2.7}
$$

Observe that $\{e_i\}_{i=0}^{2J}$ is an eigenbasis for $\mathcal{N}$ with $\mathcal{N} e_i = i \cdot e_i$. If x is a vertex in Λ, then we say that *the particle number operator at* x, $\mathcal{N}_x$, is the operator which acts as $\mathcal{N}$ on the site x and as the identity at all other factors in the tensor product.

Remark 1.2.1. When speaking about any subgraph of $\mathbb{Z}$, say Λ, we will use $\#(\Lambda)$ to mean the cardinality of Λ and we will use $|\Lambda|$ to mean the length of Λ as measured on the real line.

If Λ is a finite subgraph of $\mathbb{Z}$, we then define the *total particle number operator* $\mathcal{N}_\Lambda$ as

$$
\mathcal{N}_\Lambda := \sum_{x \in \mathcal{V}(\Lambda)} \mathcal{N}_x , \tag{1.2.8}
$$

Here $\mathcal{V}(\Lambda)$ denotes the vertex set of Λ. The eigenvalues $\sigma(\mathcal{N}_\Lambda) = \{0, 1, \ldots, 2J\#(\Lambda)\}$ of $\mathcal{N}_\Lambda$ are consequently interpreted as the total number of particles occupying the vertices of Λ. We introduce the operator

$$
\tilde{h}_{x,x+1} := h_{x,x+1} - J(\mathcal{N}_x + \mathcal{N}_{x+1}) = -\mathcal{N}_x \mathcal{N}_{x+1} - \frac{1}{2\Delta}\left(S_x^+ S_{x+1}^- + S_x^- S_{x+1}^+\right). \tag{1.2.9}
$$

The second equality can be directly verified by checking it on the basis elements of the tensor product $\mathbb{C}^{2J+1} \otimes \mathbb{C}^{2J+1}$. Now, let $\mathcal{E}_\Lambda = \{\{x, x+1\} \subset \Lambda\}$ be the set of edges when viewing Λ

as a graph. We introduce the XXZ Hamiltonian H_Λ on the graph Λ,

$$H_\Lambda = \tilde{H}_\Lambda + 2J\mathcal{N}_\Lambda + \lambda V_{\Lambda,\omega} \tag{1.2.10}$$

$$\tilde{H}_\Lambda = \sum_{\{x,x+1\}\in\mathcal{E}_\Lambda} \tilde{h}_{x,x+1} \tag{1.2.11}$$

$$V_{\Lambda,\omega} = \sum_{x\in\Lambda} \omega_x \mathcal{N}_x \,. \tag{1.2.12}$$

Here, ω_x are independently and identically distributed random variables, whose common probability distribution μ has $\{0,1\} \subset \mathrm{supp}\mu \subset [0,1]$ and is Hölder continuous of order $\alpha \in (0,1]$:

$$\sup_{a\in\mathbb{R}} \mu\{[a,a+t]\} \leq Kt^\alpha \text{ for all } t \in [0,1], \tag{1.2.13}$$

where K is a constant.

Definition 1.2.2. Let $i \in \mathbb{Z}$ and let $p \in \{0,1,2,\dots\}$. We then define the interval with radius p centered at i, $\Lambda_p(i) = \{j \in \mathbb{Z} : |i-j| \leq p\}$.

1.2.1 Projections and Identities

Given the subset Λ, we define orthogonal projections $P_\Lambda^\pm$ on $\mathcal{H}_\Lambda$ as follows:

$$P_\Lambda^+ := \bigotimes_{x\in\Lambda} \pi_{e_0}(x) \,, \tag{1.2.14}$$

where $\pi_{e_0}(x)$ is the orthogonal projection onto $\ker(\mathcal{N}_x)$. We interpret P_Λ^+ as the orthogonal projection onto the state where no particles are present in Λ (vacuum). We also define

$$P_\Lambda^- := 1 - P_\Lambda^+ \tag{1.2.15}$$

This is a projection onto the space of configurations where there is at least one particle in Λ. Next, we summarize some useful identities involving these projections.

Proposition 1.2.3. *Suppose* $\{k, k+1\} \in \mathcal{E}_\Lambda$*:*

$$\tilde{h}_{k,k+1} P^+_{\{k,k+1\}} = P^+_{\{k,k+1\}} \tilde{h}_{k,k+1} = 0 \tag{1.2.16}$$

$$\tilde{h}_{k,k+1} P^-_{\{k,k+1\}} = P^-_{\{k,k+1\}} \tilde{h}_{k,k+1} = \tilde{h}_{k,k+1} \tag{1.2.17}$$

$$P^+_{\{k\}} \tilde{h}_{k,k+1} P^+_{\{k\}} = P^+_{\{k+1\}} \tilde{h}_{k,k+1} P^+_{\{k+1\}} = 0 \tag{1.2.18}$$

These identities rest on the observation that $P^+_{\{k,k+1\}}$ is a projection onto the space of configurations where there are no particles at sites k and $k+1$, and likewise $P^-_{\{k,k+1\}}$ is a projection onto the space of configurations where there is at least one particle at site k or site $k+1$.

For any $i \in \Lambda$, we also have the following useful identity for projections:

$$P^-_\Lambda = \sum_{p=-1}^{\infty} P^+_{\Lambda_p(i)\cap\Lambda} P^-_{\{i+p+1,i-p-1\}\cap\Lambda}. \tag{1.2.19}$$

Given $q \in \mathbb{N}$, multiplying both sides by $P^+_{\Lambda_q(i)\cap\Lambda}$ we get

$$P^+_{\Lambda_q(i)\cap\Lambda} P^-_\Lambda = \sum_{p=q}^{\infty} P^+_{\Lambda_p(i)\cap\Lambda} P^-_{\{i+p+1,i-p-1\}\cap\Lambda}. \tag{1.2.20}$$

In order to read the above equations, we define that $\Lambda_{-1}(p) = \emptyset$ and $P^+_\emptyset = I$.

1.3 Equivalence to a Schrödinger operator.

1.3.1 Operators on Subintervals

Suppose that Λ and K are subintervals of $\mathbb{Z}$ and that $K \subsetneq \Lambda$. We define the edge boundary of K relative to Λ by

$$\boldsymbol{\partial}^{\Lambda} K = \{\{u, v\} \in \mathbb{Z}^2, u \in \Lambda \setminus K, v \in K, |u - v| = 1\}. \tag{1.3.1}$$

We also define a few more related objects:

$$\partial^{\Lambda} K = \bigcup_{\{u,v\} \in \boldsymbol{\partial}^{\Lambda} K} \{u, v\}, \text{ the vertex set of } \boldsymbol{\partial}^{\Lambda} K. \tag{1.3.2}$$

$$\partial^{\mathbb{Z}} K = \partial K \tag{1.3.3}$$

$$\partial^{in} K = \partial^{\Lambda} K \cap K \tag{1.3.4}$$

$$\Lambda \setminus \partial K = G \subset \mathbb{Z} \text{ a graph where } \mathcal{V}(G) = \mathcal{V}(\Lambda), \text{ and } \mathcal{E}_G = \mathcal{E}_{\Lambda} \setminus \mathcal{E}_{\boldsymbol{\partial}^{\Lambda} K} \tag{1.3.5}$$

$$\Lambda \setminus K = G \subset \mathbb{Z} \text{ a graph where } \mathcal{V}(G) = \mathcal{V}(\Lambda) \setminus \mathcal{V}(K), \text{ and } \mathcal{E}_G = \mathcal{E}_{\Lambda} \setminus \mathcal{E}_K. \tag{1.3.6}$$

Notice that $\#(\partial^{\Lambda} K) = 2$ if Λ and K share an endpoint and $\#(\partial^{\Lambda} K) = 4$ if they do not. We define the edge connecting operator:

$$\Gamma_{\Lambda \setminus \partial K} = \sum_{\{x, x+1\} \in \boldsymbol{\partial}^{\Lambda} K} \tilde{h}_{x,x+1}. \tag{1.3.7}$$

Notice in particular the identity,

$$H_{\Lambda \setminus \partial K} = H_{\Lambda} - \Gamma_{\Lambda \setminus \partial K} = H_K + H_{\Lambda \setminus K} \text{ which acts on } \mathcal{H}_{\Lambda} = \mathcal{H}_K \otimes \mathcal{H}_{\Lambda \setminus K}. \tag{1.3.8}$$

We define a few more essential spaces and operators,

$$\mathcal{H}'_\Lambda = P^-_\Lambda \mathcal{H}_\Lambda = \Omega^\perp_\Lambda \tag{1.3.9}$$

$$H'_\Lambda = P^-_\Lambda H = H P^-_\Lambda \text{ which acts on } \mathcal{H}'_\Lambda \tag{1.3.10}$$

$$H'_{\Lambda\setminus\partial K} = H'_K + H_{\Lambda\setminus K} \text{ which acts on } P^-_K \mathcal{H}_\Lambda = \mathcal{H}'_K \otimes \mathcal{H}_{\Lambda\setminus K}. \tag{1.3.11}$$

1.3.2 Spaces of Configurations

Fix Λ a finite subinterval of $\mathbb{Z}$. It can be verified that the XXZ Hamiltonian H_Λ preserves the total particle number, i.e. $[H_\Lambda, \mathcal{N}_\Lambda] = 0$. We thus decompose $\mathcal{H}_\Lambda = \bigoplus_{N=0}^{2J\#(\Lambda)} \mathcal{H}_\Lambda^{(N)}$, where $\mathcal{H}_\Lambda^{(N)}$ denotes the eigenspace of $\mathcal{N}_\Lambda$ corresponding to the eigenvalue N – the space of all N-particle configurations in Λ with the restriction that no site can be occupied by more than $2J$ particles. We also define $H_\Lambda^{(N)} := H_\Lambda \restriction_{\mathcal{H}_\Lambda^{(N)}}$. Let,

$$\mathbf{M}_\Lambda^{(N)} := \left\{ \mathbf{m} : \Lambda \to \{0, 1, \ldots, 2J\} : \sum_{x\in\Lambda} \mathbf{m}(x) = N \right\}, \tag{1.3.12}$$

be the set of all functions from Λ to $\{0, 1, \ldots, 2J\}$ whose values add up to N. For convenience, we also define $\mathbf{M}_\Lambda := \bigcup_{N=0}^{2J\#(\Lambda)} \mathbf{M}_\Lambda^{(N)}$ – the set of *all* functions from Λ to $\{0, 1, \ldots, 2J\}$.

Furthermore we define the support of $\mathbf{m} \in \mathbf{M}_\Lambda$,

$$\operatorname{supp} \mathbf{m} = \{x_i \in \Lambda \text{ such that } \mathbf{m}(x_i) \geq 1\}. \tag{1.3.13}$$

If we can write $\mathbf{m} = \mathbf{m}_1 + \cdots + \mathbf{m}_n$ where for each $i \neq j$, $\operatorname{supp} \mathbf{m}_i$ is connected and $\operatorname{dist}(\operatorname{supp} \mathbf{m}_i, \operatorname{supp} \mathbf{m}_j) \geq 2$, then $\mathbf{m}$ is an n cluster configuration with the $\mathbf{m}_i$ being the clusters of $\mathbf{m}$.

There are other important subsets of $\mathbf{M}_\Lambda^{(N)}$. Suppose that $\Phi \subset \Lambda \subset \mathbb{Z}$ then we define,

$$\mathbf{S}_\Phi^{(N)} = \{\mathbf{m} \in \mathbf{M}_\Lambda^{(N)} \text{ such that } \mathbf{m}(x_i) \geq 1 \text{ for some } x_i \in \Phi\}. \tag{1.3.14}$$

Notice in particular that,

$$\mathbf{m} \in \mathbf{S}_\Phi^{(N)} \iff \Phi \cap \operatorname{supp} \mathbf{m} \neq \emptyset. \tag{1.3.15}$$

For any $\mathbf{m} \in \mathbf{M}_\Lambda$, we then define

$$\psi_{\mathbf{m}} := \bigotimes_{x \in \Lambda} e_{\mathbf{m}(x)} \ . \tag{1.3.16}$$

This means that for any $x \in \Lambda$ we get $\mathcal{N}_x \psi_{\mathbf{m}} = \mathbf{m}(x)\psi_{\mathbf{m}}$. In other words, $\psi_{\mathbf{m}}$ describes a configuration of particles, where at each site $x \in \Lambda$, there are exactly $\mathbf{m}(x)$ particles. Since

$$\mathcal{N}_\Lambda \psi_{\mathbf{m}} = \left(\sum_{x \in \Lambda} \mathbf{m}(x) \right) \psi_{\mathbf{m}} \ , \tag{1.3.17}$$

it immediately follows that

$$\mathcal{H}_\Lambda^{(N)} = \operatorname{span} \left\{ \psi_{\mathbf{m}} : \mathbf{m} \in \mathbf{M}_\Lambda^{(N)} \right\} \ . \tag{1.3.18}$$

Now, consider the Hilbert space $\ell^2(\mathbf{M}_\Lambda^{(N)}) = \left\{ f : \mathbf{M}_\Lambda^{(N)} \to \mathbb{C} \right\}$ equipped with inner product $\langle f, g \rangle = \sum_{\mathbf{m} \in \mathbf{M}_\Lambda^{(N)}} \overline{f(\mathbf{m})} g(\mathbf{m})$ and let $\{\phi_{\mathbf{m}}\}_{\mathbf{m} \in \mathbf{M}_\Lambda^{(N)}}$ denote the canonical basis of $\ell^2(\mathbf{M}_\Lambda^{(N)})$, i.e.

$$\phi_{\mathbf{m}}(\mathbf{n}) = \begin{cases} 1 & \text{if} \quad \mathbf{m} = \mathbf{n} \\ 0 & \text{else.} \end{cases} \tag{1.3.19}$$

The Hilbert spaces $\mathcal{H}_\Lambda^{(N)}$ and $\ell^2(\mathbf{M}_\Lambda^{(N)})$ are unitarily equivalent via

$$U_\Lambda^{(N)} : \mathcal{H}_\Lambda^{(N)} \to \ell^2(\mathbf{M}_\Lambda^{(N)}), \quad U_\Lambda^{(N)} \psi_{\mathbf{m}} = \phi_{\mathbf{m}} \,. \tag{1.3.20}$$

One also naturally identifies $\ell^2(\mathbf{M}_\Lambda) = \bigoplus_{N=0}^{2J\#(\Lambda)} \ell^2(\mathbf{M}_\Lambda^{(N)})$. For any $f \in \ell^2(\mathbf{M}_\Lambda^{(N)})$, let us now define the adjacency operator $A_\Lambda^{(N)}$, given by

$$(A_\Lambda^{(N)} f)(\mathbf{m}) = \sum_{\mathbf{n}:\mathbf{n}\sim\mathbf{m}} w(\mathbf{m},\mathbf{n}) f(\mathbf{n}) \,, \tag{1.3.21}$$

where for two configurations $\mathbf{m}, \mathbf{n} \in \mathbf{M}_\Lambda^{(N)}$ to be adjacent (denoted by $\mathbf{m} \sim \mathbf{n}$) is defined as follows:

$$\begin{aligned} \exists \{x_0, x_1\} \in \mathcal{E}_\Lambda \text{ such that } & \mathbf{m}(x_0) = \mathbf{n}(x_0) + 1, \\ & \mathbf{m}(x_1) = \mathbf{n}(x_1) - 1, \\ & \text{and } \mathbf{m}(x) = \mathbf{n}(x) \text{ when } x \in \Lambda \setminus \{x_0, x_1\}. \end{aligned}$$

This definition should be interpreted in the following way: two configurations $\mathbf{m}, \mathbf{n}$ of N particles distributed over L sites (with the requirement that no site be occupied by more than $2J$ particles) are adjacent if one configuration can be obtained by moving a single particle along an edge to the other configuration.

For $\mathbf{m} \sim \mathbf{n}$, the weight function $w(\mathbf{m},\mathbf{n}) = w(\mathbf{n},\mathbf{m})$ in (1.3.21) is given by

$$w(\mathbf{m},\mathbf{n}) = \prod_{x:\mathbf{m}(x)\neq\mathbf{n}(x)} \left(J(\mathbf{m}(x) + \mathbf{n}(x) + 1) - \mathbf{m}(x)\mathbf{n}(x)\right)^{1/2} \,, \tag{1.3.22}$$

Moreover, for any $\mathbf{m}, \mathbf{n} \in \mathbf{M}_\Lambda^{(N)}$, we define their distance $d_\Lambda^{(N)}(\mathbf{m},\mathbf{n})$ to be the length of the shortest path connecting $\mathbf{m}$ and $\mathbf{n}$, which we will refer to as graph distance. If there is no path connecting the configurations $\mathbf{m}$ and $\mathbf{n}$, we adapt the convention $d_\Lambda^{(N)}(\mathbf{m},\mathbf{n}) = \infty$.

Likewise we use $d_\Lambda(\mathbf{m},\mathbf{n})$ for configurations on $\mathbf{M}_\Lambda$. Also notice that if $\mathbf{m}$ and $\mathbf{n}$ have different particle numbers then $d_\Lambda(\mathbf{m},\mathbf{n})=\infty$, and otherwise $d_\Lambda^{(N)}(\mathbf{m},\mathbf{n})=d_\Lambda(\mathbf{m},\mathbf{n})$. Hence we will omit the superscript and simply use, $d_\Lambda(\mathbf{m},\mathbf{n})$ as the distance between configurations going forward.

Next, we define the interaction potential $\mathcal{W}_\Lambda^{(N)}$, a multiplication operator on $\ell^2(\mathbf{M}_\Lambda^{(N)})$, to be given by

$$(\mathcal{W}_\Lambda^{(N)} f)(\mathbf{m}) = \mathcal{W}_\Lambda^{(N)}(\mathbf{m}) f(\mathbf{m}) = \left(2JN - \sum_{\{i,i+1\}\in\mathcal{E}_\Lambda} \mathbf{m}(i)\mathbf{m}(i+1)\right) f(\mathbf{m})\,. \tag{1.3.23}$$

By a slight abuse of notation, we also use the symbol $V_{\Lambda,\omega}^{(N)}$ to denote the random potential acting as a multiplication operator on $\ell^2(\mathbf{M}_\Lambda^{(N)})$, i.e.

$$(V_{\Lambda,\omega}^{(N)} f)(\mathbf{m}) = V_{\Lambda,\omega}^{(N)}(\mathbf{m}) f(\mathbf{m}) = \left(\sum_{x\in\Lambda} \mathbf{m}(x)\omega_x\right) f(\mathbf{m})\,. \tag{1.3.24}$$

In [42, Prop. 2.1], it was shown that the Hamiltonian $H_\Lambda^{(N)}$ is unitarily equivalent to a many–body Schrödinger-type operator $\widehat{H}_\Lambda^{(N)}$:

Proposition 1.3.1. *We have the following unitary equivalence:*

$$U_\Lambda^{(N)} H_\Lambda^{(N)} \left(U_\Lambda^{(N)}\right)^* = -\frac{1}{2\Delta} A_\Lambda^{(N)} + \mathcal{W}_\Lambda^{(N)} + \lambda V_{\Lambda,\omega}^{(N)} =: \widehat{H}_\Lambda^{(N)}\,. \tag{1.3.25}$$

Remark 1.3.2. While in the statement of the proposition, we introduced the symbol $\widehat{H}_\Lambda^{(N)}$ for the Schrödinger-type operator acting on $\ell^2(\mathbf{M}_\Lambda^{(N)})$, in what follows, we will abuse notation and just use the symbol $H_\Lambda^{(N)}$ instead.

Definition 1.3.3. We define

$$\mathbf{M}_\Lambda^{\geq 4J} := \left\{\mathbf{m} : \Lambda \to \{0,1,\ldots,2J\} : \sum_{x\in\Lambda} \mathbf{m}(x) \geq 4J\right\}. \tag{1.3.26}$$

This is the set of all mass functions with at least $4J$ particles. Likewise define $H_\Lambda^{\geq 4J}$ to be the restriction of H_Λ onto $\ell^2(\mathbf{M}_\Lambda^{\geq 4J})$.

Remark 1.3.4. For the special cases $N = 0$ and $N = 2J\#(\Lambda)$, note that

$$\dim(\ell^2(\mathbf{M}_\Lambda^{(0)})) = \dim(\ell^2(\mathbf{M}_\Lambda^{(2J\#(\Lambda))})) = 1 \, .$$

On these one-dimensional spaces, the operators $H_\Lambda^{(0)}$ and $H_\Lambda^{(2J\#(\Lambda))}$ are just given by $H_\Lambda^{(0)} = 0$ and $H_\Lambda^{(2J\#(\Lambda))} = 4J^2 + 2J\sum_{x\in\Lambda}\omega_x$.

1.4 Bounds and Minimizers

Let us also introduce the operators

$$A_\Lambda := \bigoplus_{N=0}^{2J\#(\Lambda)} A_\Lambda^{(N)} \tag{1.4.1}$$

with $\mathcal{W}_\Lambda$ and $V_{\Lambda,\omega}$ being defined analogously.

Definition 1.4.1. The Hamiltonian without random potential is given by

$$H_{0,\Lambda} = -\tfrac{1}{2\Delta}A_\Lambda + \mathcal{W}_\Lambda.$$

Lemma 1.4.2. *We have the bound:*

$$-4J\mathcal{W}_\Lambda \leq A_\Lambda \leq 4J\mathcal{W}_\Lambda \tag{1.4.2}$$

and consequently, we get

$$\left(1 - \tfrac{2J}{\Delta}\right)\mathcal{W}_\Lambda \leq H_{0,\Lambda} \leq \left(1 + \tfrac{2J}{\Delta}\right)\mathcal{W}_\Lambda \, . \tag{1.4.3}$$

Proof. In [67], the following bound was shown

$$-4J(J^2 - S_i^3 S_{i+1}^3) \le S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+ \le 4J(J^2 - S_i^3 S_{i+1}^3) \tag{1.4.4}$$

Summing over the edges in Λ yields the desired result. □

Lemma 1.4.3. *For every $i \in \mathbb{Z}$,*

$$\|\tilde{h}_{i,i+1}\| = 4J^2. \tag{1.4.5}$$

Proof. First we will prove that $\|\tilde{h}_{i,i+1}\| \le 4J^2$. Starting with the definition of $\tilde{h}_{i,i+1}$ we have,

$$\tilde{h}_{i,i+1} = -\mathcal{N}_i\mathcal{N}_{i+1} - \tfrac{1}{2\Delta}(S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+) \tag{1.4.6}$$

From the relative bound in Lemma 1.4.2.

$$-\mathcal{N}_i\mathcal{N}_{i+1} - \tfrac{1}{2\Delta}4J(J^2 - S_i^3 S_{i+1}^3) \le \tilde{h}_{i,i+1} \le -\mathcal{N}_i\mathcal{N}_{i+1} + \tfrac{1}{2\Delta}4J(J^2 - S_i^3 S_{i+1}^3) \tag{1.4.7}$$

We will use the fact that $\mathcal{N}_i = J - S_i^3$ to proceed. First we prove the lower bound:

$$\begin{aligned}\tilde{h}_{i,i+1} &\ge -\mathcal{N}_i\mathcal{N}_{i+1} - \tfrac{2J}{\Delta}(J^2 - (J-\mathcal{N}_i)(J-\mathcal{N}_{i+1})) \\ &= -\left(1 - \tfrac{2J}{\Delta}\right)\mathcal{N}_i\mathcal{N}_{i+1} - \tfrac{2J^2}{\Delta}(\mathcal{N}_i + \mathcal{N}_{i+1}) \\ &\ge -\left(1 - \tfrac{2J}{\Delta}\right)4J^2 - \tfrac{2J}{\Delta}(4J) = -4J^2.\end{aligned} \tag{1.4.8}$$

Then the upper bound,

$$\begin{aligned}\tilde{h}_{i,i+1} &\le -\mathcal{N}_i\mathcal{N}_{i+1} + \tfrac{2J}{\Delta}(J\mathcal{N}_i + J\mathcal{N}_{i+1} - \mathcal{N}_i\mathcal{N}_{i+1})) \\ &= -\left(1 + \tfrac{2J}{\Delta}\right)\mathcal{N}_i\mathcal{N}_{i+1} + \tfrac{2J^2}{\Delta}(\mathcal{N}_i + \mathcal{N}_{i+1}) \\ &\le 0 + \tfrac{2J^2}{\Delta}(4J) = \tfrac{8J^3}{\Delta} < 4J^2.\end{aligned} \tag{1.4.9}$$

Consider the unit vector $v = e_{2J,i} \otimes e_{2J,i+1}$, it is straightforward to show that $\tilde{h}_{i,i+1} v = -4J^2 v$, so we see the bound is attained. $\square$

For brevity, we introduce

$$Q_k = 4J^2 + 2Jk. \tag{1.4.10}$$

For any $k \in \mathbb{N}$ we define $\mathbf{M}_{\Lambda,k}^{(N)}$ to be the set of configurations for which the potential $\mathcal{W}_\Lambda^{(N)}$ is bounded by $Q_k - 1$, i.e.

$$\mathbf{M}_{\Lambda,k}^{(N)} := \{\mathbf{m} \in \mathbf{M}_\Lambda^{(N)} : \mathcal{W}_\Lambda^{(N)}(\mathbf{m}) \leq Q_k - 1\}. \tag{1.4.11}$$

Moreover, we define $\mathbf{M}_{\Lambda,k} = \bigcup_N \mathbf{M}_{\Lambda,k}^{(N)}$ as well as $P_{\Lambda,k}$ and $P_{\Lambda,k}^{(N)}$ to be the orthogonal projections onto $\ell^2(\mathbf{M}_{\Lambda,k})$ and $\ell^2(\mathbf{M}_{\Lambda,k}^{(N)})$, respectively. Let $\overline{P}_{\Lambda,k} := I - P_{\Lambda,k}$ and $\overline{P}_{\Lambda,k}^{(N)} := I - P_{\Lambda,k}^{(N)}$. Finally, if $\Phi \subset \mathbf{M}_\Lambda^{(N)}$ then χ_Φ is the characteristic function of Φ, it is also the orthogonal projection onto $\ell^2(\Phi)$. Using Lemma 1.4.2, we also note that

$$\overline{P}_{\Lambda,k}^{(N)} H_\Lambda^{(N)} \overline{P}_{\Lambda,k}^{(N)} = \overline{P}_{\Lambda,k}^{(N)} \left(-\tfrac{1}{2\Delta} A_\Lambda^{(N)} + \mathcal{W}_\Lambda^{(N)} + \lambda V_{\Lambda,\omega}^{(N)}\right) \overline{P}_{\Lambda,k}^{(N)} \geq \left(1 - \tfrac{2J}{\Delta}\right) \mathcal{W}_\Lambda^{(N)} \overline{P}_{\Lambda,k}^{(N)}. \tag{1.4.12}$$

Proposition 1.4.4. *Let $N \in \mathbb{N}$ and let Λ_L be a finite interval with $2L+1 \geq \lceil \frac{N}{2J} \rceil$. We define $\mathcal{W}_{\Lambda,0}^{(N)} := \min\{\mathcal{W}_\Lambda^{(N)}(\mathbf{m}) : \mathbf{m} \in \mathbf{M}_\Lambda^{(N)}\}$. Then*

$$\mathcal{W}_0^{(N)} := \mathcal{W}_{\Lambda,0}^{(N)} = \begin{cases} 2JN - \lfloor \frac{N^2}{4} \rfloor & N < 4J \\ 4J^2 & N \geq 4J \end{cases}. \tag{1.4.13}$$

Moreover, if $N \geq 4J$ then, – up to overall translations – the minimizers of $\mathcal{W}_\Lambda^{(N)}$ are given *by*

$$\mathbf{m}_j^{(N)}(x) = \begin{cases} j & \text{if} \quad x = 1 \\ 2J & \text{if} \quad x = 2, \dots, r \\ 2J - j & \text{if} \quad x = r + 1 \end{cases} \tag{1.4.14}$$

for $N = 2Jr, \;\; 2 \le r \le 2L, \; j = 0, \dots, 2J - 1$ *and*

$$\mathbf{m}_j^{(N)}(x) = \begin{cases} j & \text{if} \quad x = 1 \\ 2J & \text{if} \quad x = 2, \dots, 1 + \lfloor \frac{N}{2J} \rfloor \\ N(\text{mod } 2J) - j & \text{if} \quad x = 2 + \lfloor \frac{N}{2J} \rfloor \end{cases} \tag{1.4.15}$$

if N *is not a multiple of* $2J$, $j = 0, \dots, N(\text{mod } 2J) - 1$.

Proposition 1.4.4, for the case where $N \ge 4J$, is proved in [43, Prop. 2.5]. The proof for the case where $N < 4J$ is below.

Proof. We will prove that the minimizer has the desired form in steps. First we will show that the minimizer has one cluster. Consider a mass function $\mathbf{m}$ with two or more clusters. Since $\mathcal{W}$ has a translation symmetry, then without loss of generality we will take the $\mathbf{m}(x)$ to be supported so that the left most point in the left most gap is at the site $x = 1$ (so $\mathbf{m}(x)$ is supported at 0). We define n to be the smallest positive number for $\mathbf{m}(\mathbf{x}) > 0$. Let $\mathbf{m}'$ be the function where the portion of $\mathbf{m}$ that lies to the right of 0 is translated to remove the first gap. In particular this means that for $x \ge 1$, $\mathbf{m}'(x) = \mathbf{m}(x + n - 1)$. Then

$$\begin{aligned} \mathcal{W}(\mathbf{m}) &= 2JN - \sum_{i<0} \mathbf{m}(i)\mathbf{m}(i+1) - \sum_{i \ge n} \mathbf{m}(i)\mathbf{m}(i+1) \\ &> 2JN - \sum_{i<0} \mathbf{m}(i)\mathbf{m}(i+1) - \mathbf{m}(0)\mathbf{m}(n) - \sum_{i \ge n} \mathbf{m}(i)\mathbf{m}(i+1) = \mathcal{W}(\mathbf{m}'). \end{aligned} \tag{1.4.16}$$

We have shown that, for any N, configurations with two or more clusters are not minimal. Now let supp $(\mathbf{m}) = [0, \ldots, n]$, we will show that there is a minimal configuration with $n = 2$, and that minimal configurations never have $n \geq 4$. Suppose $n \geq 3$, since $N < 4J$ we can construct a configuration with tighter support $\mathbf{m}'$ by redistributing the mass of $\mathbf{m}(0)$ onto $\mathbf{m}(1)$ and $\mathbf{m}(2)$; notice that again since $N < 4J$, and $\mathbf{m}(x) \leq 2J$ we will never need to redistribute onto $\mathbf{m}(3)$. Let $\mathbf{m}'(x)$ be a such modified configuration. We have that:

$$\begin{aligned}\mathcal{W}(\mathbf{m}) - \mathcal{W}(\mathbf{m}') &= -\mathbf{m}(3)\mathbf{m}(2) - \mathbf{m}(2)\mathbf{m}(1) - \mathbf{m}(1)\mathbf{m}(0) + \mathbf{m}(3)\mathbf{m}'(2) + \mathbf{m}'(2)\mathbf{m}'(1) \\ &= \mathbf{m}(3)(\mathbf{m}'(2) - \mathbf{m}(2)) + \mathbf{m}'(2)(\mathbf{m}'(1) - \mathbf{m}(1)) - \mathbf{m}(1)(\mathbf{m}'(1) - \mathbf{m}(1)) \\ &= (I) + (II) - (III) \qquad (1.4.17)\end{aligned}$$

We used the fact that $\mathbf{m}(0) = \mathbf{m}'(2) - \mathbf{m}(2) + \mathbf{m}'(1) - \mathbf{m}(1)$. Now we want to show that as long as $\mathbf{m}(2), \mathbf{m}(1), \mathbf{m}(0) > 0$ we can choose $\mathbf{m}'$ such that $(I) + (II) - (III) \geq 0$. We will redistribute so that $\mathbf{m}'(2)$ is as large as possible. There are three possibilities:

- $2J \geq \mathbf{m}'(2) > \mathbf{m}(2)$ and $\mathbf{m}'(1) = \mathbf{m}(1)$. Here we have, $(II) = (III) = 0$. If $\mathbf{m}(3) > 0$ then $(I) > 0$ and we have shown that if $\mathbf{m}$ is supported on at least 4 points then $\mathbf{m}'$ is supported a set one point smaller and has strictly smaller potential. If $\mathbf{m}(3) = 0$ then $\mathbf{m}$ is supported on 3 points and $\mathbf{m}'$ is supported on 2 points with no change in $\mathcal{W}$.

- $2J = \mathbf{m}'(2) > \mathbf{m}(2)$ and $\mathbf{m}'(1) > \mathbf{m}(1)$. In this case, we have:

$$\begin{aligned}\mathcal{W}(\mathbf{m}) - \mathcal{W}(\mathbf{m}') &= \mathbf{m}(3)(2J - \mathbf{m}(2)) + (2J - \mathbf{m}(1))(\mathbf{m}'(1) - \mathbf{m}(1)) \qquad (1.4.18) \\ &= (A) + (B)\end{aligned}$$

  It is clear that since the maximum occupation number is $2J$, and $N \leq 4J$, (A) and (B) are both nonnegative.

- $2J = \mathbf{m}'(2) = \mathbf{m}(2)$ and $\mathbf{m}'(1) > \mathbf{m}(1)$. In this case we have $(I) = 0$ and then

$$\mathcal{W}(\mathbf{m}) - \mathcal{W}(\mathbf{m}') = (2J - \mathbf{m}(1))(\mathbf{m}'(1) - \mathbf{m}(1))$$

  Since $\mathbf{m}(1) < 2J$ both factors are positive which yields the desired conclusion.

We have shown that minimal configurations can be taken to be supported on two points. Applying some simple quadratic optimization we get,

$$\mathcal{W}(\mathbf{m}) = 2JN - \mathbf{m}(0)\mathbf{m}(1) \geq 2JN - \left\lfloor \frac{N}{2} \right\rfloor \left\lceil \frac{N}{2} \right\rceil = 2JN - \left\lfloor \frac{N^2}{4} \right\rfloor. \tag{1.4.19}$$

$\square$

Proposition 1.4.4 with (1.4.12) shows that for every $N \geq 4J$, we have

$$H_{\Lambda,\omega}^{(N)} \geq 4J^2 \left(1 - \tfrac{2J}{\Delta}\right) . \tag{1.4.20}$$

Theorem 1.4.5. *(Minimal configurations with two or more clusters) Suppose that $N \geq 4J$ and assume that $\mathbf{m}$ has two clusters, that is $\mathbf{m} = \mathbf{m}_1 + \mathbf{m}_2$, and $\mathbf{m}_1, \mathbf{m}_2 \neq 0$, and* $\operatorname{dist}(\operatorname{supp} \mathbf{m}_1, \operatorname{supp} \mathbf{m}_2) \geq 2$. *Then, we have the following lower bound:*

$$\mathcal{W}_\Lambda^{(N)}(\mathbf{m}) \geq \inf_{\substack{k,\ell \geq 1 \\ k+\ell=N}} \left(\mathcal{W}_0^{(k)} + \mathcal{W}_0^{(\ell)}\right) \geq 4J^2 + 2J = Q_1. \tag{1.4.21}$$

Proof. Let $N \geq 4J$. If $J = \frac{1}{2}$ the inequality (1.4.21) is trivially true. Therefore for the rest of the proof we assume that $J \geq 1$. Let $\mathbf{m} = \mathbf{m}_1 + \mathbf{m}_2$ with with $\mathbf{m}_1, \mathbf{m}_2 \neq 0$ and $\operatorname{dist}(\operatorname{supp} \mathbf{m}_1, \operatorname{supp} \mathbf{m}_2) \geq 2$, and let N_1 and N_2 be the particle numbers of $\mathbf{m}_1$ and $\mathbf{m}_2$ respectively, and note that $N_1 + N_2 = N$.

Consider $\mathbf{m_1}$. It follows from (1.4.13) that $\mathcal{W}(\mathbf{m_1}) \geq 2J$ for all $N_1 \in \mathbb{N}$ and $\mathcal{W}(\mathbf{m_1}) = 4J^2$

for $N_1 \geq 4J-1$. The same is true for $\mathbf{m_2}$. If either $N_1 \geq 4J-1$ or $N_2 \geq 4J-1$ we have

$$\mathcal{W}(\mathbf{m}) = \mathcal{W}(\mathbf{m_1}) + \mathcal{W}(\mathbf{m_2}) \geq 4J^2 + 2J. \tag{1.4.22}$$

If $N_1 \leq 4J-2$ and $N_2 \leq 4J-2$, using (1.4.13) and $N = N_1 + N_2 \geq 4J$, we get

$$\begin{aligned}
\mathcal{W}(\mathbf{m}) = \mathcal{W}(\mathbf{m_1}) + \mathcal{W}(\mathbf{m_2}) &\geq 2JN - \tfrac{N_1^2}{4} - \tfrac{N_2^2}{4} \geq JN + \sum_{i=1}^{2}(JN_i - \tfrac{N_i^2}{4}) \qquad (1.4.23)\\
&\geq 4J^2 + \sum_{i=1}^{2} N_i(J - \tfrac{N_i}{4}) \geq 4J^2 + \sum_{i=1}^{2} N_i(J - \tfrac{4J-2}{4}) = 4J^2 + \sum_{i=1}^{2} \tfrac{N_i}{2}\\
&= 4J^2 + \tfrac{N}{2} \geq 4J^2 + 2J.
\end{aligned}$$

□

Remark 1.4.6. Note that the converse is not true when $J \geq 1$. In fact, there are one-cluster configurations with arbitrary large potential $\mathcal{W}_{\Lambda}^{(N)}$. Indeed, fix a value $k \in \mathbb{N}$, let $\#(\Lambda) > k$ and choose $N = \#(\Lambda)$. Defining the one cluster configuration $\mathbf{m}(i) = 1$ for all $i \in \Lambda$, one immediately sees that $\mathcal{W}_{\Lambda}^{(N)}(\mathbf{m}) = 2J\#(\Lambda) - \#(\Lambda) + 1 = (2J-1)\#(\Lambda) + 1 \geq k$.

Remark 1.4.7. Note that if $N < 4J$, then there are configurations $\mathbf{m}$ with two or more clusters such that $\mathcal{W}_{\Lambda}^{(N)}(\mathbf{m}) \leq 4J^2 + 2J - 1$. As a simple example, take any two-cluster configuration with $N = 2$, for which we get $\mathcal{W}_{\Lambda}^{(N)}(\mathbf{m}) = 4J < 4J^2 + 2J - 1$ as long as $J \geq 1$.

For any $K \in \{1, 2, \dots\}$ and any $\delta \in (0,1)$, this motivates the definition of k-cluster energy bands I_k and $I_{k,\delta}$:

$$I_k := \left[4J^2\left(1 - \tfrac{2J}{\Delta}\right), (4J^2 + 2Jk)\left(1 - \tfrac{2J}{\Delta}\right)\right) \tag{1.4.24}$$

$$I_{k,\delta} := \left[4J^2\left(1 - \tfrac{2J}{\Delta}\right), (4J^2 + 2Jk - \delta)\left(1 - \tfrac{2J}{\Delta}\right)\right] . \tag{1.4.25}$$

When considering other energy intervals we define,

$$I(E,\theta) := [E-\theta, E+\theta]. \tag{1.4.26}$$

Remark 1.4.8. From this point forward, if it is necessary to distinguish between different particle numbers and different subintervals we will use the full decoration of $\mathcal{W}$ as $\mathcal{W}_{\Lambda_\ell}^{(N)}$; but otherwise we will always use the convention that $\mathcal{W} := \mathcal{W}_{\Lambda}^{(N)}$.

Remark 1.4.9. We will occasionally use the following notation for the resolvent,

$$R(E) = (H-E)^{-1}. \tag{1.4.27}$$

Decorations may be added or omitted depending on the context, and in all cases decorations on $R(E)$ will correspond directly to decorations on H, for example $R_{\Lambda}^{(N)}(E) = (H_{\Lambda}^{(N)} - E)^{-1}$.

1.5 Main Results

We introduce several fixed constants that will be used throughout the paper, $\xi, \zeta, \beta, \tau, \tilde{\tau}, \kappa \in (0,1)$, and $\gamma > 1$. These constants satisfy the inequalities:

$$0 < \xi < \zeta < \beta < \tfrac{1}{\gamma} < 1 < \gamma < \sqrt{\tfrac{\zeta}{\xi}} \text{ and } \max\left\{\gamma\beta, \frac{(\gamma-1)\beta+1}{\gamma}\right\} < \tau < \tilde{\tau} < 1. \tag{1.5.1}$$

Note that,

$$0 < \xi < \xi^2\gamma < \zeta < \beta < \tfrac{\tau}{\gamma} < \tfrac{1}{\gamma} < \tau < 1 < \tfrac{1-\beta}{\tau-\beta} < \gamma < \tfrac{\tau}{\beta}. \tag{1.5.2}$$

We also say that ζ'', ζ' and, ζ_1 are chosen and fixed constants such that $\zeta < \zeta_1 < \zeta' < \zeta'' < \beta$.

Notice that,

$$\gamma > \tfrac{1-\beta}{\tau-\beta} > \tfrac{1-\zeta}{\tau-\zeta} \implies \zeta < \frac{\gamma\tau - 1}{\gamma - 1}, \tag{1.5.3}$$

therefore we can select ζ_* such that

$$\zeta < \zeta_* < \tfrac{\gamma\tau-1}{\tau-1} < 1. \tag{1.5.4}$$

We will fix $\Delta_0 > 2J$, $\lambda_0 > 0$ and $\delta \in (0,1)$, and repeatedly invoke the hypothesis that the parameters are such that the following equations are satisfied.

$$\max\left\{ \frac{4J(4J^2+2J)}{\bar{\mu}} \left(1 - \tfrac{2J}{\Delta_0}\right) L_0^{-\zeta'}, e^{-\frac{1}{6}L_0^{\beta}} \right\} \leq \lambda_0 \tag{1.5.5}$$

$$e^{-L_0^{\beta}} < \frac{\delta}{2}\left(1 - \tfrac{2J}{\Delta_0}\right) \tag{1.5.6}$$

$$L_0^{-\kappa} < \frac{1}{3}\log\left(1 + \frac{\delta(\Delta_0 - 2J)}{4J(4J^2+2J)}\right) \tag{1.5.7}$$

Notice that $\Delta \geq \Delta_0$, $\lambda \geq \lambda_0$, δ and $L \geq L_0$ also satisfy (1.5.5) through (1.5.7). The following equation will also be required for the starting condition.

$$e^{L_0^{\zeta''}} \leq \Delta\lambda. \tag{1.5.8}$$

Definition 1.5.1. For a given energy interval $I \subset I_{1,\delta}$ and $m > 0$ we will say that an interval $\Lambda_L \subset \mathbb{Z}$ is (m, I)-localizing for $H^{\geq 4J}$ if an eigensystem $\{(\varphi_\nu, \nu)\}_{\nu \in \sigma(H^{\geq 4J}_{\Lambda_L})}$ is (m, I)-localized, that is, for all $\nu \in \sigma(H^{\geq 4J}_{\Lambda_L}) \cap I$ there is $j_\nu \in \Lambda_L$ such that φ_ν is (j_ν, m)-localized:

$$\|P_i^- \varphi_\nu\| \leq e^{-m|i-j_\nu|} \text{ for all } i \in \Lambda_L \text{ with } |i - j_\nu| \geq L^\tau. \tag{1.5.9}$$

We also define the following event,

$$\mathcal{Q}(m, L, I, u) = \{\Lambda_L(u) \text{ is } (m, I) - \text{localizing for } H^{\geq 4J}\}. \tag{1.5.10}$$

Theorem 1.5.2. *Fix $\Delta_0 > 2J$, $\lambda_0 > 0$, and $\delta \in (0, 1)$. Let m_0 satisfy,*

$$L_0^{-\kappa} < m_0 \leq \log\left(1 + \frac{\delta(\Delta_0 - 2J)}{4J(4J^2 + 2J)}\right). \tag{1.5.11}$$

Suppose Δ_0, λ_0, δ, and the scale L_0 satisy (1.5.5)- (1.5.7)*.Further suppose that L_0 is sufficiently large, depending on λ_0 and Δ_0. Moreover, suppose that $\Delta \geq \Delta_0$, and $\lambda \geq \lambda_0$ are chosen satisfying* (1.5.8)*. Then there exists a scale $\mathcal{L} = \mathcal{L}(\Delta, \delta, L_0)$ such that for all $L \geq \mathcal{L}$ we have*

$$\mathbb{P}\{\mathcal{Q}(m_0/4, L, I_{1,\delta}, u)\} \geq 1 - e^{-L^{\xi_2}} \text{ for all } u \in \mathbb{Z}. \tag{1.5.12}$$

Moreover if $\omega \in \mathcal{Q}(m_0/4, L, I_{1,\delta}, u)$ and $\{\varphi_\nu, \nu\}_{\nu \in \sigma(H^{\geq 4J}_{\Lambda_L(u)})}$ is an eigensystem for $H^{\geq 4J}_{\Lambda_L(u)}$, then

$$\sum_{\nu \in \sigma(H_{\Lambda_L(u)}) \cap I_{1,\delta}} \|P_i^- \varphi_\nu\| \|P_j^- \varphi_\nu\| \leq e^{-\frac{m_0}{8}|i-j|} \text{ for all } i, j \in \Lambda_L(u) \text{ with } |i - j| \geq L^{\tilde{\tau}}. \tag{1.5.13}$$

Definition 1.5.3. Given $E \in \mathbb{R}$ and $m > 0$, an interval Λ_L is said to be (m, E)-regular if

$$m > L^{-\kappa} \tag{1.5.14}$$

$$\operatorname{dist}(E, \sigma(H^{\geq 4J}_{\Lambda_L})) > e^{-L^\beta} \tag{1.5.15}$$

$$\|P_i^-(H^{\geq 4J}_{\Lambda_L} - E)^{-1} P^+_{\Lambda_R(i) \cap \Lambda_L}\| \leq e^{-m(R+1)} \text{ for all } i \in \Lambda_L \text{ and } R > L^\tau. \tag{1.5.16}$$

We also introduce the event

$$\mathcal{R}(m, L, I, u, v) = \{E \in I \implies \Lambda_L(u) \text{ or } \Lambda_L(v) \text{ is } (m, E) - \text{regular.}\} \tag{1.5.17}$$

Theorem 1.5.4. *Fix $\Delta_0 > 2J$, $\lambda_0 > 0$, and $\delta \in (0,1)$. Suppose that Δ_0, λ_0, δ, and the scale L_0 satisy* (1.5.5)- (1.5.7)*; that L_0 is sufficiently large, depending on λ_0 and Δ_0; and that $\Delta \geq \Delta_0$, and $\lambda \geq \lambda_0$ are chosen satisfying* (1.5.8)*. Then for all $E \in I_{1,\delta}$ and $L \geq L_0^{\gamma}$ we have*

$$\mathbb{P}\{\mathcal{R}(m_0/4, L, I(E,\theta) \cap I_{1,\delta}, u, v)\} \geq 1 - e^{-L^{\xi}} \text{ for all } u, v \in \mathbb{Z} \text{ with } |u - v| > 2L. \tag{1.5.18}$$

Here m_0 is as in (1.5.11)*, and $\theta = e^{-3L_0}$. It follows that there exists a scale $\mathcal{L} = \mathcal{L}(\Delta, \delta, L_0)$ such that for all $L \geq \mathcal{L}$ we have*

$$\mathbb{P}\{\mathcal{R}(\tfrac{m_0}{4}, L, I_{1,\delta}, u, v)\} \geq 1 - e^{-L^{\xi_1}} \text{ for all } u, v \in \mathbb{Z} \text{ with } |u - v| > 2L. \tag{1.5.19}$$

Notice that Theorem 1.5.2 and Theorem 1.5.4 are both statements about the operator $H^{\geq 4J}$. From looking at (1.4.13) and (1.4.2) we can see that the properties and, in particular the spectra, of $H^{\geq 4J}$ and $H^{<4J}$ are not the same. For everything until Section 1.12, *unless it is explicitly stated otherwise*, it is assumed that $N \geq 4J$ and that H_Λ is shorthand for $H_\Lambda^{\geq 4J}$. However, by replacing $H^{\geq 4J}$ with $H^{<4J}$ in Definition 1.5.3, a theorem similar to 1.5.4 can be proven for $H^{<4J}$; this will be discussed in Section 1.12.

1.6 Combes–Thomas estimate and projection bounds

For any $N \geq 1$ we define the lifted operator

$$H_{\Lambda,k}^{(N)} := H_{\Lambda}^{(N)} + (Q_k - 1)\left(1 - \tfrac{2J}{\Delta}\right) P_{\Lambda,k}^{(N)}. \tag{1.6.1}$$

Lemma 1.6.1. *We have*

$$H_{\Lambda,k}^{(N)} \geq Q_k \left(1 - \tfrac{2J}{\Delta}\right). \tag{1.6.2}$$

Proof. Applying the bound from Lemma 1.4.2 yields

$$\begin{aligned} H_{\Lambda,k}^{(N)} &\geq \left(1 - \tfrac{2J}{\Delta}\right) \mathcal{W} + (Q_k - 1)\left(1 - \tfrac{2J}{\Delta}\right) P_{\Lambda,k}^{(N)} \\ &= \left(1 - \tfrac{2J}{\Delta}\right) \left(\left(\mathcal{W} + (Q_k - 1) P_{\Lambda,k}^{(N)}\right) P_{\Lambda,k}^{(N)} + \left(\mathcal{W} + (Q_k - 1) P_{\Lambda,k}^{(N)}\right) \overline{P}_{\Lambda,k}^{(N)}\right) \\ &\geq \left(1 - \tfrac{2J}{\Delta}\right) \left(Q_k P_{\Lambda,k}^{(N)} + \mathcal{W} \overline{P}_{\Lambda,k}^{(N)}\right) \geq \left(1 - \tfrac{2J}{\Delta}\right) Q_k. \end{aligned} \tag{1.6.3}$$

□

Remark 1.6.2. Since $P_{\Lambda,k}^{(N)}$ is an operator of rank $|\mathbf{M}_{\Lambda,k}^{N}|$, note that this implies that $H_{\Lambda}^{(N)}$ has at most $|\mathbf{M}_{\Lambda,k}^{N}|$ eigenvalues inside I_k.

Corollary 1.6.3. *If $E \in I_{k,\delta}$ then*

$$H_{\Lambda,k}^{(N)} - E \geq \frac{\delta}{Q_k} \left(1 - \tfrac{2J}{\Delta}\right) \mathcal{W} \tag{1.6.4}$$

and in particular

$$\left\| \mathcal{W}^{\frac{1}{2}} (H_{\Lambda,k}^{(N)} - E)^{-1} \mathcal{W}^{\frac{1}{2}} \right\| \leq \frac{Q_k}{\delta \left(1 - \frac{2J}{\Delta}\right)}. \tag{1.6.5}$$

Proof.

$$H_{\Lambda,k}^{(N)} - E \geq (P_{\Lambda,k}^{(N)} + \overline{P}_{\Lambda,k}^{(N)}) \Big(\left(1 - \tfrac{2J}{\Delta}\right) \mathcal{W} \; + (Q_k - 1) \left(1 - \tfrac{2J}{\Delta}\right) P_{\Lambda,k}^{(N)} - E \Big) \tag{1.6.6}$$

$$= P_{\Lambda,k}^{(N)} \Big(\left(1 - \tfrac{2J}{\Delta}\right) \mathcal{W} \; + (Q_k - 1) \left(1 - \tfrac{2J}{\Delta}\right) P_{\Lambda,k}^{(N)} - E \Big) \tag{1.6.7}$$

$$+ \overline{P}_{\Lambda,k}^{(N)} \Big(\left(1 - \tfrac{2J}{\Delta}\right) \mathcal{W} \; + (Q_k - 1) \left(1 - \tfrac{2J}{\Delta}\right) P_{\Lambda,k}^{(N)} - E \Big) = (I) + (II) \,, \tag{1.6.8}$$

where we used Lemma 1.4.2 and the fact that the projections $P_{\Lambda,k}^{(N)}$ and $\overline{P}_{\Lambda,k}^{(N)}$ commute with $\mathcal{W}$. Now we estimate both terms,

$$\begin{aligned}
(I) &\geq \left(1 - \tfrac{2J}{\Delta}\right) P_{\Lambda,k}^{(N)} \left(\mathcal{W} + (Q_k - 1) - (Q_k - \delta)\right) && (1.6.9)\\
&= \left(1 - \tfrac{2J}{\Delta}\right) P_{\Lambda,k}^{(N)} \left(\mathcal{W} - (1 - \delta)\right)\\
&\geq \left(1 - \tfrac{2J}{\Delta}\right) P_{\Lambda,k}^{(N)} (\mathcal{W} - (1-\delta)\mathcal{W}) = \left(1 - \tfrac{2J}{\Delta}\right) \delta \mathcal{W} P_{\Lambda,k}^{(N)}\\
&\geq \tfrac{\delta}{Q_k} \left(1 - \tfrac{2J}{\Delta}\right) \mathcal{W} P_{\Lambda,k}^{(N)}
\end{aligned}$$

and

$$\begin{aligned}
(II) &\geq \overline{P}_{\Lambda,k}^{(N)} (\left(1 - \tfrac{2J}{\Delta}\right) \mathcal{W} - \left(1 - \tfrac{2J}{\Delta}\right) (Q_k - \delta) \tfrac{\mathcal{W}}{Q_k}) && (1.6.10)\\
&= \left(1 - \tfrac{2J}{\Delta}\right) \mathcal{W} \left(1 - \tfrac{Q_k - \delta}{Q_k}\right) \overline{P}_{\Lambda,k}^{(N)} = \tfrac{\delta}{Q_k} \left(1 - \tfrac{2J}{\Delta}\right) \mathcal{W} \overline{P}_{\Lambda,k}^{(N)} .
\end{aligned}$$

Adding the separate inequalites gives the desired result.

□

Proposition 1.6.4. *Let Λ be a finite subgraph of $\mathbb{Z}$. For any $N \in \{1, 2, \ldots, 2J\#(\Lambda)\}$, let $Y_\Lambda^{(N)}$ be an arbitrary multiplication operator on $\ell^2(\mathbf{M}_\Lambda^{(N)})$ and $z \notin \sigma(H_\Lambda^{(N)} + Y_\Lambda^{(N)})$ such that there exists $\kappa_z > 0$ for which*

$$\left\| \mathcal{W}^{\frac{1}{2}} (H_\Lambda^{(N)} + Y_\Lambda^{(N)} - z)^{-1} \mathcal{W}^{\frac{1}{2}} \right\| \leq \frac{1}{\kappa_z} < \infty \,. \tag{1.6.11}$$

Then for all subsets $\Phi, \Psi \subseteq \mathbf{M}_\Lambda^{(N)}$, *we have*

$$\begin{aligned}\left\|\chi_\Phi \left(H_\Lambda^{(N)} + Y_\Lambda^{(N)} - z\right)^{-1} \chi_\Psi\right\| \leq & \frac{1}{\mathcal{W}_0^{(N)}} \left\|\chi_\Phi \mathcal{W}^{\frac{1}{2}} \left(H_\Lambda^{(N)} + Y_\Lambda^{(N)} - z\right)^{-1} \mathcal{W}^{\frac{1}{2}} \chi_\Psi\right\| \\ \leq & \frac{2}{\mathcal{W}_0^{(N)} \kappa_z} e^{-\eta_z d_\Lambda(\Phi,\Psi)}\,,\end{aligned} \tag{1.6.12}$$

where

$$\eta_z = \log\left(1 + \frac{\Delta\kappa_z}{4J}\right). \tag{1.6.13}$$

Proof. Consider $\theta : \mathbf{M}_\Lambda^{(N)} \to \mathbb{R}$ such that

$$\theta_\infty := \sup_{\substack{y \in \mathbf{M}_\Lambda^{(N)} \\ x \sim y}} |\theta(x) - \theta(y)| < \infty \tag{1.6.14}$$

Let $A_\Lambda^{(N)}$, $Y_\Lambda^{(N)}$, $H_\Lambda^{(N)}$, and $V_\Lambda^{(N)}$ be A, Y, H, and V respectively. Let

$$H_\theta = e^\theta H e^{-\theta} = -\frac{1}{2\Delta} A_\theta + V \tag{1.6.15}$$

We have that if $\psi \in \ell^2(\mathbf{M}_\Lambda^{(N)})$ then

$$A_\theta \psi(x) = \sum_{y \sim x} e^{\theta(x) - \theta(y)} A(x,y) \psi(y). \tag{1.6.16}$$

Define

$$B_\theta = -A_\theta + A. \tag{1.6.17}$$

Then we have that

$$|(B_\theta\psi)(x)| = \left| -\sum_{y \sim x} \left(e^{\theta(x)-\theta(y)} - 1\right) A(x,y)\psi(y)\right| \leq \left(e^{\theta_\infty} - 1\right)(A|\psi|)(x) \tag{1.6.18}$$

From this and Lemma 1.4.2 we have that

$$\|\mathcal{W}^{-\frac{1}{2}}B_\theta\mathcal{W}^{-\frac{1}{2}}\psi\|^2 = \sum_x\left|\sum_{y\sim x}\mathcal{W}^{-\frac{1}{2}}(x)\mathcal{W}^{-\frac{1}{2}}(y)(e^{\theta(x)-\theta(y)}-1)A(x,y)\psi(y)\right|^2 \tag{1.6.19}$$
$$\leq \left(e^{\theta_\infty}-1\right)^2\sum_x\left(\sum_{y\sim x}\mathcal{W}^{-\frac{1}{2}}(x)\mathcal{W}^{-\frac{1}{2}}(y)A(x,y)|\psi(y)|\right)^2$$
$$\leq \left(e^{\theta_\infty}-1\right)^2\|\mathcal{W}^{-\frac{1}{2}}A\mathcal{W}^{-\frac{1}{2}}\|^2\cdot\|\psi\|^2 \leq (4J)^2(e^{\theta_\infty}-1)^2\cdot\|\psi\|^2.$$

This implies

$$\|\mathcal{W}^{-\frac{1}{2}}B_\theta\mathcal{W}^{-\frac{1}{2}}\| \leq 4J(e^{\theta_\infty}-1). \tag{1.6.20}$$

as well,

$$\|\mathcal{W}^{-\frac{1}{2}}(H_\theta-Y-z)\mathcal{W}^{-\frac{1}{2}}-\mathcal{W}^{-\frac{1}{2}}(H-Y-z)\mathcal{W}^{-\frac{1}{2}}\| = \frac{1}{2\Delta}\|W^{-\frac{1}{2}}B_\theta\mathcal{W}^{-\frac{1}{2}}\|. \tag{1.6.21}$$

Let κ_z be as in Equation (1.6.11). Then suppose we select θ such that θ_∞ satisfies the following

$$\frac{4J}{2\Delta}(e^{\theta_\infty}-1)\kappa_z^{-1} \leq \frac{1}{2} \text{ i.e. } \theta_\infty \leq \log\left(1+\frac{\Delta\kappa_z}{4J}\right) = \eta_z. \tag{1.6.22}$$

Using Neumann Series we get,

$$\|\mathcal{W}^{\frac{1}{2}}(H_\theta-Y-z)^{-1}\mathcal{W}^{\frac{1}{2}}\| = \|(I)(II)\| \leq \frac{2}{\kappa_z}, \tag{1.6.23}$$

where

$$(I) = \mathcal{W}^{\frac{1}{2}}(H-Y-z)^{-1}\mathcal{W}^{\frac{1}{2}} \tag{1.6.24}$$
$$(II) = \left(1-\frac{1}{2\Delta}(\mathcal{W}^{-\frac{1}{2}}B_\theta\mathcal{W}^{-\frac{1}{2}})(\mathcal{W}^{\frac{1}{2}}(H-Y-z)^{-1}\mathcal{W}^{\frac{1}{2}})\right)^{-1}. \tag{1.6.25}$$

Suppose Φ and Ψ are subsets of $\mathbf{M}_\Lambda^{(N)}$ and $R > 1$. Let $\theta_R : \mathbf{M}_\Lambda^{(N)} \to \mathbb{R}$ be given by

$$\theta_R(x) = -\log\left(1 + \frac{\Delta\kappa_z}{4J}\right)(d_\Lambda(x, \Phi) \wedge R). \tag{1.6.26}$$

Notice that θ_R satisfies (1.6.14) and also that $\chi_\Phi\theta_R = 0$. This implies

$$\begin{aligned} \|\chi_\Phi \mathcal{W}^{\frac{1}{2}}(H + Y - z)^{-1}\mathcal{W}^{\frac{1}{2}}\chi_\Psi\| &= \|\chi_\Phi \mathcal{W}^{\frac{1}{2}}(H_{\theta_R} + Y - z)^{-1}\mathcal{W}^{\frac{1}{2}} e^{\theta_R}\chi_\Psi\| \\ &\leq \|\mathcal{W}^{\frac{1}{2}}(H_{\theta_R} + Y - z)^{-1}\mathcal{W}^{\frac{1}{2}}\| \cdot \|e^{\theta_R}\chi_\Psi\| \\ &\leq \frac{2}{\kappa_z} e^{-\log\left(1+\frac{\Delta\kappa_z}{4J}\right)(d_\Lambda(\Phi,\Psi)\wedge R)} \end{aligned} \tag{1.6.27}$$

This estimate holds for all R. So taking $R \to \infty$ completes the proof. □

Corollary 1.6.5. *Suppose $E \in I_{k,\delta}$ and that Φ and Ψ are subsets of $\mathbf{M}_\Lambda^{(N)}$. We have that*

$$\left\|\chi_\Phi \mathcal{W}^{\frac{1}{2}}(H_{\Lambda,k}^{(N)} - E)^{-1}\mathcal{W}^{\frac{1}{2}}\chi_\Psi\right\| \leq \frac{2Q_k}{\delta\left(1 - \frac{2J}{\Delta}\right)\mathcal{W}_0^{(N)}} e^{-\log\left(1+\frac{\delta(\Delta-2J)}{4JQ_k}\right)d_\Lambda(\Phi,\Psi)}. \tag{1.6.28}$$

Proof. In light of Corollary 1.6.3 we can apply Proposition 1.6.4 with $\kappa_z = \delta\left(1 - \frac{2J}{\Delta}\right)/Q_k$
. □

Lemma 1.6.6. *Let $N \geq 1$ and let $k \leq N$ be a natural number, let $\Lambda = \Lambda_L(i)$, let $E \in I_{k,\delta} \setminus \sigma(H_\Lambda^{(N)})$, let $\Theta \subset \mathbf{M}_\Lambda^{(N)}$, and let $0 \leq q \leq s \leq L$, so $\Lambda_q(i) \subset \Lambda_s(i)$. Further suppose that $\mathbf{S}_{\Lambda_q(i)} \cap \Theta \subset \overline{\mathbf{M}_{\Lambda,k}^{(N)}}$, and that*

$$q < s \leq d_\Lambda(\mathbf{S}_{\Lambda_q(i)} \cap \Theta, \mathbf{M}_{\Lambda,k}^{(N)}) + q. \tag{1.6.29}$$

Then for all $\Psi \subset \mathbf{M}_{\Lambda}^{(N)}$ *we have*

$$\|P^-_{\Lambda_q(i)}\chi_\Theta(H_\Lambda^{(N)}-E)^{-1}\chi_\Psi\| \le \frac{2\sqrt{Q_k}}{\delta\left(1-\frac{2J}{\Delta}\right)\mathcal{W}_0^{(N)}} e^{-\log\left(1+\frac{\delta(\Delta-2J)}{4JQ_k}\right)d_\Lambda(\mathbf{S}_{\Lambda_q(i)}\cap\Theta,\Psi)}$$
$$+\frac{2(Q_k-1)\sqrt{Q_k}}{\delta\mathcal{W}_0^{(N)}}\sum_{r\in\Lambda} e^{-\log\left(1+\frac{\delta(\Delta-2J)}{4JQ_k}\right)\left(1-\frac{q}{s}\right)\max\{|r-i|,s\}}\left\|P_r^-\left(H_\Lambda^{(N)}-E\right)^{-1}\chi_\Psi\right\| \quad (1.6.30)$$

Proof. We introduce the shorthand $\Lambda_q = \Lambda_q(i)\cap\Lambda$ and likewise for Λ_s, we also use $R_\Lambda^{(N)}(E) = (H_\Lambda^{(N)}-E)^{-1}$ and $R_{\Lambda,k}^{(N)}(E) = (H_{\Lambda,k}^{(N)}-E)^{-1}$. From the resolvent identity, we get

$$R_\Lambda^{(N)}(E) = R_{\Lambda,k}^{(N)}(E) - (Q_k-1)\left(1-\tfrac{2J}{\Delta}\right)R_{\Lambda,k}^{(N)}(E)P_{\Lambda,k}^{(N)}R_\Lambda^{(N)}(E) \quad (1.6.31)$$

It follows that

$$P^-_{\Lambda_q}\chi_\Theta\mathcal{W}^{\frac12}R_\Lambda^{(N)}(E)\chi_\Psi = P^-_{\Lambda_q}\chi_\Theta\mathcal{W}^{\frac12}R_{\Lambda,k}^{(N)}(E)\chi_\Psi$$
$$-(Q_k-1)\left(1-\tfrac{2J}{\Delta}\right)P^-_{\Lambda_q}\chi_\Theta\mathcal{W}^{\frac12}R_{\Lambda,k}^{(N)}(E)P_{\Lambda,k}^{(N)}R_\Lambda^{(N)}(E)\chi_\Psi \quad (1.6.32)$$

We will focus on a portion of the second term.

$$\mathcal{W}^{\frac12}R_{\Lambda,k}^{(N)}(E)P_{\Lambda,k}^{(N)}R_\Lambda^{(N)}(E) = \mathcal{W}^{\frac12}R_{\Lambda,k}^{(N)}(E)\mathcal{W}^{\frac12}P^+_{\Lambda_s}P_{\Lambda,k}^{(N)}\mathcal{W}^{-\frac12}R_\Lambda^{(N)}(E)$$
$$+\mathcal{W}^{\frac12}R_{\Lambda,k}^{(N)}(E)\mathcal{W}^{\frac12}P^-_{\Lambda_s}P_{\Lambda,k}^{(N)}\mathcal{W}^{-\frac12}R_\Lambda^{(N)}(E) \quad (1.6.33)$$

Observe that

$$\mathcal{W}P^-_{\Lambda_q}\chi_\Theta \ge Q_kP^-_{\Lambda_q}\chi_\Theta \text{ since we assumed that } \mathbf{S}_{\Lambda_q}\cap\Theta\subset\overline{\mathbf{M}_{\Lambda,k}^{(N)}}\,. \quad (1.6.34)$$

Therefore, we get

$$
\begin{aligned}
\sqrt{Q_k}\left\|P^-_{\Lambda_q}\chi_\Theta R^{(N)}_\Lambda(E)\chi_\Psi\right\| &\le \left\|P^-_{\Lambda_q}\chi_\Theta \mathcal{W}^{\frac{1}{2}} R^{(N)}_\Lambda(E)\chi_\Psi\right\| \\
&\le \left\|P^-_{\Lambda_q}\chi_\Theta \mathcal{W}^{\frac{1}{2}} R^{(N)}_{\Lambda,k}(E)\chi_\Psi\right\| \\
&+ (Q_k-1)\left(1-\tfrac{2J}{\Delta}\right)\left\|P^-_{\Lambda_q}\chi_\Theta \mathcal{W}^{\frac{1}{2}} R^{(N)}_{\Lambda,k}(E)\mathcal{W}^{\frac{1}{2}}P^{(N)}_{\Lambda,k}\right\|\left\|P^{(N)}_{\Lambda,k}P^-_{\Lambda_s}\mathcal{W}^{-\frac{1}{2}}\mathbb{R}^{(N)}_\Lambda(E)\chi_\Psi\right\| \\
&+ (Q_k-1)\left(1-\tfrac{2J}{\Delta}\right)\left\|P^-_{\Lambda_s}\chi_\Theta \mathcal{W}^{\frac{1}{2}} R^{(N)}_{\Lambda,k}(E)\mathcal{W}^{\frac{1}{2}}P^{(N)}_{\Lambda,k}P^+_{\Lambda_s}\mathcal{W}^{-\frac{1}{2}}R^{(N)}_\Lambda(E)\chi_\Psi\right\| \\
&= (I)+(II)+(III)\,.
\end{aligned} \tag{1.6.35}
$$

Using Corollary 1.6.5, we have

$$
(I) \le \frac{2Q_k}{\delta\left(1-\frac{2J}{\Delta}\right)\mathcal{W}^{(N)}_0} e^{-\log\left(1+\frac{\delta(\Delta-2J)}{4JQ_k}\right)\mathrm{dist}_\Lambda(\mathbf{S}_{\Lambda_q}\cap\Theta,\Psi)} \tag{1.6.36}
$$

Also,

$$
(II) \le \frac{2(Q_k-1)Q_k}{\delta\mathcal{W}^{(N)}_0} e^{-\log\left(1+\frac{\delta(\Delta-2J)}{4JQ_k}\right)\mathrm{dist}_\Lambda(\mathbf{S}_{\Lambda_q}\cap\Theta,\mathbf{M}^{(N)}_{\Lambda,k})}\left\|P^-_{\Lambda_s}R^{(N)}_\Lambda(E)\chi_\Psi\right\|\,. \tag{1.6.37}
$$

To estimate (III), we use (1.2.20) in order to get

$$
\begin{aligned}
&\frac{(III)}{(Q_k-1)\left(1-\frac{2J}{\Delta}\right)} \leq \qquad (1.6.38)\\
&\left\| P^{-}_{\Lambda_q}\mathcal{W}^{\frac{1}{2}}R^{(N)}_{\Lambda,k}(E)\mathcal{W}^{\frac{1}{2}}P^{(N)}_{\Lambda,k}\left(\sum_{t=0}^{\infty}P^{+}_{\Lambda_{t+s}}P^{-}_{\{i+t+s+1,i-t-s-1\}}\right)\mathcal{W}^{-\frac{1}{2}}R^{(N)}_{\Lambda}(E)\chi_{\Psi}\right\|\\
\leq&\sum_{t=0}^{\infty}\left\|P^{-}_{\Lambda_q}\mathcal{W}^{\frac{1}{2}}R^{(N)}_{\Lambda,k}(E)\mathcal{W}^{\frac{1}{2}}P^{+}_{\Lambda_{s+t}}\right\|\left\|P^{-}_{\{i+t+s+1,i-t-s-1\}}R^{(N)}_{\Lambda}(E)\chi_{\Psi}\right\|\\
\leq&\frac{2Q_k\delta^{-1}}{\left(1-\frac{2J}{\Delta}\right)\mathcal{W}^{(N)}_0}\sum_{t=0}^{\infty}\mathrm{e}^{-\log\left(1+\frac{\delta(\Delta-2J)}{4JQ_k}\right)\mathrm{dist}_{\Lambda}(\mathbf{S}_{\Lambda_q},\mathbf{M}^{(N)}_{\Lambda\setminus\Lambda_{s+t}})}\\
\times&\left(\left\|P^{-}_{i+s+t+1}R^{(N)}_{\Lambda}(E)\chi_{\Psi}\right\|+\left\|P^{-}_{i-s-t-1}R^{(N)}_{\Lambda}(E)\chi_{\Psi}\right\|\right)\\
\leq&\frac{2Q_k}{\delta\left(1-\frac{2J}{\Delta}\right)\mathcal{W}^{(N)}_0}\sum_{t=0}^{\infty}\mathrm{e}^{-\log\left(1+\frac{\delta(\Delta-2J)}{4JQ_k}\right)(s+t+1-q)}\\
\times&\left(\left\|P^{-}_{i+s+t+1}R^{(N)}_{\Lambda}(E)\chi_{\Psi}\right\|+\left\|P^{-}_{i-s-t-1}R^{(N)}_{\Lambda}(E)\chi_{\Psi}\right\|\right)\\
\leq&\frac{2Q_k}{\delta\left(1-\frac{2J}{\Delta}\right)\mathcal{W}^{(N)}_0}\sum_{r\in\Lambda\setminus\Lambda_s}\mathrm{e}^{-\log\left(1+\frac{\delta(\Delta-2J)}{4JQ_k}\right)(|r-i|-q)}\left\|P^{-}_{r}R^{(N)}_{\Lambda}(E)\chi_{\Psi}\right\|
\end{aligned}
$$

Combining our bounds for (I), (II), and (III) we get that

$$
\begin{aligned}
\|P^{-}_{\Lambda_q}\chi_{\Theta}R^{(N)}_{\Lambda}(E)\chi_{\Psi}\|\leq&\frac{2\sqrt{Q_k}}{\delta\left(1-\frac{2J}{\Delta}\right)\mathcal{W}^{(N)}_0}e^{-\log\left(1+\frac{\delta(\Delta-2J)}{4JQ_k}\right)\mathrm{dist}_{\Lambda}(\mathbf{S}_{\Lambda_{q(i)}}\cap\Theta,\Psi)} \qquad (1.6.39)\\
&+\frac{2(Q_k-1)\sqrt{Q_k}}{\delta\mathcal{W}^{(N)}_0}e^{-\log\left(1+\frac{\delta(\Delta-2J)}{4JQ_k}\right)\mathrm{dist}_{\Lambda}(\mathbf{S}_{\Lambda_q}\cap\Theta,\mathbf{M}^{(N)}_{\Lambda,k})}\left\|P^{-}_{\Lambda_s}R^{(N)}_{\Lambda}(E)\chi_{\Psi}\right\|\\
&+\frac{2(Q_k-1)\sqrt{Q_k}}{\delta\mathcal{W}^{(N)}_0}\sum_{r\in\Lambda\setminus\Lambda_s}\mathrm{e}^{-\log\left(1+\frac{\delta(\Delta-2J)}{4JQ_k}\right)(|r-i|-q)}\left\|P^{-}_{r}R^{(N)}_{\Lambda}(E)\chi_{\Psi}\right\|\\
&\leq\frac{2\sqrt{Q_k}}{\delta\left(1-\frac{2J}{\Delta}\right)\mathcal{W}^{(N)}_0}e^{-\log\left(1+\frac{\delta(\Delta-2J)}{4JQ_k}\right)\mathrm{dist}_{\Lambda}(\mathbf{S}_{\Lambda_q}\cap\Theta,\Psi)}\\
&+\frac{2(Q_k-1)\sqrt{Q_k}}{\delta\mathcal{W}^{(N)}_0}\sum_{r\in\Lambda}\mathrm{e}^{-\log\left(1+\frac{\delta(\Delta-2J)}{4JQ_k}\right)\left(1-\frac{q}{s}\right)\max\{|r-i|,s\}}\left\|P^{-}_{r}R^{(N)}_{\Lambda}(E)\chi_{\Psi}\right\|
\end{aligned}
$$

$\square$

Corollary 1.6.7. *Assume the hypotheses of Lemma 1.6.6 and suppose that* (ψ, E) *is an eigenpair for* $H_\Lambda^{(N)}$*. Then*

$$\|P^-_{\Lambda_q(i)}\chi_\theta\psi\| \leq \frac{2(Q_k-1)\sqrt{Q_k}}{\delta\mathcal{W}_0^{(N)}}\sum_{r\in\Lambda} e^{-\log\left(1+\frac{\delta(\Delta-2J)}{4JQ_k}\right)(1-\frac{q}{s})\max\{|r-i|,s\}}\|P_r^-\psi\|. \tag{1.6.40}$$

Proof. Recalling (1.6.1) we have that

$$\left(H_{\Lambda,k}^{(N)} - E\right)\psi = (Q_k-1)\left(1-\tfrac{2J}{\Delta}\right)P_{\Lambda,k}^{(N)}\psi \tag{1.6.41}$$

Since $E \in I_{k,\delta}$ we have (1.6.4), so

$$\psi = (Q_k-1)\left(1-\tfrac{2J}{\Delta}\right)R_{\Lambda,k}^{(N)}(E)P_{\Lambda,k}^{(N)}\psi. \tag{1.6.42}$$

It follows that

$$\begin{aligned} P^-_{\Lambda_q}\chi_\theta\mathcal{W}^{\frac{1}{2}}\psi_E &= (Q_k-1)\left(1-\tfrac{2J}{\Delta}\right)P^-_{\Lambda_q}\chi_\theta\mathcal{W}^{\frac{1}{2}}R_{\Lambda,k}^{(N)}(E)\mathcal{W}^{\frac{1}{2}}\mathcal{W}^{-\frac{1}{2}}P_{\Lambda,k}^{(N)}\psi \\ &= (Q_k-1)\left(1-\tfrac{2J}{\Delta}\right)P^-_{\Lambda_q}\chi_\theta\mathcal{W}^{\frac{1}{2}}R_{\Lambda,k}^{(N)}(E)\mathcal{W}^{\frac{1}{2}}\mathcal{W}^{-\frac{1}{2}}P_{\Lambda,k}^{(N)}(P^-_{\Lambda_s}+P^+_{\Lambda_s})\psi. \end{aligned} \tag{1.6.43}$$

Compare (1.6.43) to (1.6.32); they are identical except that the first term of (1.6.32) is not present in (1.6.43), and ψ has been substituted for $(H_\Lambda^{(N)} - E)^{-1}\chi_\psi$. In order to estimate (1.6.43) we will proceed in the same way as for (1.6.32) and arrive at a version of (1.6.39)

$$\begin{aligned} \|P^-_{\Lambda_q}\chi_\theta\psi\| \leq{}& \frac{2(Q_k-1)\sqrt{Q_k}}{\delta\mathcal{W}_0^{(N)}}e^{-\log\left(1+\frac{\delta(\Delta-2J)}{4JQ_k}\right)\operatorname{dist}_\Lambda(\mathbf{S}_{\Lambda_q}\cap\Theta,\mathbf{M}_{\Lambda,k}^{(N)})}\left\|P^-_{\Lambda_s}\psi\right\| \\ &+\frac{2(Q_k-1)\sqrt{Q_k}}{\delta\mathcal{W}_0^{(N)}}\sum_{r\in\Lambda\setminus\Lambda_s}\mathrm{e}^{-\log\left(1+\frac{\delta(\Delta-2J)}{4JQ_k}\right)(|r-i|-q)}\left\|P_r^-\psi\right\|. \end{aligned} \tag{1.6.44}$$

□

1.7 Large Deviation and Wegner Estimates

1.7.1 Large deviation

Let $N \in \{4J, 4J+1, \ldots, 2J\#(\Lambda)\}$, $k \in \{1, 2, \ldots, N\}$, and $\Lambda = \Lambda_\ell$. Let $\bar{\mu} = \mathcal{E}\{\omega_0\}$, assume

$$\lceil N/2J \rceil \lambda \bar{\mu} \geq 2Q_k \left(1 - \tfrac{2J}{\Delta}\right). \tag{1.7.1}$$

Let N_k be defined as,

$$N_k = \frac{4JQ_k \left(1 - \frac{2J}{\Delta}\right)}{\lambda \bar{\mu}} \tag{1.7.2}$$

Notice that if $N \geq N_k$ then (1.7.1) holds, which can be seen from a calculation. For any $\mathbf{m} \in \mathbf{M}_\Lambda^{(N)}$ note that $V_\omega(\mathbf{m}) = \sum_{i \in \Lambda} \omega(i)\mathbf{m}(i) \geq \sum_{i:\mathbf{m}(i) \neq 0} \omega(i) =: \tilde{V}_\omega(\mathbf{m})$. Now, since $\mathbf{m} \in \mathbf{M}_\Lambda^{(N)}$, which implies that $\sum_{i \in \Lambda} \mathbf{m}(i) = N$ and since $0 \leq \mathbf{m}(i) \leq 2J$, this implies that $|\operatorname{supp}(\mathbf{m})| \geq \lceil N/2J \rceil$ and thus $V_\omega(\mathbf{m})$ can be bounded from below by a sum of at least $\lceil N/2J \rceil$ i.i.d. random variables. From (1.7.1), the standard large deviation estimate gives

$$\begin{aligned} \mathbb{P}\left\{\lambda V_\omega(\mathbf{m}) < Q_k \left(1 - \tfrac{2J}{\Delta}\right)\right\} &\leq \mathbb{P}\left\{\lambda \tilde{V}_\omega(\mathbf{m}) < Q_k \left(1 - \tfrac{2J}{\Delta}\right)\right\} \\ &\leq \mathbb{P}\left\{\tilde{V}_\omega(\mathbf{m}) < \lceil N/2J \rceil \tfrac{\bar{\mu}}{2}\right\} \leq \mathrm{e}^{-c_\mu N} \text{ for all } \mathbf{m} \in \mathbf{M}_\Lambda^{(N)}, \end{aligned} \tag{1.7.3}$$

where c_μ is a constant depending only on the probability distribution μ and J. It implies that we have

$$\mathbb{P}\{\lambda V_\omega(\mathbf{m}) < Q_k \left(1 - \tfrac{2J}{\Delta}\right)\} \leq e^{c_\mu N_k} \mathrm{e}^{-c_\mu N} \text{ for all } N \in \mathbb{N}. \tag{1.7.4}$$

Consequently,

$$\mathbb{P}\left\{\lambda V_\omega(\mathbf{m}) < Q_k\left(1-\tfrac{2J}{\Delta}\right) \text{ for some } \mathbf{m}\in \mathbf{M}_{\Lambda,k}^{(N)}\right\} \le \left|\mathbf{M}_{\Lambda,k}^{(N)}\right| e^{c_\mu N_k} \mathrm{e}^{-c_\mu N}\ . \tag{1.7.5}$$

In order to estimate $|\mathbf{M}_{\Lambda,k}^{(N)}|$, we use an upper bound that is implied from following several results in [43]. From Equations (2.1.2), (3.29), and (3.37) in [43] we get the following bound:

$$|\mathbf{M}_{\Lambda,k}^{(N)}| \le (4J\mathrm{e})^{\frac{Q_k}{J}-2}(2\ell+1)^{\frac{2Q_k}{J}-3} \le \ell^{6Q_k}. \tag{1.7.6}$$

Where $\Lambda = \Lambda_\ell$ and the last inequality holds for ℓ large enough. We therefore conclude

$$\begin{aligned} &\mathbb{P}\left\{\lambda V_\omega(\mathbf{m}) < Q_k\left(1-\tfrac{2J}{\Delta}\right) \text{ for some } \mathbf{m}\in \mathbf{M}_{\Lambda,k}^{(N)}\right\} \\ &\quad \le e^{c_\mu N_k} \ell^{6Q_k} \mathrm{e}^{-c_\mu N}\ . \end{aligned} \tag{1.7.7}$$

This implies that with probability greater than $1 - e^{c_\mu N_0}\ell^{6Q_k}\mathrm{e}^{-c_\mu N}$, we have

$$\lambda V_\omega P_{\Lambda,k}^{(N)} \ge Q_k\left(1-\tfrac{2J}{\Delta}\right) P_{\Lambda,k}^{(N)}\ , \tag{1.7.8}$$

which implies

$$\begin{aligned} H_\Lambda^{(N)} &\ge \left(1-\tfrac{2J}{\Delta}\right)\mathcal{W} + \lambda V_\omega \\ &= \left[\left(1-\tfrac{2J}{\Delta}\right)\mathcal{W} + \lambda V_\omega\right] P_{\Lambda,k}^{(N)} + \left[\left(1-\tfrac{2J}{\Delta}\right)\mathcal{W} + \lambda V_\omega\right] \bar{P}_{\Lambda,k}^{(N)} \\ &\ge \left(1-\tfrac{2J}{\Delta}\right)\left(4J^2+Q_k\right) P_{\Lambda,k}^{(N)} + \left(1-\tfrac{2J}{\Delta}\right) Q_k \bar{P}_{\Lambda,k}^{(N)} \\ &\ge Q_k\left(1-\tfrac{2J}{\Delta}\right)\ , \end{aligned} \tag{1.7.9}$$

where we used the fact that for $N \geq 4J$, one has by Proposition 1.4.4 that $\mathcal{W} \geq 4J^2$. In addition, for any $E \in I_{k,\delta}$, it was shown in [43, Eqns. 2.34-2.36] that

$$\mathcal{W}^{-\frac{1}{2}} \left(H_\Lambda^{(N)} + \lambda V_\omega - E \right) \mathcal{W}^{-\frac{1}{2}} \geq \frac{\delta \left(1 - \frac{2J}{\Delta}\right)}{Q_k} \,. \tag{1.7.10}$$

Definition 1.7.1. We say that an interval Λ is (k, N)-reduced if (1.7.8) holds. That is,

$$\lambda V_\omega P_{\Lambda,k}^{(N)} \geq Q_k \left(1 - \tfrac{2J}{\Delta}\right) P_{\Lambda,k}^{(N)}. \tag{1.7.11}$$

Note that being (k, N)-reduced is an event, which we will call $\mathcal{Y}_{\Lambda,k}^{(N)}$ from here on. Also if Λ is (k, N)-reduced for all $N > \ell^{\zeta'}$ then we say that Λ is $(k, \mathcal{N})$ reduced; this event is $\mathcal{Y}_{\Lambda,k}$. Notice that in particular we have

$$\mathcal{Y}_{\Lambda,k} \subset \{ I_k \cap \sigma(H_\Lambda^{(N)}) = \emptyset \text{ for all } N > \ell^{\zeta'} \}. \tag{1.7.12}$$

Lemma 1.7.2. *Suppose that Λ is $(1, \mathcal{N})$-reduced. Then for any $N > \ell^{\zeta'}$, any energy $E \in I_{1,\delta}$ and any two sets of configurations $\Phi, \Psi \in \mathbf{M}_\Lambda^{(N)}$, we have that the Combes-Thomas Estimate, Proposition 1.6.4, holds:*

$$\left\| \chi_\Phi \left(H_\Lambda^{(N)} + \lambda V_\omega - E \right)^{-1} \chi_\Psi \right\| \leq \frac{2Q_1}{4J^2 \delta \left(1 - \frac{2J}{\Delta}\right)} e^{-\log\left(1 + \frac{\delta(\Delta - 2J)}{4JQ_1}\right) \operatorname{dist}_1(\Phi,\Psi)}. \tag{1.7.13}$$

Notice that the bound is uniform for $E \in I_{1,\delta}$. We also have the following probability estimate

$$\mathbb{P}\left\{ \Lambda \textit{ is } (1, \mathcal{N})\textit{-reduced} \right\} = \mathbb{P}\{\mathcal{Y}_{\Lambda,1}\} \geq 1 - e^{-\frac{c_\mu}{4} \ell^{\zeta'}}. \tag{1.7.14}$$

Proof. Equation (1.7.13) follows from Proposition 1.6.4 using (1.7.10) and noting that for $N > 4J$, we have by Proposition 1.4.4 that $\mathcal{W}_0^{(N)} = 4J^2$.

When we consider (1.7.7) we can estimate the probability that Λ is $(1,\mathcal{N})$-reduced. The equation is rephrased

$$\mathbb{P}\{\Lambda \text{ is not } (1,N)-\text{reduced}\} \leq e^{c_\mu N_1}\ell^{6Q_1}\mathrm{e}^{-c_\mu N} < e^{-\frac{c_\mu}{2}N}\,. \tag{1.7.15}$$

Which is true assuming that (1.7.1) holds, $N > \ell^{\zeta'}$, and that,

$$\ell^{\zeta'} > \frac{2}{c_\mu}\left(\frac{4JQ_1\left(1-\frac{2J}{\Delta}\right)}{\lambda\bar{\mu}} + 6Q_1\log(\ell)\right) = \tfrac{2}{c_\mu}\left(N_1 + 6Q_1\log(\ell)\right). \tag{1.7.16}$$

This immediately yields the estimate

$$\mathbb{P}\left\{\mathcal{Y}^c_{1,\Lambda}\right\} \leq \sum_{N\geq\ell^{\zeta'}}\mathbb{P}\{\Lambda \text{ is not } (1,N)-\text{reduced}\} \leq \frac{e^{-\frac{c_\mu}{2}\ell^{\zeta'}}}{1-e^{-\frac{c_\mu}{2}\ell^{\zeta'}}} < e^{-\frac{c_\mu}{4}\ell^{\zeta'}}\,. \tag{1.7.17}$$

This finishes the proof. □

1.7.2 Wegner estimates

Let μ be a probability measure on $\mathbb{R}$ with concentration, $S_\mu(t) := \sup_{a\in\mathbb{R}}\mu\{[a,a+t]\}$, $t \geq 0$. Note that in our application $S_\mu(t) \leq Kt^\alpha$ for some $\alpha \in (0,1]$. Given a finite index set A, then denote by μ^A the A-fold product measure of μ on $\mathbb{R}^A$.

Lemma 1.7.3. *[81] Given a finite index set A, consider a function Φ on $\mathbb{R}^A$ which satisfies $\Phi(q+te)-\Phi(q) \geq t$ for $e=(1,1,\ldots,1)\in\mathbb{R}^A$ and all $t>0$. Then, for any open interval $I\subset\mathbb{R}$, we have*

$$\mu^A\{q : \Phi(q)\in I\} \leq |A|S_\mu(|I|). \tag{1.7.18}$$

Given an operator A and an interval I, we set $\sigma_I(A) = \sigma(A)\cap I$.

Lemma 1.7.4. *Consider an open interval $I \subset I_k$. Then*

$$\mathbb{P}\left\{\sigma_I(H_\Lambda^{(N)}) \neq \emptyset\right\} \leq K\,|I|^\alpha\,(N\lambda)^{-\alpha}(2\ell+1)\left|\mathbf{M}_{\Lambda,k}^{(N)}\right| \tag{1.7.19}$$

Proof. Let $E_1 \leq E_2 \leq \ldots$ be the eigenvalues of $H_\Lambda^{(N)}$ in I, counted with multiplicity, which we consider as functions of ω_Λ. By Remark 1.6.2, there are at most $\left|\mathbf{M}_{\Lambda,k}^{(N)}\right|$ of them. Each $E_n(\omega_\Lambda)$ is a monotone function on $\mathbb{R}^{\#(\Lambda)}$. Let $e = (1, 1, \ldots, 1)$ as in Lemma 1.7.3 with $A = \Lambda$. We have $E_n(\tilde\omega + te) - E_n(\tilde\omega) = N\lambda t$ for all $t > 0$ and all n by the min-max principle and can apply Lemma 1.7.3 with $\Phi = (N\lambda)^{-1}\,E_n$, to get – using that $S_\mu(|I|) \leq K|I|^\alpha$ –

$$\mathbb{P}\{\omega_\Lambda : E_n(\omega_\Lambda) \in I\} \leq K\,|I|^\alpha\,(N\lambda)^{-\alpha}(2\ell+1). \tag{1.7.20}$$

Using (1.7.20) for each one of the eigenvalues E_n yields (1.7.19). □

In particular, we have the following Wegner estimates for $H_\Lambda^{(N)}$.

Lemma 1.7.5. *Consider an open interval $I \subset I_k$. Then*

$$\mathbb{P}\left\{\sigma_I(H_\Lambda^{(N)}) \neq \emptyset\right\} \leq K\,|I|^\alpha\,\lambda^{-\alpha}\ell^{8Q_k}. \tag{1.7.21}$$

Proof. Lemma 1.7.4 and Equation (1.7.6) give

$$\begin{aligned}\mathbb{P}\left\{\sigma_I(H_\Lambda^{(N)}) \neq \emptyset\right\} &\leq K\,|I|^\alpha\,(N\lambda)^{-\alpha}(2\ell+1)\left|\mathbf{M}_{\Lambda,k}^{(N)}\right| \\ &\leq K|I|^\alpha(N\lambda)^{-\alpha}(2\ell+1)\ell^{6Q_k} \leq K|I|^\alpha\lambda^{-\alpha}\ell^{8Q_k}\end{aligned} \tag{1.7.22}$$

This holds as long as ℓ is larger than $N^{-\alpha/6Q_k}$. □

This result, with (1.7.12) and (1.7.16), yields the following lemma.

Lemma 1.7.6. *Let I be an open interval such that $I \subset I_1$. Then*

$$\mathbb{P}\left\{\sigma_I(H_\Lambda^{(N)}) \neq \emptyset \text{ for some } N \leq \ell^{\zeta'}\right\} \leq K\,|I|^\alpha\,\lambda^{-\alpha}\ell^{8Q_1+1} \tag{1.7.23}$$

In addition, if (1.7.16) *holds we have*

$$\mathbb{P}\left\{\sigma_I(H_\Lambda) \neq \emptyset\right\} \leq K\,|I|^\alpha\,\lambda^{-\alpha}\ell^{8Q_1+1} + e^{-\frac{c_\mu}{4}\ell^{\zeta'}}. \tag{1.7.24}$$

Proof. It follows from Lemma 1.7.5 that

$$\begin{aligned}\mathbb{P}\left\{\sigma_I(H_\Lambda^{(N)}) \neq \emptyset \text{ for some } N \leq \ell^{\zeta'}\right\} &\leq \ell^{\zeta'} K\,|I|^\alpha\,\lambda^{-\alpha}\ell^{8Q_1} \\ &\leq K\,|I|^\alpha\,\lambda^{-\alpha}\ell^{8Q_1+1}.\end{aligned} \tag{1.7.25}$$

Using also (1.7.14) we get

$$\begin{aligned}\mathbb{P}\left\{\sigma_I(H_\Lambda) \neq \emptyset\right\} \leq\; &\mathbb{P}\left\{\sigma_I(H_\Lambda^{(N)}) \neq \emptyset \text{ for some } N \leq \ell^{\zeta'}\right\} \\ &+ \mathbb{P}\left\{\Lambda \text{ is not } (1,\mathcal{N}) - \text{reduced}\right\} \\ &\leq K\,|I|^\alpha\,\lambda^{-\alpha}\ell^{8Q_1+1} + e^{-\frac{c_\mu}{4}\ell^{\zeta'}}.\end{aligned} \tag{1.7.26}$$

$\square$

Lemma 1.7.7. *Consider two disjoint intervals Λ_1 and Λ_2, and let*

$$0 < \eta < \delta\left(1 - \tfrac{2J}{\Delta}\right). \tag{1.7.27}$$

Then (recall (1.7.12)*)*

$$\mathbb{P}\left\{\left\{\operatorname{dist}(\sigma_{I_{1,\delta}}(H_{\Lambda_1}), \sigma_{I_{1,\delta}}(H_{\Lambda_2})) \leq \eta\right\} \cap \mathcal{Y}_{\Lambda_1,1} \cap \mathcal{Y}_{\Lambda_2,1}\right\} \leq K\,(2\eta)^\alpha\,\lambda^{-\alpha}(\ell_1\ell_2)^{8Q_1+1}.$$

Proof. Since the two intervals are disjoint, the random variables ω_{Λ_1} and ω_{Λ_2} are independent. Thus, given $\nu \in \sigma_{I_1,\delta}(H_{\Lambda_1})$, as a random variable ν is independent of ω_{Λ_2}, so it follows from (1.7.23) in Lemma 1.7.6 that

$$\begin{aligned} \mathbb{P}\left\{\{\operatorname{dist}(\nu, \sigma(H_{\Lambda_2}) \le \eta\} \cap \mathcal{Y}_{\Lambda_2,1}\right\} &= \mathbb{P}\left\{\{[\nu-\eta, \nu+\eta] \cap \sigma(H_{\Lambda_2}) \neq \emptyset\} \cap \mathcal{Y}_{\Lambda_2,1}\right\} \\ &\le \mathbb{P}\left\{\sigma_{[\nu-\eta,\nu+\eta]}(H_{\Lambda_2}^{(N)}) \neq \emptyset \text{ for some } N \le \ell^{\zeta'}\right\} \\ &\le K\,(2\eta)^{\alpha}\,\lambda^{-\alpha}\ell_2^{8Q_1+1}. \end{aligned} \tag{1.7.28}$$

Therefore we can estimate the desired event

$$\mathbb{P}\left\{\left\{\operatorname{dist}(\sigma_{I_1,\delta}(H_{\Lambda_1}), \sigma_{I_1,\delta}(H_{\Lambda_2})) \le \eta\right\} \cap \mathcal{Y}_{\Lambda_1,1} \cap \mathcal{Y}_{\Lambda_2,1}\right\} \tag{1.7.29}$$

$$\le \mathbb{P}\left\{\bigcup_{\nu} \{\{[\nu-\eta, \nu+\eta] \cap \sigma(H_{\Lambda_2}) \neq \emptyset\} \cap \mathcal{Y}_{\Lambda_2,1}\}\right\}$$

$$\le \sum_{N \le 2J\ell_1} \left|\mathbf{M}_{\Lambda_1,1}^{(N)}\right| K\,(2\eta)^{\alpha}\,\lambda^{-\alpha}\ell_2^{8Q_1+1} \le 2J\ell_1\ell_1^{6Q_1} K\,(2\eta)^{\alpha}\,\lambda^{-\alpha}\ell_2^{8Q_1+1}$$

$$\le K(2\eta)^{\alpha}\lambda^{-\alpha}(\ell_1\ell_2)^{8Q_1+1}. \tag{1.7.30}$$

Assuming that both ℓ_1 and ℓ_2 are large enough. □

Remark 1.7.8. We will need (1.7.27) with $\eta = 2\mathrm{e}^{-L^{\beta}}$, that is,

$$\mathrm{e}^{-L^{\beta}} < \tfrac{\delta}{2}\left(1 - \tfrac{2J}{\Delta}\right). \tag{1.7.31}$$

1.8 The Starting Condition

Definition 1.8.1. Given $E \in \mathbb{R}$ and $m > 0$, an interval Λ_L is said to be (m, E)-regular2 if

$$m > 2L^{-\kappa} \tag{1.8.1}$$

$$\operatorname{dist}(E, \sigma(H_{\Lambda_L})) > 2e^{-L^{\beta}} \tag{1.8.2}$$

$$\|P_i^-(H_{\Lambda_L} - E)^{-1} P^+_{\Lambda_R(i)\cap\Lambda_L}\| \leq e^{-m(R+1)} \text{ for all } i \in \Lambda_L \text{ and } R > L^{\tau}. \tag{1.8.3}$$

Theorem 1.8.2. *Fix $\Delta_0 > 2J$, $\lambda_0 > 0$, and $\delta \in (0,1)$. Suppose Δ_0, λ_0, δ, and the scale L satisy* (1.5.5)- (1.5.7)*. Further suppose that L is sufficiently large, depending on λ_0, δ, and Δ_0. Moreover, suppose that $\Delta \geq \Delta_0$, and $\lambda \geq \lambda_0$ are chosen satisfying* (1.5.8) *for L. Let*

$$m = \tfrac{1}{4} \min\left\{1, \log\left(1 + \frac{\delta(\Delta_0 - 2J)}{4JQ_1}\right)\right\}. \tag{1.8.4}$$

Then setting $\theta_L = e^{-3L}$, we have for all $E \in I_{1,\delta}$ that

$$\mathbb{P}\{\mathcal{R}(m, L, I(E, \theta_L), u, v)\} \geq 1 - e^{-L^{\zeta}} \textit{ for all } u, v \in \mathbb{Z} \textit{ with } |u - v| > 2L. \tag{1.8.5}$$

Lemma 1.8.3. *Fix $\Delta_0 > 2J$, $\lambda_0 > 0$, and $\delta \in (0,1)$. Suppose Δ_0, λ_0, δ, and the scale L satisy* (1.5.5)- (1.5.7)*. Further suppose that L is sufficiently large, depending on λ_0, δ, and Δ_0. Moreover, suppose that $\Delta \geq \Delta_0$, and $\lambda \geq \lambda_0$ are chosen satisfying* (1.5.8) *for L. Let $\Lambda = \Lambda_L$ be a subinterval of $\mathbb{Z}$ of length $2L$, and for $E \notin \sigma(H_\Lambda)$, let $R_\Lambda(E) = (H_\Lambda - E)^{-1}$. Then for all $E \in I_{1,\delta}$, we have*

$$\mathbb{P}\left\{\Lambda \textit{ is } \left(\frac{1}{2}\min\left\{1, \log\left(1 + \frac{\delta(\Delta_0 - 2J)}{4JQ_1}\right)\right\}, E\right) - \textit{regular2}\right\} \geq 1 - e^{-L^{\zeta_1}} \tag{1.8.6}$$

Proof. Let $\mathcal{Y}_{\Lambda,1}$, be the event that Λ_L is $(1, \mathcal{N})$-reduced. In view of (1.5.5) and the choice of

$\zeta < \zeta_1 < \zeta'$, Equation (1.7.14) holds with ζ_1 substituted for ζ so for sufficiently large L,

$$\mathbb{P}\left\{\mathcal{Y}_{\Lambda,1}\right\} \geq 1 - \mathrm{e}^{-\frac{c_\mu L^{\zeta'}}{4}} \geq 1 - \tfrac{1}{10} e^{-L^{\zeta_1}}. \tag{1.8.7}$$

Let $E \in I_{1,\delta}$ and $\omega \in \mathcal{Y}_{\Lambda,1}$, it follows from Proposition 1.6.4 that for all $N > L^{\zeta'}$, $i \in \Lambda$ and $R > L^\tau$ we have

$$\begin{aligned}
\left\| P_i^- R_\Lambda^{(N)}(E) P_{i,R}^+ \right\| &\leq \frac{Q_1}{2J^2\delta\left(1 - \frac{2J}{\Delta}\right)} e^{-\log\left(1+\frac{\delta(\Delta_0-2J)}{4JQ_1}\right)(R+1)}. \qquad (1.8.8)\\
&\leq \mathrm{e}^{-\log\left(1+\frac{\delta(\Delta_0-2J)}{4JQ_1}\right)\left(1-\frac{\log\frac{Q_1}{\delta\left(1-\frac{2J}{\Delta}\right)2J^2}}{3L^{-\kappa}L^\tau}\right)(R+1)}\\
&\leq \mathrm{e}^{-\log\left(1+\frac{\delta(\Delta_0-2J)}{4JQ_1}\right)\left(1-\frac{L^\beta+\log(Q_1/4J^2)}{3L^{\tau-\kappa}}\right)(R+1)} \leq \mathrm{e}^{-\log\left(1+\frac{\delta(\Delta_0-2J)}{4JQ_1}\right)\left(1-\frac{1}{L^{\tau-\kappa-\beta}}\right)(R+1)}\\
&\leq \mathrm{e}^{-\frac{1}{2}\log\left(1+\frac{\delta(\Delta_0-2J)}{4JQ_1}\right)(R+1)},
\end{aligned}$$

for sufficiently large L, where we used (1.5.7).

We now consider the operators $H_\Lambda^{(N)}$, $\mathcal{W}_\Lambda^{(N)}$, etc. If $N \leq L^{\zeta'}$, given $\eta > 0$, it follows from the Holder Continuity of μ, that for all $\mathbf{m} \in \mathbf{M}_\Lambda^{(N)}$ we have

$$\mathbb{P}\left\{|\mathcal{W}(\mathbf{m}) + \lambda V_\omega(\mathbf{m}) - E| \leq \eta\right\} \leq K\left(\tfrac{2\eta}{\lambda}\right)^\alpha, \tag{1.8.9}$$

For a particular configuration inside of Λ, let $m_1 = \mathbf{m}(x_1)$ be the number of particles of the first nonzero entry from the left.

$$\begin{aligned}
&|\mathcal{W}(\mathbf{m}) + \lambda V_\omega(\mathbf{m}) - E| \leq \eta\\
&\implies \lambda\omega_1 m_1 \in [\mathcal{W} + \lambda V_\omega(\mathbf{m}(x_2, x_3 \dots)) - E - \eta, \mathcal{W} + \lambda V_\omega(\mathbf{m}(x_2, x_3 \dots)) - E + \eta]\\
&\mathbb{P}\left\{|\mathcal{W}(\mathbf{m}) + \lambda V_\omega(\mathbf{m}) - E| \leq \eta\right\} \leq K\left(\frac{2\eta}{m_1\lambda}\right)^\alpha \leq K\left(\tfrac{2\eta}{\lambda}\right)^\alpha
\end{aligned}$$

Consider the event $\mathcal{Q}$:

$$\mathcal{Q} = \left\{ \left\| (\mathcal{W} + \lambda V_\omega - E)^{-1} \right\| \geq \tfrac{1}{\eta} \right\}. \tag{1.8.10}$$

Which we estimate by

$$\mathbb{P}\left\{\mathcal{Q}\right\} \leq K \left(\tfrac{2\eta}{\lambda}\right)^{\alpha} (2L+2)^{N} \leq \tfrac{1}{10}\mathrm{e}^{-L^{\zeta_1}}, \tag{1.8.11}$$

which holds if

$$\eta \leq \frac{\lambda}{2}\left(\frac{(2L+2)^{-N}}{10K} e^{-L^{\zeta_1}}\right)^{1/\alpha}. \tag{1.8.12}$$

We choose

$$\eta = \lambda \mathrm{e}^{-\frac{1}{2}L^{\zeta''}} \quad (\text{recall} \quad \zeta_1 < \zeta' < \zeta'' < \beta), \tag{1.8.13}$$

so (1.8.11) holds for L sufficiently large. For every $\omega \in \mathcal{Q}^{\mathrm{c}}$, we have,

$$\begin{aligned} R_\Lambda^{(N)}(E) &= \left(H_\Lambda^{(N)} - E\right)^{-1} = \left(-\tfrac{1}{2\Delta}A_\Lambda^{(N)} + \mathcal{W} + \lambda V - E\right)^{-1} \\ &= (\mathcal{W} + \lambda V - E)^{-1}\left(1 - \tfrac{1}{2\Delta}A_\Lambda^{(N)}(\mathcal{W} + \lambda V - E)^{-1}\right)^{-1} \\ &= (\mathcal{W} + \lambda V - E)^{-1}\left(1 + \sum_{n=1}^{\infty}\left(\tfrac{1}{2\Delta}A_\Lambda^{(N)}(\mathcal{W} + \lambda V - E)^{-1}\right)^{n}\right), \end{aligned} \tag{1.8.14}$$

where the series converges absolutely in operator norm if we have

$$\begin{aligned} \left\| \tfrac{1}{2\Delta}A_\Lambda^{(N)}(\mathcal{W} + \lambda V - E)^{-1} \right\| &\leq \tfrac{1}{2\eta\Delta}\left\|A_\Lambda^{(N)}\right\| \leq \tfrac{2J}{\eta\Delta}\left\|\mathcal{W}\right\| \leq \tfrac{4J^2}{\eta\Delta}L^{\zeta'} \\ &= \tfrac{4J^2}{\Delta\lambda}L^{\zeta'}\mathrm{e}^{\frac{1}{2}L^{\zeta''}} \leq \tfrac{1}{2}, \end{aligned} \tag{1.8.15}$$

which we obtain by requiring (1.5.8) with L sufficiently large. Then

$$P_i^- (\mathcal{W} + \lambda V - E)^{-1} \left(\tfrac{1}{2\Delta} A_\Lambda^{(N)} (\mathcal{W} + \lambda V - E)^{-1} \right)^n P_{i,R}^+ = 0 \tag{1.8.16}$$

if $n < R$. Hence

$$\begin{aligned} &\left\| P_i^- R_\Lambda^{(N)}(E) P_{i,R}^+ \right\| \\ &\quad \leq \sum_{n \geq R} \left\| P_i^- (\mathcal{W} + \lambda V - E)^{-1} \left(\tfrac{1}{2\Delta} A_\Lambda^{(N)} (\mathcal{W} + \lambda V - E)^{-1} \right)^n P_{i,R}^+ \right\| \\ &\quad \leq \sum_{n \geq R} \tfrac{1}{\eta} \tfrac{1}{2^n} = \tfrac{1}{\lambda} \mathrm{e}^{\frac{1}{2} L^{\zeta''}} \tfrac{2}{2^R} \leq \mathrm{e}^{\frac{1}{6} L^\beta} \mathrm{e}^{\frac{1}{2} L^{\zeta''}} 2 \mathrm{e}^{-(\log 2) R} \\ &\quad \leq \mathrm{e}^{-(\log 2) R \left(1 - \frac{\frac{1}{6} L^\beta + \frac{1}{2} L^{\zeta''} + \log 2}{(\log 2) L^\tau} \right)} \\ &\quad \leq \mathrm{e}^{-(\log 2) \left(1 - 3 L^{-\tau + \beta} \right) R} \leq \mathrm{e}^{-\frac{1}{2} R}, \end{aligned} \tag{1.8.17}$$

for L sufficiently large, where we used (1.5.5). Combining (1.8.8) and (1.8.17), and our assumptions, we conclude that for $\omega \in \mathcal{Y}_{\Lambda,1} \cap \mathcal{Q}^{\mathrm{c}}$ we have

$$\left\| P_i^- R_\Lambda^{(N)}(E) P_{i,R}^+ \right\| \leq \mathrm{e}^{-\frac{1}{2} \min\left\{ 1, \log\left(1 + \frac{\delta(\Delta_0 - 2J)}{4 J Q_1} \right) \right\} R} \tag{1.8.18}$$

with probability

$$\mathbb{P}\left\{ \mathcal{Y}_{\Lambda,1} \cap \mathcal{Q}^{\mathrm{c}} \right\} \geq 1 - \tfrac{1}{10} \mathrm{e}^{-L^{\zeta_1}} - \tfrac{1}{10} \mathrm{e}^{-L^{\zeta_1}} \geq 1 - \tfrac{3}{4} \mathrm{e}^{-L^{\zeta_1}}. \tag{1.8.19}$$

In addition, it follows from (1.7.24) in Lemma 1.7.6

$$\mathbb{P}\left\{ \operatorname{dist}(E, \sigma(H_\Lambda)) < 2 e^{-L^\beta} \right\} \leq K \left(\tfrac{4 e^{-L^\beta}}{\lambda} \right)^\alpha L^{8 Q_1 + 1} + e^{-\frac{c_\mu}{4} L^{\zeta'}} \leq \tfrac{1}{10} \mathrm{e}^{-L^{\zeta_1}}, \tag{1.8.20}$$

for sufficiently large L, using (1.5.5).

Thus, if (1.5.8), (1.5.5), and (1.5.7) hold and L is sufficiently large (in particular, $L^{-\kappa} \leq \frac{1}{4}$), we proved that

$$\mathbb{P}\left\{\Lambda_L \text{ is } \left(\tfrac{1}{2}\min\left\{1, \log\left(1+\frac{\delta(\Delta_0-2J)}{4JQ_1}\right)\right\}, E\right)\text{-regular2}\right\} \geq 1-\mathrm{e}^{-L^{\zeta_1}}. \tag{1.8.21}$$

□

Lemma 1.8.4. *Fix $\Delta_0 > 2J$, $\lambda_0 > 0$, and $\delta \in (0,1)$. Suppose Δ_0, λ_0, δ, and the scale L satisy* (1.5.5)- (1.5.7). *Further suppose that L is sufficiently large, depending on λ_0, δ, and Δ_0. Moreover, suppose that $\Delta \geq \Delta_0$, and $\lambda \geq \lambda_0$ are chosen satisfying* (1.5.8) *for L. If the interval Λ_L is (m, E)-regular2 for some $E \in I_{1,\delta}$ and some $m > L^{-\kappa}$, then the interval Λ_L is $(m/2, E')$-regular for all $E' \in I(E,\theta)$, where $\theta = e^{-3L}$.*

Proof. Assume $\Lambda = \Lambda_L$ is (m, E)-regular2 for some $E \in I_{1,\delta}$. Let $E' \in I(E,\theta)$. By the resolvent identity we have that

$$R_\Lambda(E') = R_\Lambda(E) + (E'-E)R_\Lambda(E')R_\Lambda(E) \tag{1.8.22}$$

Therefore

$$\|R_\Lambda(E')\| \leq \frac{1}{2}e^{L^\beta} + \frac{1}{2}\theta e^{L^\beta}\|R_\Lambda(E')\| \tag{1.8.23}$$

So taking $\theta \leq e^{-L^\beta}$, we get

$$\|R_\Lambda(E')\| \leq e^{L^\beta} \tag{1.8.24}$$

Now, given that $L^{-\kappa} < \frac{m}{2} < m$, then if $i \in \Lambda_L$ and $R > L^\tau$ we get the following

$$\|P_i^- R_\Lambda(E') P^+_{\Lambda_R(i)}\| \leq \|P_i^- R_\Lambda(E) P^+_{\Lambda_R(i)}\| + \frac{1}{2}\theta e^{2L^\beta} \leq e^{-m(R+1)} + \frac{1}{2}\theta e^{2L^\beta} \leq e^{-\frac{m}{2}(R+1)}. \tag{1.8.25}$$

The final inequality must hold for all R, with $L^\tau < R \leq 2L-1$. In this case we need

$$\theta \leq \inf_{L^\tau < R \leq 2L-1} 2e^{-m(R+1)-2L^\beta}\left(e^{\frac{m}{2}(R+1)}-1\right) \tag{1.8.26}$$

Notice that since $e^x - 1 \geq x$ we have,

$$2e^{-m(R+1)-2L^\beta}\left(e^{\frac{m}{2}(R+1)}-1\right) \geq m(R+1)e^{-m(R+1)-2L^\beta} \geq mL^\tau e^{-2mL-2L^\beta} \tag{1.8.27}$$

The inequality will be satisfied if we choose

$$\theta = e^{-3L} < \min\{mL^\tau e^{-2mL-2L^\beta}, e^{-L^\beta}\}. \tag{1.8.28}$$

□

Proof of Theorem 1.8.2. From Lemma 1.8.3 and Lemma 1.8.4 we have that for all $y \in \mathbb{Z}$,

$$\mathbb{P}\{\mathcal{A}(y)\} := \mathbb{P}\left\{\Lambda_L(y) \text{ is } (m, E') - \text{regular for all } E' \in I(E, \theta_L)\right\} \geq 1 - e^{-L^{\zeta_1}} \tag{1.8.29}$$

Therefore

$$\mathbb{P}\{\mathcal{R}(m, L, I(E, \theta_L), u, v)\} \geq \mathbb{P}\{\mathcal{A}(u) \cap \mathcal{A}(v)\} \geq \left(1 - e^{-L^{\zeta_1}}\right)^2 \geq 1 - e^{-L^{\zeta}}. \tag{1.8.30}$$

□

1.9 Preliminaries for the Multiscale Analysis

1.9.1 Subintervals

Lemma 1.9.1. *Let $K \subsetneq \Lambda$ be finite intervals in $\mathbb{Z}$. Let (ψ, θ) be an an eigenpair for H_Λ with $\theta \geq 2J\left(1 - \frac{2J}{\Delta}\right)$, and let $0 < \eta < \delta\left(1 - \frac{2J}{\Delta}\right)$. Suppose*

$$\operatorname{dist}\left(\theta, \sigma\left(H_{\Lambda\setminus\partial K}\right)\right) \geq \eta. \tag{1.9.1}$$

Then for all $x \in K$ we have

$$\left\|P_x^- \psi\right\| \leq \left\|P_K^- \psi\right\| \leq \tfrac{1}{\eta} \sum_{b \in \boldsymbol{\partial}^\Lambda K} \left\|P_b^- \psi\right\| \leq \frac{8J^2}{\eta} \left\|P_r^- \psi\right\| \quad \textit{for some} \quad r \in \partial^\Lambda K. \tag{1.9.2}$$

Proof. Note that for $\theta \geq 2J\left(1 - \frac{2J}{\Delta}\right)$ and $0 < \eta < \delta\left(1 - \frac{2J}{\Delta}\right)$ we have

$$\operatorname{dist}\left(\theta, \sigma\left(H_{\Lambda\setminus\partial K}\right)\right) \geq \eta \iff \operatorname{dist}\left(\theta, \sigma\left(H'_{\Lambda\setminus\partial K}\right)\right) \geq \eta. \tag{1.9.3}$$

Let $\{\varphi_\nu, \nu\}_{\nu \in \sigma(H_K)}$ be an eigensystem for H_K and $\{\eta_\kappa, \kappa\}_{\kappa \in \sigma(H_{\Lambda\setminus\partial K})}$ be an eigensystem for $H_{\Lambda\setminus\partial K}$, so $\{\varphi_\nu \otimes \eta_\kappa, \nu + \kappa\}_{\nu \in \sigma(H_K), \kappa \in \sigma(H_{\Lambda\setminus K})}$ is an eigensystem for $H_{\Lambda\setminus\partial K}$. Then, since $P_K^- = 0$ on $\mathbb{C}\Omega_K \otimes \mathcal{H}_{\Lambda\setminus K}$,

$$
\begin{aligned}
P_K^- \psi = P_K^- & \sum_{\nu \in \sigma(H'_K), \kappa \in \sigma(H_{\Lambda\setminus K})} \langle \varphi_\nu \otimes \eta_\kappa, \psi \rangle \, \varphi_\nu \otimes \eta_\kappa \\
= P_K^- & \sum_{\nu \in \sigma(H'_K), \kappa \in \sigma(H_{\Lambda\setminus K})} (\theta - (\nu + \kappa))^{-1} \left\langle \varphi_\nu \otimes \eta_\kappa, \left(H_\Lambda - H_{\Lambda \setminus \partial K} \right) \psi \right\rangle \varphi_\nu \otimes \eta_\kappa \\
= P_K^- & \sum_{\nu \in \sigma(H'_K), \kappa \in \sigma(H_{\Lambda\setminus K})} (\theta - (\nu + \kappa))^{-1} \left\langle \varphi_\nu \otimes \eta_\kappa, \Gamma_{\Lambda \setminus \partial K} \psi \right\rangle \varphi_\nu \otimes \eta_\kappa. \\
= P_K^- & \left(\theta - H'_{\Lambda \setminus \partial K} \right)^{-1} \sum_{\nu \in \sigma(H'_K), \kappa \in \sigma(H_{\Lambda\setminus K})} \left\langle \varphi_\nu \otimes \eta_\kappa, \Gamma_{\Lambda \setminus \partial K} \psi \right\rangle \varphi_\nu \otimes \eta_\kappa \\
= & \left(\theta - H'_{\Lambda \setminus \partial K} \right)^{-1} P_K^- \sum_{\nu \in \sigma(H'_K), \kappa \in \sigma(H_{\Lambda\setminus K})} \left\langle \varphi_\nu \otimes \eta_\kappa, \Gamma_{\Lambda \setminus \partial K} \psi \right\rangle \varphi_\nu \otimes \eta_\kappa \\
= & \left(\theta - H'_{\Lambda \setminus \partial K} \right)^{-1} P_K^- \sum_{\nu \in \sigma(H_K), \kappa \in \sigma(H_{\Lambda\setminus K})} \left\langle \varphi_\nu \otimes \eta_\kappa, \Gamma_{\Lambda \setminus \partial K} \psi \right\rangle \varphi_\nu \otimes \eta_\kappa \\
= & \left(\theta - H'_{\Lambda \setminus \partial K} \right)^{-1} P_K^- \Gamma_{\Lambda \setminus \partial K} \psi = \left(\theta - H'_{\Lambda \setminus \partial K} \right)^{-1} P_K^- \sum_{b \in \boldsymbol{\partial}^\Lambda K} \tilde{h}_b P_b^- \psi,
\end{aligned} \tag{1.9.4}
$$

Thus, using (1.9.1) and (1.9.3) we get

$$
\left\| P_K^- \psi \right\| \leq \tfrac{1}{\eta} \sum_{b \in \boldsymbol{\partial}^\Lambda K} \|\tilde{h}_b\| \left\| P_b^- \psi \right\| \leq \frac{8J^2}{\eta} \| P_r^- \psi \|. \tag{1.9.5}
$$

Where we used from Lemma 1.4.3 that $\|\tilde{h}_b\| = 4J^2$.

$\square$

We now want to look at the Hamiltonian restricted to subspaces of sufficiently high particle number. To this end, let $H_\Lambda^{[N_0,N_1]} := \chi_{[N_0,N_1]}(\mathcal{N}_\Lambda) H_\Lambda$ and moreover $H_\Lambda^{[N_0]} := \chi_{[N_0,\infty)}(\mathcal{N}_\Lambda) H_\Lambda$.

Lemma 1.9.2. *Let $K \subsetneq \Lambda$ be finite intervals in $\mathbb{Z}$ and assume that $N_0 \geq 4J$. Let $\theta \in I_{1,\delta}$ and $0 < \eta < \delta\left(1 - \frac{2J}{\Delta}\right)$. Then*

$$\operatorname{dist}\left(\theta, \sigma\left(H_K^{[N_0]}\right)\right) \geq \eta \quad \iff \quad \operatorname{dist}\left(\theta, \sigma\left(H_{\Lambda\setminus\partial K}^{[N_0]}\right)\right) \geq \eta. \tag{1.9.6}$$

Proof. Note first that for $\theta \geq 2J\left(1 - \frac{2J}{\Delta}\right)$ and $0 < \eta < \delta\left(1 - \frac{2J}{\Delta}\right)$,

$$\operatorname{dist}\left(\theta, \sigma\left(H_K^{[N_0]}\right)\right) \geq \eta \iff \operatorname{dist}\left(\theta, \sigma\left(H_K'^{[N_0]}\right)\right) \geq \eta. \tag{1.9.7}$$

In view of (1.9.3), we only need to show

$$\operatorname{dist}\left(\theta, \sigma\left(H_K^{[N_0]}\right)\right) < \eta \iff \operatorname{dist}\left(\theta, \sigma\left(H_{\Lambda\setminus\partial K}^{[N_0]'}\right)\right) < \eta. \tag{1.9.8}$$

Note that

$$\begin{aligned} H_{\Lambda\setminus\partial K}^{[N_0]'} &= \bigoplus_{k,l:l\geq 1,k+l\geq N_0} \left[H_{\Lambda\setminus K}^{(k)} \otimes I_K + I_{\Lambda\setminus K} \otimes H_K^{(l)}\right] && (1.9.9)\\ &= \left(I_{\Lambda\setminus K} \otimes H_K^{[N_0]}\right) \oplus \left(\bigoplus_{k\geq 1,l\geq 1:k+l\geq N_0} \left[H_{\Lambda\setminus K}^{(k)} \otimes I_K + I_{\Lambda\setminus K} \otimes H_K^{(l)}\right]\right) && (1.9.10) \end{aligned}$$

It follows that $\sigma\left(H_K^{[N_0]}\right) \subset \sigma\left(H_{\Lambda\setminus\partial K}^{[N_0]'}\right)$, and hence

$$\operatorname{dist}\left(\theta, \sigma\left(H_K^{[N_0]}\right)\right) < \eta \implies \operatorname{dist}\left(\theta, \sigma\left(H_{\Lambda\setminus\partial K}^{[N_0]'}\right)\right) < \eta. \tag{1.9.11}$$

Conversely, since $\theta \in I_{1,\delta}$, if $\operatorname{dist}\left(\theta, \sigma\left(H_{\Lambda\setminus\partial K}'^{[N_0]}\right)\right) < \eta$ there exist $l \geq 1$ and $k \geq 0$, satisfying $l + k \geq N_0$, such that for some $\nu \in \sigma(H_K^{(l)})$, and $\kappa \in \sigma\left(H_{\Lambda\setminus K}^{(k)}\right)$ we have $|\theta - (\nu + \kappa)| < \eta$.

Moreover we have that

$$H_{\Lambda\setminus K}^{(k)} \otimes I_K + I_{\Lambda\setminus K} \otimes H_K^{(l)} \geq \left(1 - \tfrac{2J}{\Delta}\right) \left(\mathcal{W}_0^{(k)} + \mathcal{W}_0^{(l)}\right). \tag{1.9.12}$$

Now if $k \geq 1$ then we get by Theorem 1.4.5, the bound

$$\nu + \kappa \geq \inf \sigma(H_{\Lambda\setminus K}^{(k)} \otimes I_K + I_{\Lambda\setminus K} \otimes H_K^{(l)}) \geq Q_1 \left(1 - \tfrac{2J}{\Delta}\right). \tag{1.9.13}$$

This implies a contradiction. We would have that $\nu + \kappa \geq Q_1 \left(1 - \frac{2J}{\Delta}\right)$ and consequently $\theta > (Q_1 - \delta)\left(1 - \frac{2J}{\Delta}\right)$, but we assumed that $\theta \in I_{1,\delta}$. Hence, $\kappa = 0$, which is only possible for $k = 0$. Therefore $|\theta - \nu| < \eta$, thus implying $\operatorname{dist}(\theta, \sigma(H_K^{[N_0]})) < \eta$.

$\square$

Corollary 1.9.3. *Let $K \subsetneq \Lambda$ be finite intervals in $\mathbb{Z}$ and assume that $N_0 \geq 4J$. Let $\theta \in I_{1,\delta}$ and suppose that ψ is an eigenfunction of ${H'}_{\Lambda\setminus\partial K}^{[N_0]}$ with eigenvalue θ. Then $\psi = \Omega_{\Lambda\setminus K} \otimes \psi_K$, where $\psi_K \in \mathcal{H}_K^{[N_0]}$.*

Proof. This follows from the proof of Lemma 1.9.2. Let A be the set of all tuples $\{(\alpha_1, \alpha_2) : \alpha_1 \geq 0, \alpha_2 \geq 1, \alpha_1 + \alpha_2 \geq N_0\}$. Since ψ is an eigenfunction of $H_{\Lambda\setminus\partial K}^{[N_0]'}$, it can be written as a linear combination of eigenpairs $(\xi_\alpha, \kappa_\alpha)$ for $H_{\Lambda\setminus K}^{(\alpha_1)}$ and $(\phi_\alpha, \nu_\alpha)$ for $H_K^{(\alpha_2)}$ where for each α, $\kappa_\alpha + \nu_\alpha = \theta$. From (1.9.10) we can write

$$\psi = \sum_{\alpha\in A} \xi_\alpha \otimes \phi_\alpha = \left(\Omega_{\Lambda\setminus K} \otimes \sum_{\alpha_2 \geq N_0} \phi_{(0,\alpha_2)}\right) + \sum_{\alpha\in A:\alpha_1\geq 1} \xi_\alpha \otimes \phi_\alpha = (I) + (II). \tag{1.9.14}$$

Suppose for some α that $\nu_\alpha \neq 0$, by an analogous argument we could show that θ is too large to be in $I_{1,\delta}$. Therefore κ_α is identically 0, which implies that $\xi_\alpha = \Omega_{\Lambda\setminus K}$.

$\square$

Lemma 1.9.4. *Assume the interval Λ_L is $(1,\mathcal{N})$-reduced and suppose that $N \geq 4J$. Suppose the interval $\Lambda_\ell := \Lambda_\ell(j) \subset \Lambda_L$ is (m, E)-regular, where $E \in I_{1,\delta}$ and*

$$\ell^{-\kappa} < m \leq \log\left(1 + \frac{\delta(\Delta - 2J)}{4JQ_1}\right). \tag{1.9.15}$$

Then, for sufficiently large ℓ, we have

$$\operatorname{dist}\left(E, \sigma(H'_{\Lambda_L\setminus\partial\Lambda_\ell})\right) > \mathrm{e}^{-\ell^\beta}, \tag{1.9.16}$$

and

$$\left\| P_i^- R^{(N)}_{\Lambda_L\setminus\partial\Lambda_\ell}(E) P^+_{\Lambda_R(i)} \right\| \leq e^{-m'(R+1)} \quad \textit{for all} \quad i \in \Lambda \quad \textit{and} \quad R > \ell^\tau, \tag{1.9.17}$$

where

$$m' \geq m\left(1 - \tfrac{C_J}{\ell^{\tau-\kappa-\gamma\beta}}\right). \tag{1.9.18}$$

Proof. The estimate (1.9.16) follows from (1.5.3) and Lemma 1.9.2. Our proof is divided into two cases the first is when $N > L^{\zeta'}$, and the second is when $N \in [4J, L^{\zeta'}]$.

Case 1: Since Λ_L is $(1,\mathcal{N})$-reduced and $N > L^{\zeta'}$ we have Lemma 1.7.2 for $H^{(N)}_{\Lambda_L\setminus\partial\Lambda_\ell}$. So for all $N > L^{\zeta'}$ and $i \in \Lambda$ we have

$$\begin{aligned}\left\| P_i^- R^{(N)}_{\Lambda_L\setminus\partial\Lambda_\ell}(E) P^+_{\Lambda_R(i)} \right\| &\leq \frac{2Q_1}{4J^2\delta\left(1 - \frac{2J}{\Delta}\right)} \exp\left(-\log\left(1 + \frac{\delta(\Delta - 2J)}{4JQ_1}\right)(R+1)\right) \\ &\leq \frac{Q_1}{2J^2\delta\left(1 - \frac{2J}{\Delta}\right)} \mathrm{e}^{-m(R+1)} \leq \mathrm{e}^{-m'(R+1)},\end{aligned} \tag{1.9.19}$$

for all $R \geq \ell^\tau$. Here, we used (1.9.15), and

$$m' \geq m\left(1 - \frac{\log \frac{1}{2J^2\xi}}{\ell^\tau \ell^{-\kappa}}\right) = m\left(1 - \frac{\left|\log \delta\left(1-\frac{2J}{\Delta}\right)\right| + \log(2+\frac{1}{J})}{\ell^{\tau-\kappa}}\right) \tag{1.9.20}$$

$$\geq m\left(1 - \frac{\ell^\beta + \log(2+\frac{1}{J})}{\ell^{\tau-\kappa}}\right) \geq m\left(1 - \frac{C_{1,J}}{\ell^{\tau-\kappa-\beta}}\right),$$

where we used (1.5.6). This finishes Case 1.

Case 2: Let $\{\varphi_\nu, \nu\}_{\nu\in\sigma(H^{(N)}_{\Lambda_\ell(j)})}$ be an eigensystem for H_{Λ_ℓ} and $\{\eta_\kappa, \kappa\}_{\kappa\in\sigma(H_{\Lambda_L\setminus\Lambda_\ell})}$ be an eigensystem for $H_{K\setminus\Lambda}$, so $\{\varphi_\nu \otimes \eta_\kappa, \nu+\kappa\}_{\nu\in\sigma(H_\Lambda),\kappa\in\sigma(H_{K\setminus\Lambda})}$ is an eigensystem for $H_{K\setminus\partial\Lambda}$.

For ϕ, a unit vector, let $\pi_\phi = \langle\phi,\cdot\rangle\,\phi$ denotes the orthogonal projection onto $\mathbb{C}\phi$. Since there are N particles in Λ_L in total, we will need to consider all possibilities for how the N particles can be divided between Λ_L and $\Lambda_L \setminus \Lambda_\ell$. Let N_1 be the number of particles in Λ_ℓ and N_2 be the number of particles in $\Lambda_L \setminus \Lambda_\ell$. Then,

$$R^{(N)'}_{\Lambda_L\setminus\partial\Lambda}(E) = \sum_{\substack{N_1+N_2=N\\ N_1\geq 1, N_2\geq 0}} \sum_{\substack{\nu\in\sigma(H^{(N_1)}_{\Lambda_\ell}),\\ \kappa\in\sigma(H^{(N_2)}_{\Lambda_L\setminus\Lambda_\ell})}} (\nu+\kappa-E)^{-1}\,\pi_{\varphi_\nu}\otimes\pi_{\eta_\kappa} \tag{1.9.21}$$

$$= \sum_{\substack{N_1+N_2=N\\ N_1\geq 1, N_2\geq 0}} \sum_{\kappa\in\sigma(H^{(N_2)}_{\Lambda_L\setminus\Lambda_\ell})} \left(H^{(N_1)}_{\Lambda_\ell} + \kappa - E\right)^{-1} \otimes \pi_{\eta_\kappa},$$

so, for $i \in \Lambda_\ell(j)$,

$$P_i^- R^{(N)}_{\Lambda_L\setminus\partial\Lambda_\ell}(E) P^+_{\Lambda_R(i)} = P_i^- R^{(N)'}_{\Lambda_L\setminus\partial\Lambda_\ell}(E) P^+_{\Lambda_R(i)} \tag{1.9.22}$$

$$= \left(\sum_{\substack{N_1+N_2=N\\ N_1\geq 1, N_2\geq 0}} \sum_{\kappa\in\sigma(H^{(N_2)}_{\Lambda_L\setminus\Lambda_\ell})} \left(P_i^- \left(H^{(N_1)}_{\Lambda_\ell} + \kappa - E\right)^{-1} P^+_{\Lambda_R(i)\cap\Lambda_\ell(j)}\right) \otimes \pi_{\eta_\kappa}\right) P^+_{\Lambda_R(i)},$$

and hence

$$
\begin{aligned}
&\left\|P_i^- R^{(N)}_{\Lambda_L\setminus\partial\Lambda_\ell}(E) P^+_{\Lambda_i(R)}\right\| = \left\|P_i^- P^-_{\Lambda_\ell} R^{(N)'}_{\Lambda_L\setminus\partial\Lambda_\ell}(E) P^+_{\Lambda_i(R)}\right\| \qquad (1.9.23)\\
&\leq \sum_{\substack{N_1+N_2=N\\ N_1\geq 1, N_2\geq 0}} \sum_{\kappa\in\sigma(H^{(N_2)}_{\Lambda_L\setminus\Lambda_\ell})} \left\|\left(\left(P_i^- \left(H^{(N_1)}_{\Lambda_\ell}+\kappa-E\right)^{-1} P^+_{\Lambda_R(i)\cap\Lambda_\ell(j)}\right)\otimes \pi_{\eta_\kappa}\right) P^+_{\Lambda_R(i)}\right\|\\
&\leq \sum_{\substack{N_1+N_2=N\\ N_1\geq 1, N_2\geq 0}} \sum_{\kappa\in\sigma(H^{(N_2)}_{\Lambda_L\setminus\Lambda_\ell})} \left\|P_i^- \left(H^{(N_1)}_{\Lambda_\ell}+\kappa-E\right)^{-1} P^+_{\Lambda_R(i)\cap\Lambda_\ell(j)}\right\|\\
&\leq \left\|P_i^- \left(H^{(N)}_{\Lambda_\ell}-E\right)^{-1} P^+_{\Lambda_R(i)\cap\Lambda_\ell(j)}\right\|\\
&\qquad + \sum_{\substack{N_1+N_2=N\\ N_1\geq 1, N_2\geq 1}} \sum_{\kappa\in\sigma(H^{(N_2)}_{\Lambda_L\setminus\Lambda_\ell})} \left\|P_i^- \left(H^{(N_1)}_{\Lambda_\ell}+\kappa-E\right)^{-1} P^+_{\Lambda_R(i)\cap\Lambda_\ell(j)}\right\|\\
&\leq e^{-m(R+1)} + \sum_{\substack{N_1+N_2=N\\ N_1\geq 1, N_2\geq 1}} \sum_{\kappa\in(H^{(N_2)}_{\Lambda_L\setminus\Lambda_\ell})} \left\|P_i^- \left(H^{(N_1)}_{\Lambda_\ell}+\kappa-E\right)^{-1} P^+_{\Lambda_R(i)\cap\Lambda_\ell(j)}\right\|,
\end{aligned}
$$

where we used the hypothesis that the interval $\Lambda = \Lambda_\ell(j)$ is (m, E)-regular to get the last inequality. We have the estimate:

$$
\begin{aligned}
H^{(N_1)}_{\Lambda_\ell(j)} - E + \kappa &\geq \left(1-\tfrac{2J}{\Delta}\right)\mathcal{W}_0^{(N_1)} - E + \left(1-\tfrac{2J}{\Delta}\right)\mathcal{W}_0^{(N_2)} \qquad (1.9.24)\\
&\geq \left(1-\tfrac{2J}{\Delta}\right)\left(\mathcal{W}_0^{(N_1)}+\mathcal{W}_0^{(N_2)}\right) - (4J^2+2J-\delta)\left(1-\tfrac{2J}{\Delta}\right)\\
&\geq \left(1-\tfrac{2J}{\Delta}\right) Q_1 - (4J^2+2J-\delta)\left(1-\tfrac{2J}{\Delta}\right) = \delta\left(1-\tfrac{2J}{\Delta}\right) \qquad (1.9.25)
\end{aligned}
$$

Inequality (1.9.24) follows from the fact that $\kappa\in\sigma(H^{(N_2)}_{\Lambda_L\setminus\Lambda_\ell})$, so that $\kappa \geq \left(1-\frac{2J}{\Delta}\right)\mathcal{W}_0^{(N_1)}$, and Inequality (1.9.25) follows from Theorem 1.4.5. We have two subcases: first suppose that $N_1 < 4J$; this implies that $\mathcal{W}_{\Lambda_\ell} \leq 8J^2$. In this case we have

$$
H^{(N_1)}_{\Lambda_\ell} - E + \kappa \geq \delta\left(1-\tfrac{2J}{\Delta}\right) \geq \frac{\delta}{8J^2}\left(1-\tfrac{2J}{\Delta}\right)\mathcal{W}_{\Lambda_\ell}. \qquad (1.9.26)
$$

Now if $N_1 \geq 4J$, then $\mathcal{W}_\Lambda \geq 4J^2$ and we have that

$$\begin{aligned} H_{\Lambda_\ell}^{(N_1)} - E + \kappa &\geq \left(1 - \tfrac{2J}{\Delta}\right) \mathcal{W}_{\Lambda_\ell} - E + \kappa \geq \left(1 - \tfrac{2J}{\Delta}\right) \mathcal{W}_{\Lambda_\ell} - (4J^2 - \delta)\left(1 - \tfrac{2J}{\Delta}\right) \quad (1.9.27) \\ &\geq \left(1 - \tfrac{2J}{\Delta}\right) \mathcal{W}_{\Lambda_\ell} - (4J^2 - \delta)\left(1 - \tfrac{2J}{\Delta}\right) \frac{\mathcal{W}_{\Lambda_\ell}}{4J^2} \geq \frac{\delta}{4J^2}\left(1 - \tfrac{2J}{\Delta}\right) \mathcal{W}_{\Lambda_\ell}. \end{aligned}$$

In either case Inequality (1.9.26) holds. Then we have the following

$$\begin{aligned} H_{\Lambda_\ell}^{(N_1)} - (E - \kappa) &\geq \frac{\delta}{8J^2}\left(1 - \tfrac{2J}{\Delta}\right) \mathcal{W}_{\Lambda_\ell} \quad (1.9.28) \\ &\implies \left\| \mathcal{W}_{\Lambda_\ell}^{\frac{1}{2}} \left(H_{\Lambda_\ell}^{(N_1)} + \kappa - E \right)^{-1} \mathcal{W}_{\Lambda_\ell}^{\frac{1}{2}} \right\| \leq \frac{8J^2}{\delta\left(1 - \frac{2J}{\Delta}\right)}. \end{aligned}$$

Moreover, using $L \geq 1$,

$$\begin{aligned} \left| \sigma^{[4J, L^{\zeta'}]}(H_{\Lambda_L \setminus \Lambda_\ell(j)}) \right| &= \left| \bigcup_{N=4J}^{L^{\zeta'}} \sigma(H_{\Lambda_L \setminus \Lambda_\ell(j)}^{(N)}) \right| \leq \sum_{N=1}^{L^{\zeta'}} \binom{(2L + 1 - (2\ell + 1))2J}{N} \quad (1.9.29) \\ &\leq ((1 + 2(L - \ell))2J)^{L^{\zeta'}} - 1 \leq (4JL)^{L^{\zeta'}}. \end{aligned}$$

Thus it follows from Proposition 1.6.4 (as in (1.6.13)) that

$$\begin{aligned} &\sum_{\substack{N_1 + N_2 = N \\ N_1 \geq 1, N_2 \geq 1}} \sum_{\kappa \in \sigma(H_{\Lambda_L \setminus \Lambda}^{(N_2)})} \left\| P_i^- \left(H_{\Lambda_\ell}^{(N_1)} + \kappa - E \right)^{-1} P_{\Lambda_R(i) \cap \Lambda_\ell(j)}^+ \right\| \quad (1.9.30) \\ &\leq \frac{Q_1}{2J^2 \delta \left(1 - \frac{2J}{\Delta}\right)} \exp\left(-\log\left(1 + \frac{\delta(\Delta - 2J)}{4JQ_1}\right)(R+1)\right)(4JL)^{L^{\zeta'}} \leq \mathrm{e}^{-m''(R+1)}, \end{aligned}$$

for all $R \geq \ell^\tau$. In the previous equation we used (1.9.15), and

$$\begin{aligned} m'' &\geq m\left(1 - \frac{\log\frac{1}{2J^2\xi} + L^{\zeta'}\log(2L)}{\ell^\tau \ell^{-\kappa}}\right) \\ &= m\left(1 - \frac{\left|\log\delta\left(1-\frac{2J}{\Delta}\right)\right| + \log(2+\frac{1}{J}) + \ell^{\gamma\zeta'}\log(2\ell^\gamma)}{\ell^{\tau-\kappa}}\right) \\ &\geq m\left(1 - \frac{\ell^\beta + \log(2+\frac{1}{J}) + \ell^{\gamma\zeta'}\log(2\ell^\gamma)}{\ell^{\tau-\kappa}}\right) \geq m\left(1 - \frac{3}{\ell^{\tau-\kappa-\gamma\beta}}\right), \end{aligned} \tag{1.9.31}$$

which holds because of (1.5.6), $\tau > \kappa + \gamma\beta$, $\zeta' < \beta$, and because L is taken to be sufficiently large. Combining (1.9.19) and (1.9.23)-(1.9.30) we get

$$\left\|P_i^- R^{[4J]}_{\Lambda_L\setminus\partial\Lambda_\ell}(E) P^+_{\Lambda_R(i)}\right\| = \sup_{N\geq 4J}\left\|P_i^- R^{(N)}_{\Lambda_L\setminus\partial\Lambda_\ell}(E) P^+_{\Lambda_R(i)}\right\| \leq 2\mathrm{e}^{-m''(R+1)} \leq \mathrm{e}^{-m'''(R+1)}, \tag{1.9.32}$$

for $R > \ell^\tau$, where m''' is as in (1.9.18).

$\square$

Remark 1.9.5. In what follows, we will occasionally substitute i_* for i. Given two box sizes $\ell < L$, and the center j of Λ_L, we define i_* as a function of i,

$$i_* = \begin{cases} i & \text{if } |j-i| < L - 2\ell \\ j - L + \ell & \text{if } j - i > L - 2\ell \\ j + L - \ell & \text{if } i - j > L - 2\ell \end{cases} \tag{1.9.33}$$

This guarantees that $\Lambda_\ell(i_*) \subset \Lambda_L(j)$ for all $i \in \Lambda_L(j)$.

Lemma 1.9.6. *Let $E \in I_{1,\delta}$. Assume that the interval Λ_L is $(1,\mathcal{N})$-reduced. Let m satisfy (1.9.15). Let $i,j \in \Lambda_L$ with $|j-i| < R-2\ell$ so that $\Lambda_\ell(i_*) \subset \Lambda_R(j)$, and suppose that the interval $\Lambda_\ell(i_*)$ is (m,E)-regular. Then*

$$\begin{aligned} &\|P_i^- R_{\Lambda_L}(E) P^+_{\Lambda_R(j)}\| \\ &\quad \leq \max\left\{ e^{-\tilde{m}(R+1-|j-i|)}, \max_{r\in\Lambda_L} e^{-\tilde{m}\max\{|r-i|,\ell^\tau\}} \left\| P_r^- R_{\Lambda_L}(E) P^+_{\Lambda_R(j)} \right\| \right\} \end{aligned} \tag{1.9.34}$$

where

$$\tilde{m} \geq m(1 - C\ell^{\kappa+\beta-\tau}) \tag{1.9.35}$$

for sufficiently large L.

Proof. Let $E \in I_{1,\delta}$. Let $i,j \in \Lambda_L$ with $|j-i| < R-2\ell$ so that $\Lambda_\ell := \Lambda_\ell(i_*) \subset \Lambda_R(j)$. Consider when $N > L^{\zeta'}$. We apply Proposition 1.6.4,

$$\begin{aligned} \|P_i^- R^{(N)}_{\Lambda_L}(E) P^+_{\Lambda_R(j)}\| &\leq \frac{Q_1}{2J^2\delta\left(1-\frac{2J}{\Delta}\right)} \\ &\quad \times \exp\left(-\log\left(1+\frac{\delta(\Delta-2J)}{4JQ_1}\right)(R+1-|j-i|\right) \\ &\leq \frac{Q_1}{2J^2\delta\left(1-\frac{2J}{\Delta}\right)} e^{-m(R+1-|j-i|)} \leq e^{-m'(R+1-|j-i|)} \end{aligned} \tag{1.9.36}$$

since $R > \ell$ and (1.9.15). Also

$$\begin{aligned} m' &\geq m\left(1 - \frac{\log\frac{1}{2J^2\xi}}{\ell\ell^{-\kappa}}\right) = m\left(1 - \frac{\left|\log\delta\left(1-\frac{2J}{\Delta}\right)\right| + \log(2+\frac{1}{J})}{\ell^{\tau-\kappa}}\right) \\ &\geq m\left(1 - \frac{\ell^\beta + \log(2+\frac{1}{J})}{\ell^{\tau-\kappa}}\right) \geq m\left(1 - \frac{C_{1,J}}{\ell^{\tau-\kappa-\beta}}\right), \end{aligned} \tag{1.9.37}$$

The nontrivial case is when $4J \leq N \leq L^{\zeta'}$. We will consider the Hilbert space $\mathcal{H}^{(N)}_{\Lambda_L}$. It follows from the resolvent identity that

$$\begin{aligned}
P_i^- R_{\Lambda_L}^{(N)} P_{\Lambda_R(j)}^+ &= P_i^- R_{\Lambda_L\backslash\partial\Lambda_\ell}^{(N)} P_{\Lambda_R(j)}^+ - P_i^- R_{\Lambda_L\backslash\partial\Lambda_\ell}^{(N)} \Gamma_{\Lambda_L\backslash\partial\Lambda_\ell} R_{\Lambda_L}^{(N)}(E) P_{\Lambda_R(j)}^+ \\
&= P_i^- R_{\Lambda_L\backslash\partial\Lambda_\ell}^{(N)} P_{\Lambda_\ell}^+ P_{\Lambda_R(j)}^+ - P_i^- R_{\Lambda_L\backslash\partial\Lambda_\ell}^{(N)} \Gamma_{\Lambda_L\backslash\partial\Lambda_\ell} R_{\Lambda_L}^{(N)}(E) P_{\Lambda_R(j)}^+ \\
&= 0 - P_i^- R_{\Lambda_L\backslash\partial\Lambda_\ell}^{(N)} \Gamma_{\Lambda_L\backslash\partial\Lambda_\ell} R_{\Lambda_L}^{(N)}(E) P_{\Lambda_R(j)}^+ \\
&= - P_i^- R_{\Lambda_L\backslash\partial\Lambda_\ell}^{(N)} P_{\Lambda_{\ell^\tau}(i_*)}^+ \Gamma_{\Lambda_L\backslash\partial\Lambda_\ell} R_{\Lambda_L}^{(N)}(E) P_{\Lambda_R(j)}^+ \\
&\quad - P_i^- R_{\Lambda_L\backslash\partial\Lambda_\ell}^{(N)} P_{\Lambda_{\ell^\tau}(i_*)}^- \Gamma_{\Lambda_L\backslash\partial\Lambda_\ell} R_{\Lambda_L}^{(N)}(E) P_{\Lambda_R(j)}^+ \\
&= - A - B
\end{aligned} \tag{1.9.38}$$

To estimate A, first consider the case where $i = i_*$. Then we use (1.2.20) to rewrite

$$A = \sum_{p=\ell^\tau}^{\ell-1} P_i^- R_{\Lambda_L\backslash\partial\Lambda_\ell}^{(N)} P_{\Lambda_p(i)}^+ P_{\{i+p+1,i-p-1\}}^- \Gamma_{\Lambda_L\backslash\partial\Lambda_\ell} R_{\Lambda_L}^{(N)}(E) P_{\Lambda_R(j)}^+. \tag{1.9.39}$$

From here we apply the fact that $\Lambda_\ell(i)$ is (m, E)-regular to get that

$$\begin{aligned}
\|A\| &\le \sum_{p=\ell^\tau}^{\ell-1} \|P_i^- R_{\Lambda_L\backslash\partial\Lambda_\ell}^{(N)} P_{\Lambda_p(i)}^+\| \|P_{\{i+p+1,i-p-1\}}^- \Gamma_{\Lambda_L\backslash\partial\Lambda_\ell} R_{\Lambda_L}^{(N)}(E) P_{\Lambda_R(j)}^+\| && (1.9.40) \\
&\le \sum_{p=\ell^\tau}^{\ell-1} e^{-m'(p+1)} \|P_{\{i+p+1,i-p-1\}}^- \Gamma_{\Lambda_L\backslash\partial\Lambda_\ell} R_{\Lambda_L}^{(N)}(E) P_{\Lambda_R(j)}^+\| \\
&\le e^{-m'\ell} \|P_{\{i+\ell,i-\ell\}}^- \Gamma_{\Lambda_L\backslash\partial\Lambda_\ell} R_{\Lambda_L}^{(N)}(E) P_{\Lambda_R(j)}^+\| && (1.9.41) \\
&\quad + 4J^2 \sum_{p=\ell^\tau}^{\ell-2} e^{-m'(p+1)} \|P_{\{i+p+1,i-p-1\}}^- R_K^{(N)}(E) P_{\Lambda_R(j)}^+\|.
\end{aligned}$$

We proceed by focusing on the first term of (1.9.41).

$$\begin{aligned}
\|P^-_{\{i+\ell,i-\ell\}}\Gamma_{\Lambda_L\setminus\partial\Lambda_\ell}(\cdots)\| &\le \|\Gamma_{\Lambda_L\setminus\partial\Lambda_\ell}(\cdots)\| \\
&\le \|\tilde{h}_{i+\ell,i+\ell+1}P^-_{\{i+\ell,i+\ell+1\}}(\cdots)\| + \|\tilde{h}_{i-\ell,i-\ell-1}P^-_{\{i-\ell,i-\ell-1\}}(\cdots)\| \\
&\le 4J^2\|P^-_{\{i+\ell,i+\ell+1\}}(\cdots)\| + 4J^2\|P^-_{\{i-\ell,i-\ell-1\}}(\cdots)\| \\
&= 4J^2\|(P^-_{i+\ell} + P^+_{i+\ell}P^-_{i+\ell+1})(\cdots)\| + 4J^2\|(P^-_{i-\ell} + P^+_{i-\ell}P^-_{i-\ell-1})(\cdots)\| \\
&\le 4J^2 \sum_{s\in\partial^{\Lambda_L}\Lambda_\ell} \|P^-_s(\cdots)\|
\end{aligned} \tag{1.9.42}$$

Returning to (1.9.41), we see that:

$$\begin{aligned}
\|A\| \le{}& e^{-m'\ell}4J^2 \sum_{s\in\partial^{\Lambda_L}\Lambda_\ell} \|P^-_s R^{(N)}_{\Lambda_L}(E)P^+_{\Lambda_R(j)}\| && (1.9.43)\\
&+ 4J^2 \sum_{p=\ell^\tau}^{\ell-2} e^{-m'(p+1)}\|P^-_{\{i+p+1,i-p-1\}}R^{(N)}_{\Lambda_L}(E)P^+_{\Lambda_R(j)}\| \\
\le{}& \sum_{r\in\Lambda_{\ell+1}(i)\setminus\Lambda_{\ell^\tau}(i)} e^{-m''|r-i|}\|P^-_r R^{(N)}_{\Lambda_L}(E)P^+_{\Lambda_R(j)}\| && (1.9.44)
\end{aligned}$$

Where we used $m'' \ge m'(1-C\ell^{\kappa-1})$ and $\|\tilde{h}_{n,n+1}\| = 4J^2$. If $i \neq i_*$ then a similar estimate holds. In either case we get

$$\|A\| \le \sum_{r\in\Lambda_{\ell+1}(i_*)\setminus\Lambda_{\ell^\tau}(i)} e^{-m''|r-i|}\|P^-_r R^{(N)}_{\Lambda_L}(E)P^+_{\Lambda_R(j)}\|. \tag{1.9.45}$$

Now we estimate B.

$$\begin{aligned}
\|B\| &= \|P_i^- R^{(N)}_{\Lambda_L\setminus\partial\Lambda_\ell} P^-_{\Lambda_{\ell^\tau}(i_*)} \Gamma_{\Lambda_L\setminus\partial\Lambda_\ell} R^{(N)}_{\Lambda_L}(E) P^+_{\Lambda_R(j)}\| \\
&\le \|P_i^- R^{(N)}_{\Lambda_L\setminus\partial\Lambda_\ell} P^-_{\Lambda_{\ell^\tau}(i_*)}\| \|P^-_{\Lambda_{\ell^\tau}(i_*)} \Gamma_{\Lambda_L\setminus\partial\Lambda_\ell} R^{(N)}_{\Lambda_L}(E) P^+_{\Lambda_R(j)}\| \\
&= \|P_i^- R'^{(N)}_{\Lambda_L\setminus\partial\Lambda_\ell} P^-_{\Lambda_{\ell^\tau}(i*)}\| \|\Gamma_{\Lambda_L\setminus\partial\Lambda_\ell} P^-_{\Lambda_{\ell^\tau}(i_*)} R^{(N)}_{\Lambda_L}(E) P^+_{\Lambda_R(j)}\| \\
&\le 4J^2 e^{\ell\beta} \sum_{s\in\partial^{\Lambda_L}\lambda_\ell} \|P^-_{\Lambda_{\ell^\tau}(i_*)} P_s^- R^{(N)}_{\Lambda_L}(E) P^+_{\Lambda_R(j)}\|
\end{aligned} \tag{1.9.46}$$

Next we estimate $\|P^-_{\Lambda_{\ell^\tau}(i_*)} P_s^- R^{(N)}_{\Lambda_L}(E) P^+_{\Lambda_R(j)}\|$, which we will do by using Lemma 1.6.6.

$$\begin{aligned}
\operatorname{dist}\left(\mathbf{S}_{\Lambda_{\ell^\tau}(i)} \cap \mathbf{S}_{\partial^{\Lambda_L}\Lambda_\ell}, \mathbf{M}^{(N)}_{\Lambda_\ell} \setminus \mathbf{S}_{\Lambda_R(j)}\right) &\ge \operatorname{dist}\left(\mathbf{S}_{\Lambda_{\ell^\tau}(i)}, \mathbf{M}^{(N)}_{\Lambda_\ell} \setminus \mathbf{S}_{\Lambda_R(j)}\right) \qquad (1.9.47)\\
&\ge R + 1 - |j-i| - \ell^\tau \\
\operatorname{dist}\left(\Lambda_{\ell^\tau}(i), \partial^{\Lambda_L}\Lambda_\ell(i)\right) &\ge \ell - \ell^\tau \\
\operatorname{dist}\left(\mathbf{S}_{\Lambda_{\ell^\tau}(i)} \cap \mathbf{S}_{\partial^{\Lambda_L}\Lambda_\ell}, \mathbf{M}^{(N)}_{\Lambda_\ell,1}\right) &\ge \ell - \ell^\tau - (N-2) \ge \ell - \ell^\tau - L^{\zeta'}. \qquad (1.9.48)
\end{aligned}$$

The inequality (1.9.48) is not immediately obvious. Consider $\mathbf{m}^*$, a configuration in $\mathbf{S}_{\Lambda_{\ell^\tau}(i)} \cap \mathbf{S}_{\partial^{\Lambda_L}\Lambda_\ell} \subset \mathbf{M}^{(N)}_{\Lambda_\ell}$. Suppose $\mathbf{m}^*$ is as close as possible to $\mathbf{M}^{(N)}_{\Lambda_\ell,1}$. It must be the case that $\mathbf{m}^*$ consists of a single particle in $\partial\Lambda_\ell(i)$, as well as a single particle in $\partial^{in}\Lambda_{\ell^\tau}$, the remaining $N-2$ particles are arranged in a connected configuration somewhere inside $\Lambda_\ell \setminus \Lambda_{\ell^\tau}$ and that component is connected to either one of the boundary particles. In this case it is clear that the distance to move the disconnected particle at the boundary to the connected component in the interior is at least the desired inequality. Continuing where we left off, if we let s from Lemma 1.6.6 be $3\ell/4$, then $\ell^\tau \le s \le (\ell - \ell^\tau - L^{\zeta'}) + \ell^\tau = \ell - L^{\zeta'}$.

$$\begin{aligned}
\|P^-_{\Lambda_{\ell^\tau}(i_*)} P_s^- R^{(N)}_{\Lambda_L}(E) P^+_{\Lambda_R(j)}\| &\le \frac{\sqrt{2}}{J^2\delta\left(1-\frac{2J}{\Delta}\right)} e^{-\log\left(1+\frac{\delta(\Delta-2J)}{32J}\right)(R+1-|j-i|-\ell^\tau)} \\
&+ \frac{\sqrt{2}}{J^2\delta\left(1-\frac{2J}{\Delta}\right)} \sum_{r\in\Lambda_L} \mathrm{e}^{-\log\left(1+\frac{\delta(\Delta-2J)}{32J}\right)\left(1-\frac{4}{3}\ell^{\tau-1}\right)\max\{|r-i|,3\ell/4\}} \left\|P_r^- R^{(N)}_{\Lambda_L}(E) P^+_{\Lambda_R(j)}\right\|
\end{aligned} \tag{1.9.49}$$

Combining all the previous estimates,

$$
\begin{aligned}
\|P_i^- R_{\Lambda_L}^{(N)}(E) P_{\Lambda_R(j)}^+\| \leq & \sum_{r\in\Lambda_{\ell+1}(i_*)\setminus\Lambda_{\ell^\tau}(i)} e^{-m''|r-i|} \|P_r^- R_{\Lambda_L}^{(N)}(E) P_{\Lambda_R(j)}^+\| \\
& + \frac{4\sqrt{2}e^{\ell^\beta}}{J^2\delta\left(1-\frac{2J}{\Delta}\right)} e^{-\log\left(1+\frac{\delta(\Delta-2J)}{32J}\right)(R+1-|j-i|-\ell^\tau)} \\
& + \frac{4\sqrt{2}e^{\ell^\beta}}{J^2\delta\left(1-\frac{2J}{\Delta}\right)} \sum_{r\in\Lambda_L} \mathrm{e}^{-\log\left(1+\frac{\delta(\Delta-2J)}{32J}\right)\left(1-\frac{4}{3}\ell^{\tau-1}\right)\max\{|r-i|,3\ell/4\}} \left\|P_r^- R_{\Lambda_L}^{(N)}(E) P_{\Lambda_R(j)}^+\right\| \\
\leq & \sum_{r\in\Lambda_{\ell+1}(i_*)\setminus\Lambda_{\ell^\tau}(i)} e^{-m''|r-i|} \|P_r^- R_{\Lambda_L}^{(N)}(E) P_{\Lambda_R(j)}^+\| \\
& + e^{2\ell^\beta} e^{-\log\left(1+\frac{\delta(\Delta-2J)}{32J}\right)(R+1-|j-i|-\ell^\tau)} \\
& + e^{2\ell^\beta} \sum_{r\in\Lambda_L} \mathrm{e}^{-\log\left(1+\frac{\delta(\Delta-2J)}{32J}\right)(1-\frac{4}{3}\ell^{\tau-1})\max\{|r-i|,3\ell/4\}} \left\|P_r^- R_{\Lambda_L}^{(N)}(E) P_{\Lambda_R(j)}^+\right\|.
\end{aligned}
\tag{1.9.50}
$$

This holds assuming ℓ is large enough. We use the hypothesis on m, (1.9.15) and obtain

$$
\begin{aligned}
\|P_i^- R_{\Lambda_L}^{(N)}(E) P_{\Lambda_R(j)}^+\| \leq & \, e^{-m'''(R+1-|j-i|)} \\
& + \sum_{r\in\Lambda_L} \mathrm{e}^{-m'''\max\{|r-i|,\ell^\tau\}} \left\|P_r^- R_K^{(N)}(E) P_{\Lambda_R(j)}^+\right\| \\
\leq & \, e^{-m'''(R+1-|j-i|-\ell^\tau)} + (2L+1)\max_{r\in\Lambda_L} \mathrm{e}^{-m'''\max\{|r-i|,\ell^\tau\}} \left\|P_r^- R_{\Lambda_L}^{(N)}(E) P_{\Lambda_R(j)}^+\right\| \\
\leq & \max\left\{e^{-m''''(R+1-|j-i|)}, \max_{r\in\Lambda_L} \mathrm{e}^{-m''''\max\{|r-i|,\ell^\tau\}} \left\|P_r^- R_{\Lambda_L}^{(N)}(E) P_{\Lambda_R(j)}^+\right\|\right\}
\end{aligned}
\tag{1.9.51}
$$

When we incorporate (1.9.36), we arrive at the desired result.

$$
\begin{aligned}
\|P_i^- R_{\Lambda_L}(E) P_{\Lambda_R(j)}^+\| = & \sup_N \|P_i^- R_{\Lambda_L}^{(N)}(E) P_{\Lambda_R(j)}^+\| \\
\leq & \max\left\{e^{-m^{(v)}(R+1-|j-i|)}, \max_{r\in\Lambda_L} \mathrm{e}^{-m^{(v)}\max\{|r-i|,\ell^\tau\}} \left\|P_r^- R_{\Lambda_L}^{(N)}(E) P_{\Lambda_R(j)}^+\right\|\right\}.
\end{aligned}
\tag{1.9.52}
$$

It is not too difficult to show that only a small amount of mass is lost: $m^{(v)} \geq m(1-C\ell^{\kappa+\beta-\tau})$.

□

1.9.2 Buffer Intervals

Definition 1.9.7. An interval $\Upsilon \subset \Lambda_L$ is called an (m, E)-buffer if for all $s \in \partial^{\Lambda_L}\Upsilon$ we have that $s_* = s$ and $\Lambda_\ell(s)$ is an (m, E)-regular interval. In this case we set $\Upsilon' = \Upsilon \setminus \partial^{\Lambda_L}\Upsilon$.

Definition 1.9.8. Given an interval $\Upsilon \subset \Lambda_L$, we set $\widehat{\Upsilon} = \Upsilon \cup (\cup_{s\in\partial^{\Lambda_L}\Upsilon}\Lambda_\ell(s))$

Lemma 1.9.9. *Let $E \in I_{1,\delta}$, and let $\tilde{m}$ be as in* (1.9.35)*. Assume that Λ_L is $(1, \mathcal{N})$-reduced. Let $\Upsilon \subset \Lambda_L$ be an (m, E)-buffer, where m satisfies* (1.9.15)*, $j \in \Lambda_L$ and $\widehat{\Upsilon} \subset \Lambda_R(j)$. Assume that dist $\Big(E, \sigma(H'_{\Lambda_L\setminus\partial\Upsilon})\Big) > e^{-L^\beta}$. There exist $s_\Upsilon \in \partial_{\Lambda_L}\Upsilon$ such that for all $q \in \Upsilon'$ we have*

$$\begin{aligned}
&\|P_q^- R_L(E)P^+_{\Lambda_R(j)}\| \\
&\quad \le 16J^2e^{L^\beta} \max\left\{e^{-\tilde{m}(R+1-|s_\Upsilon-j|)}, \max_{r\in\Lambda_L} e^{-\tilde{m}\max\{|r-s_\Upsilon|,\ell^\tau\}}\|P_r^- R_L(E)P^+_{\Lambda_R(j)}\|\right\}.
\end{aligned} \tag{1.9.53}$$

Proof. Let $\Upsilon \subset \Lambda_L$ be an (m, E)-buffer, where $E \in I_{1,\delta}$, and let $j \in \Lambda_L$, $\hat{\Upsilon} \subset \Lambda_R(j)$, and $q \in \Upsilon'$. It follows from the resolvent identity that,

$$P_q^- R_L(E)P^+_{\Lambda_R(j)} = -P_q^- R_{\Lambda_L\setminus\partial\Upsilon}(E)\Gamma_{\Lambda_L\setminus\partial\Upsilon}R_L(E)P^+_{\Lambda_R(j)}, \tag{1.9.54}$$

Hence,

$$\begin{aligned}
\|P_q^- R_L(E)P^+_{\Lambda_R(j)}\| &\le \|R'_{\Lambda_L\setminus\partial\Upsilon}(E)\|\|\Gamma_{\partial\Upsilon}R_L(E)P^+_{\Lambda_R(j)}\| \\
&\le 4J^2e^{L^\beta}\sum_{s\in\partial^{\Lambda_L}\Upsilon}\|P_s^- R_L(E)P^+_{\Lambda_R(j)}\| \le 16J^2e^{L^\beta}\|P^-_{s_\Upsilon}R_L(E)P^+_{\Lambda_R(j)}\|,
\end{aligned} \tag{1.9.55}$$

where $s = s_\Upsilon$ is selected from $\partial^{\Lambda_L}\Upsilon$ so that $\|P_s^- R_L(E)P^+_{\Lambda_R(j)}\|$ is at a maximum. For $s \in \partial^{\Lambda_L}(\Upsilon)$ we have, since Υ is a buffer, that $\Lambda_\ell(s)$ is an (m, E)-regular interval. Since Λ_L is $(1, \mathcal{N})$-reduced, $E \in I_{1,\delta}$ and (1.9.15) holds, it follows from Lemma 1.9.6 that (1.9.34) holds with $i = s_\Upsilon$. The conclusion follows from (1.9.55) and (1.9.34).

□

1.10 The Multiscale Analysis

Theorem 1.10.1. *Fix $\Delta_0 > 2J$, $\lambda_0 > 0$, and $\delta \in (0,1)$. Suppose Δ_0, λ_0, δ, and the scale L_0 satisy* (1.5.5)- (1.5.7) *Let $\Delta \geq \Delta_0$, and $\lambda \geq \lambda_0$, consider an interval $I \subset I_{1,\delta}$, and suppose that with the parameters Δ and λ we have*

$$\mathbb{P}\{\mathcal{R}(m_0, L_0, I, u, v)\} \geq 1 - e^{-L_0^\zeta} \text{ for all } u, v \in \mathbb{Z} \text{ with } |u - v| > 2L_0, \tag{1.10.1}$$

where m_0 satisfy (1.5.11).

Then, if L_0 is sufficiently large, depending on λ_0 and Δ_0, setting $L_{k+1} = L_k^\gamma$, we have

$$\mathbb{P}\{\mathcal{R}(m_k, L_k, I, u, v)\} \geq 1 - e^{-L_k^\zeta} \text{ for all } u, v \in \mathbb{Z} \text{ with } |u - v| > 2L_k, \tag{1.10.2}$$

for all $k = 0, 1, \ldots$ Also m_k is a decreasing sequence with $m_k \geq m_0/2$.

Proof. The theorem is proven by induction. We start with a scale $\ell \geq L_0$ and move to scale $L = \ell^\gamma$. Let $\ell^{-\kappa} < m \leq m_0$, so that m satisfies (1.9.15) with Δ_0 and λ_0 at scale ℓ, and suppose we have

$$\mathbb{P}\{\mathcal{R}(m, \ell, I, u, v)\} \geq 1 - e^{-\ell^\zeta} \text{ for all } u, v \in \mathbb{Z} \text{ with } |u - v| > 2\ell. \tag{1.10.3}$$

We will show that, if ℓ is sufficiently large, the same statement holds at scale L with a new mass m_1 with

$$L^{-\kappa} \leq m(1 - C\ell^{-q}) \leq m_1 \leq \log\left(1 + \frac{\delta(\Delta_0 - 2J)}{4JQ_1}\right) \tag{1.10.4}$$

Here $C > 0$ and $q > 0$, in particular this means that m_1 satisfies (1.9.15) at scale L. Select $u \in \mathbb{Z}$ and consider and interval $\Lambda_L = \Lambda_L(u)$. Let $\mathcal{Y}$ be the event that Λ_L is $(1, \mathcal{N})$-reduced.

We know from Lemma 1.7.2 that

$$\mathbb{P}\{\mathcal{Y}\} \geq 1 - e^{-\frac{c_\mu}{4} L^{\zeta'}} > 1 - \tfrac{1}{10} e^{-L^\zeta} \tag{1.10.5}$$

Let $\Xi = \Xi_{L,\ell}(u)$ be the collection of intervals $\Lambda_\ell \subset \Lambda_L$, that is

$$\Xi = \{\Lambda_\ell(y) : y \in \Lambda_{L-\ell}(u)\} \text{ so } |\Xi| \leq 2L. \tag{1.10.6}$$

We set (recall (1.5.4))

$$S_\ell = 2\lfloor \ell^{(\gamma-1)\zeta_*} \rfloor - 1. \tag{1.10.7}$$

For each $n \in \mathbb{N}$ we consider the event $\mathcal{B}(n)$, that for some $E \in I$ there exist at least n (m, E)-nonregular disjoint intervals in Ξ. Note that

$$\mathcal{B}(2) = \bigcup_{\substack{v,v' \in \Xi \\ |v-v'|>2\ell}} (\mathcal{R}(m, \ell, I, v, v'))^c \tag{1.10.8}$$

It follows from (1.10.3) that

$$\mathbb{P}(\mathcal{B}_2) \leq (2L)^2 e^{-\ell^\zeta} \tag{1.10.9}$$

Observe that $S_\ell + 1$ is an even integer and that events based on disjoint intervals are independent. From this, we have

$$\mathbb{P}\{\mathcal{B}(S_\ell + 1)\} \leq (\mathbb{P}\{\mathcal{B}(2)\})^{\frac{S_\ell+1}{2}} \leq \left(4\ell^{2\gamma} e^{-\ell^\zeta}\right)^{\lfloor \ell^{(\gamma-1)\zeta_*} \rfloor} \leq \tfrac{1}{10} e^{-L^\zeta}. \tag{1.10.10}$$

Let $\omega \in (\mathcal{B}(S_\ell + 1))^c$ and $E \in I$. Then there exists $A_\omega(E) = \{a_1, a_2, \ldots, a_S\} \subset \Xi$, where $S \leq S_\ell$, $|a_s - a_{s'}| > 2\ell$ for $s \neq s'$ and $\Lambda_\ell(b)$ is (m, E)-regular for all $\Lambda_\ell(b) \in \Xi \setminus \bigcup_s \Lambda_{2\ell}(a_s)$.

We will construct a collection of disjoint (m, E)-buffer intervals, $\{\Upsilon_t\}_{t=1,\ldots,T}$, where $T \leq S$, which contain the S disjoint (m, E)-nonregular intervals. From this we know that if $b \in Y = \Lambda_L \setminus \cup_t \Upsilon'_t$ then $\Lambda_\ell(b)$ is (m, E)-regular.

We need to estimate the size of $\Upsilon = \bigcup_{t=1}^{T} \Upsilon_t$ from above. Notice that the worst case is when $T = S$, this means that the disjoint nonregular intervals are far enough away from each other such that when each one is covered with an interval of length $2(2\ell + 1)$, and there is no overlap between the buffers. Also suppose that one nonregular interval, say B, is close enough to (say) the left end point of Λ_L that when B is covered by an interval, Υ_B, with $2\ell + 1$ points on each side there is not enough space for an interval of size ℓ to fit on the left hand side; in this case we expand Υ_B all the way to the left endpoint of Λ_L. Here we have that $|\Upsilon_B| \leq 2(2\ell + 1) + \ell + 1$. In total we have

$$\begin{aligned} |\Upsilon| = \sum_{t=1}^{T} |\Upsilon_t| &\leq 2(2\ell + 1)S + 2(\ell + 1) \leq 6\ell(S_\ell + 1) \\ &\leq 12\ell^{(\gamma-1)\zeta_*-1} < \ell^{\gamma_*} < \ell^{\gamma\tau} = L^\tau. \end{aligned} \tag{1.10.11}$$

Where, using (1.5.1), γ_* is chosen to satisfy

$$(\gamma - 1)\zeta_* + 1 < \gamma_* < \gamma\tau. \tag{1.10.12}$$

For $E \in I$ we consider the event

$$\mathcal{G}_{u,L}(E) = \{\operatorname{dist}(E, \sigma(H'_{\Lambda_L(u),K})) > e^{-L^\beta} \text{ for all intervals } K \in \mathcal{K}_{u,L}\}, \tag{1.10.13}$$

where $\mathcal{K}_{u,L}$ denotes the collection of all intervals $K \subset \Lambda_L(u)$ such that either $\ell \leq |K| \leq \ell^{\gamma_*}$ or $K = \Lambda_L$ (note $H_{\Lambda_L,\Lambda_L} = H_{\Lambda_L}$). Then for all $\omega \in (\mathcal{B}(S_\ell + 1))^c \cap \mathcal{G}_{u,L}(E)$ the disjoint (m, E)-buffers $\{\Upsilon_t\}_{t=1,\ldots,T}$ satisfy the hypothesis of Lemma 1.9.9.

Let $E \in I$ and $\omega \in \mathcal{Y} \cap (\mathcal{B}(S_\ell + 1))^c \cap \mathcal{G}_{u,L}(E)$ and set

$$G(r) = \|P_r^-(H_L - E)^{-1} P^+_{\Lambda_R(i)}\| \text{ for } r \in \Lambda_L \tag{1.10.14}$$

So, for all $r \in \Lambda_L$ we have the estimate $G(r) \leq \|(H_L - E)^{-1}\| \leq e^{L^\beta}$. For $R > L^\tau$, we will use previous lemmas to obtain the estimate $G(i) \leq e^{-m_1(R+1)}$. The following two estimates are essential:

1. Let $j \in \Lambda_L$ with $\Lambda_\ell(j_*) \subset \Lambda_R(i)$ and $\Lambda_\ell(j_*)$ is (m, E)-regular. Then from Lemma 1.9.6 we have that

$$G(j) \leq \max\left\{ e^{-m^{(1)}(R+1-|j-i|)}, \max_{r \in \Lambda_L} e^{-m^{(1)} \max\{|r-j|, \ell^\tau\}} G(r) \right\}. \tag{1.10.15}$$

2. Suppose for some t, that $q \in \Upsilon_t'$ such that $d(i, \Upsilon_t) + |\Upsilon_t| + 1 + \ell \leq R$, so that $\widehat{\Upsilon}_t \subset \Lambda_R(i)$. Then it follows from Lemma 1.9.9 that there exists $s_t = s_{\Upsilon_t} \in \partial^{\Lambda_L} \Upsilon_t$ such that for all $q \in \Upsilon_t'$ we have

$$\begin{aligned} G(q) &\leq 8J^2 e^{L^\beta} \max\left\{ e^{-m^{(1)}(R+1-|s_t-j|)}, \max_{r \in \Lambda_L} e^{-m^{(1)} \max\{|r-s_t|, \ell^\tau\}} G(r) \right\} \\ &\leq \max\left\{ e^{-m^{(2)}(R+1-|s_t-j|)}, \max_{r \in \Lambda_L} e^{-m^{(2)} \max\{|r-s_t|, \ell^\tau\}} G(r) \right\} \end{aligned} \tag{1.10.16}$$

where

$$m^{(2)} \geq m^{(1)}(1 - C\ell^{\gamma\beta+\kappa-\tau}) \geq m(1 - C'\ell^{\gamma\beta+\kappa-\tau}). \tag{1.10.17}$$

We notice that the right hand of the estimate does not depend on q, therefore we can say

$$G(\Upsilon_t') = \max_{t \in \Upsilon_t'} G(q) \leq \max\left\{ e^{-m^{(2)}(R+1-|s_t-j|)}, \max_{r \in \Lambda_L} e^{-m^{(2)} \max\{|r-s_t|, \ell^\tau\}} G(r) \right\}. \tag{1.10.18}$$

We will frequently change m, the rate of decay, throughout the proof. Each time the rate of decay changes we will increase the number in the superscript. It can be shown that the following inequality holds at each step

$$m^{(n)} \geq m^{(n-1)} \left(1 - C_n \ell^{-q_n}\right), \tag{1.10.19}$$

where for every n, C_n and q_n are positive constants which can be calculated explicitly. We will adjust the mass finitely many times in the following procedure. The goal of the following portion of the proof is to attain an estimate of the form

$$G(i) \leq e^{-m^{(n)}(R+1)} \tag{1.10.20}$$

for any choice of i. If $G(i) = 0$ we have the estimate and we stop immediately; henceforth assume $G(i) \neq 0$. Suppose first that $i \in Y$, then it follows from (1.10.15) that

$$G(i) \leq \max\left\{e^{-m^{(1)}(R+1)}, \max_{r\in\Lambda_L} e^{-m^{(1)}\max\{|r-i|,\ell^\tau\}} G(r)\right\}. \tag{1.10.21}$$

If the first argument of the max is larger then we have the desired estimate and we stop. Suppose not, then we have that

$$G(i) \leq e^{-m^{(1)}\max\{|r_1-i|,\ell^\tau\}} G(r_1) \text{ for some } r_1 \in \Lambda_L. \tag{1.10.22}$$

If $|r_1 - i| > R - 2\ell$, then

$$G(i) \leq e^{-m^{(1)}(R-2\ell)} e^{L^\beta} \leq e^{-m^{(2)}(R+1)}, \tag{1.10.23}$$

and again we are done. If $|r_1 - i| \leq R - 2\ell$ and $r_1 \in Y$, we can apply (1.10.15) again, this time we have

$$G(i) \le e^{-m^{(1)}\max\{|r_1-i|,\ell^\tau\}} \max\left\{e^{-m^{(1)}(R+1-|r_1-i|)}, \max_{r\in\Lambda_L} e^{-m^{(1)}\max\{|r-r_1|,\ell^\tau\}}G(r)\right\}$$

$$\le \max\left\{e^{-m^{(1)}(R+1)}, \max_{r\in\Lambda_L} e^{-m^{(1)}\max\{|r-i|,2\ell^\tau\}}G(r)\right\} \tag{1.10.24}$$

Proceeding as above, if either (1.10.22) or (1.10.23) hold then we stop. Or we have

$$G(i) \le e^{-m^{(1)}\max\{|r_2-i|,2\ell^\tau\}}G(r_2) \text{ for some } r_2 \in \Lambda_L \text{ with } |r_2 - i| \le R - 2\ell. \tag{1.10.25}$$

If $r_2 \in Y$ we can repeat the procedure. In fact we can continue to repeat the procedure as long as r_n is in Y. Eventually we will either have, for some n_* the estimate,

$$\max\{|r_{n_*} - i|, n_*\ell^\tau\} \ge R - 2\ell^{\gamma_*} \tag{1.10.26}$$

in which case we arrive at

$$G(i) \le e^{-m^{(1)}\max\{|r_{n_*}-i|,n_*\ell^\tau\}}G(r_{n_*}) \le e^{-m^{(1)}(R-2\ell^{\gamma_*})}e^{L^\beta} \le e^{-m^{(2)}(R+1)} \tag{1.10.27}$$

and we are done, or $r_n \notin Y$ and (1.10.26) does not hold. In the second case we must have $r_n = r_{n_1} \in \Upsilon'_{t_1}$ for some t_1, and

$$G(i) \le e^{-m^{(1)}|r_{n_1}-i|}G(r_{n_1}) \le e^{-m^{(1)}d(i,\Upsilon'_{t_1})}G(\Upsilon'_{t_1}). \tag{1.10.28}$$

Notice that since (1.10.26) is false we have that

$$d(i, \Upsilon_{t_1}) + |\Upsilon_{t_1}| + 1 + \ell \le |r_{n_1} - i| + 2\ell^{\gamma_*} \le R. \tag{1.10.29}$$

In order to cover all possibilities we take the convention that $n_1 \geq 0$ and $r_0 = i$. To handle the future situations where $\{r_n\}$ leaves Y we say that $\{r_{n_j}\}$ is the subsequence of $\{r_n\}$ which contains only the points which lie inside $\cup_i \Upsilon'_{t_i}$ (without satisfying (1.10.26)). To continue we will apply (1.10.18) and obtain

$$G(i) \leq e^{-m^{(1)} d(i,\Upsilon'_{t_1})} \max\left\{ e^{-m^{(2)}(R+1-|s_t-j|)}, \max_{r\in\Lambda_L} e^{-m^{(2)} \max\{|r-s_t|,\ell^\tau\}} G(r) \right\}. \tag{1.10.30}$$

If the first argument is maximal then we have

$$\begin{aligned} G(i) &\leq e^{-m^{(2)}(d(i,\Upsilon'_{t_1})+(R+1-|s_t-j|))} \leq e^{-m^{(2)}(R+1-|\Upsilon_{t_1}|)} \\ &\leq e^{-m^{(2)}(R+1-\ell^{\gamma_*})} \leq e^{-m^{(3)}(R+1)} \end{aligned} \tag{1.10.31}$$

and we are done. If the second portion is maximal then we move to another step

$$G(i) \leq e^{-m^{(1)} d(i,\Upsilon'_{t_1})} e^{-m^{(2)} \max\{|r_{n_1+1}-s_t|,\ell^\tau\}} G(r_{n_1+1}) \leq e^{-m^{(2)}(|r_{n_1+1}-i|-|\Upsilon_{t_1}|)} G(r_{n_1+1}). \tag{1.10.32}$$

If $|r_{n_1+1} - i| \geq R - 2\ell^{\gamma_*}$, then we can obtain the estimate $G(i) \leq e^{-m^{(3)}(R+1)}$. If $|r_{n_1+1} - i| < R - 2\ell^{\gamma_*}$, then we will apply the same strategy again. Notice however that if $r_{n_1+1} \in \Upsilon'_{t_1}$ then from the first inequality in (1.10.32) we get that

$$G(i) \leq e^{-m^{(2)}\ell^\tau} e^{-m^{(1)} d(i,\Upsilon'_{t_1})} G(\Upsilon'_{t_1}). \tag{1.10.33}$$

If this is the case then the preceding argument can be summarized by the following implication:

$$G(i) \leq e^{-m^{(1)} d(i,\Upsilon'_{t_1})} G(\Upsilon'_{t_1}) \implies G(i) \leq e^{-m^{(2)}\ell^\tau} e^{-m^{(1)} d(i,\Upsilon'_{t_1})} G(\Upsilon'_{t_1}). \tag{1.10.34}$$

The chain of implications can iterate indefinitely, so $G(i) = 0$. This possibility was already ruled out from the beginning; therefore $r_{n_1+1} \notin \Upsilon'_{t_1}$. From this point on we continue the iterating process as long as necessary, we move along some sequence of points in Λ_L:

$$\{r_n\} = \{r_0, r_1, \ldots, r_{n_1}, r_{n_1+1}, \ldots r_{n_2}, r_{n_2+1} \ldots r_{n_b}, \ldots r_{n_b+k}\}. \tag{1.10.35}$$

Notice that if $r_{n_i}, r_{n_j} \in \Upsilon'_{t_i}$ and $n_j > n_i$ then an argument along the same lines as in (1.10.34) will imply that $G(i) = 0$. This means that the stepping process visits each buffer interval at most once before terminating. Either the process terminates (as in (1.10.21), (1.10.23), or, (1.10.31)) or we attain an estimate of this form:

$$\begin{aligned}&G(i) \leq e^{-m^{(2)} d(i,\Upsilon'_{t_1})} e^{-m^{(2)} \max\{d(s_{t_1},\Upsilon'_{t_2}),\ell^\tau\}} \cdots e^{-m^{(2)} \max\{d(s_{t_{b-1}},\Upsilon'_{t_b}),\ell^\tau\}} e^{-m^{(2)}|r_{n_b+k}-s_{t_b},|} G(r_{n_b+k})\\ &\text{with, } |r_{n_b+k} - i| > R - 2\ell^{\gamma_*}.\end{aligned} \tag{1.10.36}$$

This is still satisfactory though:

$$\begin{aligned}G(i) &\leq e^{-m^{(2)} d(i,\Upsilon'_{t_1})} e^{-m^{(2)} \max\{d(s_{t_1},\Upsilon'_{t_2}),\ell^\tau\}} \cdots e^{-m^{(2)} \max\{d(s_{t_{b-1}},\Upsilon'_{t_b}),\ell^\tau\}} e^{-m^{(2)}|r_{n_b+k}-s_{t_b}|} e^{L^\beta}\\ &\leq e^{-m^{(2)}\left(d(i,\Upsilon_{t'_1})+\left(d(\Upsilon'_{t_1},\Upsilon_{t'_2})-|\Upsilon_{t_1}|-\ell\right)+\cdots+\left(d(\Upsilon'_{t_{b-1}},\Upsilon_{t'_b})-|\Upsilon_{t_{b-1}}|-\ell\right)\right)} e^{-m^{(2)}|r_{n_b+k}-s_{t_b}|} e^{L^\beta}\\ &\leq e^{-m^{(2)}(|r_{n_b+k}-i|-2\ell^{\gamma_*})} e^{L^\beta} \leq e^{-m^{(2)}(R-4\ell^{\gamma_*})} e^{L^\beta} \leq e^{-m^{(3)}(R+1)}.\end{aligned} \tag{1.10.37}$$

We have shown that for $E \in I$ and $\omega \in \mathcal{Y} \cap (\mathcal{B}(S_\ell+1))^c \cap \mathcal{G}(E)$ that

$$G(i) = \|P_i^-(H_L - E)^{-1} P^+_{\Lambda_R(i)}\| \leq e^{-m^{(3)}(R+1)} \text{ for all } i \in \Lambda_L \text{ and } R > L^\tau. \tag{1.10.38}$$

In this case we see that $\Lambda_L(u)$ is $(m^{(3)}, E)$-regular.

Now let $u, v \in \mathbb{Z}$ with $|u - v| > 2L$, and set $\mathcal{Y}_{u,v,L} = \mathcal{Y}_{u,L} \cap \mathcal{Y}_{v,L}$. If $K \in \mathcal{K}_{u,L}$ and $K' \in \mathcal{K}_{v,L}$ then from Lemma 1.7.7 we have

$$\mathbb{P}\{\{d(\sigma_{I_1}(H_K), \sigma_{I_1}(H_{K'})) \leq 2e^{-L^\beta}\} \cap \mathcal{Y}_{u,v,L}\} \leq C4^\alpha e^{-\alpha L^\beta} \lambda_0^{-\alpha}(2L+1)^{16Q_1+2} < e^{-\frac{2}{3}\alpha L^\beta}.$$

$$(1.10.39)$$

Define the event

$$\mathcal{G}_{u,v,L} = \left\{ d(\sigma_{I_1}(H_K), \sigma_{I_1}(H_{K'})) > 2e^{-L^\beta} \text{ for all } K \in \mathcal{K}_{u,L}, K' \in \mathcal{K}_{v,L} \right\}. \tag{1.10.40}$$

It follows that

$$\mathbb{P}\{\mathcal{G}^c_{u,v,L} \cap \mathcal{Y}_{u,v,L}\} \leq (2L\ell^{\gamma_*})^2 e^{-\frac{2}{3}\alpha L^\beta} < e^{-\frac{1}{2}\alpha L^\beta}, \tag{1.10.41}$$

for sufficiently large L. Also,

$$\mathcal{G}_{u,v,L} \subset \mathcal{G}_{u,L}(E) \cup \mathcal{G}_{v,L}(E) \text{ for all } E \in I. \tag{1.10.42}$$

This holds for the following reason: if $\omega \in \mathcal{G}_{u,v,L} \cap \mathcal{G}_{u,L}(E)^c$ we must have $\omega \in \mathcal{G}_{v,L}(E)$, and similarly for the same statement with u and v interchanged. We now define the event $\mathcal{E}_{u,v,L} = (\mathcal{B}(S_\ell + 1))^c \cap \mathcal{G}_{u,v,L} \cap \mathcal{Y}_{u,v,L}$. From (1.10.41), (1.10.10), and (1.10.5) we get

$$\mathbb{P}\{\mathcal{E}^c_{u,v,L}\} \leq \mathbb{P}\{\mathcal{B}(S_\ell + 1)\} + \mathbb{P}\{\mathcal{G}^c_{u,v,L} \cap \mathcal{Y}_{u,v,L}\} + \mathbb{P}\{\mathcal{Y}^c_{u,v,L}\} \tag{1.10.43}$$

$$\leq \tfrac{1}{10} e^{-L^\zeta} + e^{-\frac{1}{2}\alpha L^\beta} + \tfrac{1}{10} e^{-L^\zeta} \leq e^{-L^\zeta}, \tag{1.10.44}$$

for a sufficiently large L. Moreover from (1.10.38) and (1.10.42),

$$\mathcal{E}_{u,v,L} \subset \mathcal{R}(m^{(3)}, L, I, u, v). \tag{1.10.45}$$

Therefore we conclude that

$$\mathbb{P}\{\mathcal{R}(m^{(3)}, L, I, u, v)\} \geq 1 - e^{-L^{\zeta}}. \tag{1.10.46}$$

We can finish the proof by induction. Let $L_{k+1} = L_k^{\gamma}$ for $k = 0, 1, 2 \ldots.$ For $k = 0$ the conclusion holds with mass m_0 by the hypothesis. If it holds for k with mass m_k then it holds for $k + 1$ with mass $m_{k+1} = m_k^{(3)}$, that is

$$m_{k+1} \geq m_k(1 - C_1 L_k^{-q_1})(1 - C_2 L_k^{-q_2})(1 - C_3 L_k^{-q_3}) \geq m_k(1 - C_* L_k^{-q_*}) \tag{1.10.47}$$

It is apparent that as long as C_* and q_* are greater than 0, and L_0 is large enough then the product of terms $(1 - C_* L_k^{-q_*})$ will converge to a positive number. Simply taking L_0 large enough can guarantee the inequality $m_k > m_0/2$ for all k. □

Corollary 1.10.2. *Assume the hypotheses of Theorem 1.10.1. Then, if L_0 is sufficiently large, for all $L \geq L_0^{\gamma}$ we have*

$$\mathbb{P}\{\mathcal{R}(\tfrac{m_0}{4}, L, I, u, v)\} \geq 1 - e^{-L^{\xi}} \text{ for all } u, v \in \mathbb{Z} \text{ with } |u - v| > 2L. \tag{1.10.48}$$

The proof follows from the next lemma.

Lemma 1.10.3. *Given $\Delta_0 > 1$, $\lambda_0 > 0$ and $\delta \in (0, 1)$, let the scale ℓ satisfy* (1.5.6). *Let $E \in I \subset I_{1,\delta}$ and let $L = \ell^{\gamma'}$, where $\gamma \leq \gamma' < \gamma^2$. Consider a $(1, \mathcal{N})$-reduced interval $\Lambda_L(u)$ and suppose every interval $\Lambda_\ell(u')$ is (m, E)-regular for all $u' \in \Xi_{L, L_{k-1}}(u)$, and m satisfies* (1.9.15). *Then for sufficiently large ℓ,*

$$\|(H_{\Lambda_L(u)} - E)^{-1}\| \leq e^{L^{\beta}}. \tag{1.10.49}$$

Proof. Let $\Lambda = \Lambda_L(u)$, $H_\Lambda = H_{\Lambda_L(u)}$, and $R_\Lambda(E) = (H_\Lambda - E)^{-1}$. Suppose Λ is $(1, \mathcal{N})$-reduced and every interval $\Lambda_\ell(u')$ is (m, E)-regular for all $u' \in \Xi_{L,L_{k-1}}(u)$. Since Λ is $(1, \mathcal{N})$-reduced it follows from Lemma 1.7.2 that for $N > L^{\zeta'}$,

$$\|(H_\Lambda^{(N)} - E)^{-1}\| \leq \frac{2Q_1}{4J^2\delta\left(1 - \frac{2J}{\Delta_0}\right)} \leq e^{L^\beta}. \tag{1.10.50}$$

We now fix $4J \leq N \leq L^{\zeta'}$, and omit N from the notation. Let $i \in \Lambda_L$. With some basic computations we have:

$$\begin{aligned} P_i^- R_\Lambda(E) &= P_i^- R_{\Lambda/\partial\Lambda_\ell}(E)(H_{\Lambda/\partial\Lambda_\ell} - E)R_\Lambda(E) \\ &= P_i^- R_{\Lambda/\partial\Lambda_\ell}(E)(H_\Lambda - E - \Gamma_{\Lambda\backslash\partial\Lambda_\ell})R_\Lambda(E) \\ &= P_i^- R_{\Lambda/\partial\Lambda_\ell}(E) - P_i^- R_{\Lambda/\partial\Lambda_\ell}(E)\Gamma_{\Lambda\backslash\partial\Lambda_\ell}R_\Lambda(E). \end{aligned} \tag{1.10.51}$$

Since $\Lambda_\ell(i_*)$ is (m, E)-regular, we have from Lemma 1.9.4

$$\begin{aligned} \|P_i^- R_\Lambda(E)\| &\leq \|R'_{\Lambda/\partial\Lambda_\ell}(E)\| + \|P_i^- R_{\Lambda/\partial\Lambda_\ell}(E)\Gamma_{\Lambda\backslash\partial\Lambda_\ell}R_\Lambda(E)\| \\ &\leq e^{\ell^\beta} + \|P_i^- R_{\Lambda/\partial\Lambda_\ell}(E)\Gamma_{\Lambda\backslash\partial\Lambda_\ell}R_\Lambda(E)\|. \end{aligned} \tag{1.10.52}$$

We estimate the second term,

$$\begin{aligned} &\|P_i^- R_{\Lambda/\partial\Lambda_\ell}(E)\Gamma_{\Lambda\backslash\partial\Lambda_\ell}R_\Lambda(E)\| \\ &\quad\leq \|P_i^- R_{\Lambda/\partial\Lambda_\ell}(E)P^+_{\Lambda_{\ell^\tau}(i)}\Gamma_{\Lambda\backslash\partial\Lambda_\ell}R_\Lambda(E)\| + \|P_i^- R_{\Lambda/\partial\Lambda_\ell}(E)P^-_{\Lambda_{\ell^\tau}(i)}\Gamma_{\Lambda\backslash\partial\Lambda_\ell}R_\Lambda(E)\| \\ &\quad\leq \|P_i^- R_{\Lambda/\partial\Lambda_\ell}(E)P^+_{\Lambda_{\ell^\tau}(i)}\|\|\Gamma_{\Lambda\backslash\partial\Lambda_\ell}\|\|R_\Lambda(E)\| + \|P_i^- R_{\Lambda/\partial\Lambda_\ell}(E)\|\|P^-_{\Lambda_{\ell^\tau}(i)}\Gamma_{\Lambda\backslash\partial\Lambda_\ell}R_\Lambda(E)\| \\ &\quad\leq 8J^2 e^{-m\ell^\tau}\|R_\Lambda(E)\| + e^{\ell^\beta}\|P^-_{\Lambda_{\ell^\tau}(i)}\Gamma_{\Lambda\backslash\partial\Lambda_\ell}R_\Lambda(E)\|, \end{aligned} \tag{1.10.53}$$

Next we use that $R_{\Lambda,1}^{(N)}(E) = (H_{\Lambda,1}^{(N)} - E)^{-1}$, recalling (1.6.1),

$$H_{\Lambda,1}^{(N)} = H_\Lambda^{(N)} + (Q_1 - 1)\left(1 - \frac{2J}{\Delta_0}\right)P_{\Lambda,1}^{(N)}. \tag{1.10.54}$$

We continue our calculation with the resolvent identity, the dependence on E is suppressed for brevity.

$$\begin{aligned}
\|P^-_{\Lambda_{\ell^\tau}(i)}\Gamma_{\Lambda\backslash\partial\Lambda_\ell}R_\Lambda\| &\le \|P^-_{\Lambda_{\ell^\tau}(i)}\Gamma_{\Lambda\backslash\partial\Lambda_\ell}R^{(N)}_{\Lambda,1}\| + Q_1\left(1-\tfrac{2J}{\Delta_0}\right)\|P^-_{\Lambda_{\ell^\tau}(i)}\Gamma_{\Lambda\backslash\partial\Lambda_\ell}R^{(N)}_{\Lambda,1}P^{(N)}_{\Lambda,1}R_\Lambda\| \\
&\le 8J^2\|P^-_{\Lambda_{\ell^\tau}(i)}R^{(N)}_{\Lambda,1}\| + Q_1\left(1-\tfrac{2J}{\Delta_0}\right)\|P^-_{\Lambda_{\ell^\tau}(i)}\Gamma_{\Lambda\backslash\partial\Lambda_\ell}R^{(N)}_{\Lambda,1}P^{(N)}_{\Lambda,1}\|\|R_\Lambda\| \\
&\le \frac{8J^2Q_1}{\delta\left(1-\frac{2J}{\Delta_0}\right)} + 4J^2Q_1\left(1-\tfrac{2J}{\Delta_0}\right)\|R_\Lambda\|\sum_{b\in\boldsymbol{\partial}\Lambda_\ell(i_*)}\|P^-_bP^-_{\Lambda_{\ell^\tau}(i)}R^{(N)}_{\Lambda,1}P^{(N)}_{\Lambda,1}\| \\
&\le \frac{8J^2Q_1}{\delta\left(1-\frac{2J}{\Delta_0}\right)} + \frac{8J^2Q_1^2}{\delta}\|R_\Lambda\|\sum_{b\in\boldsymbol{\partial}\Lambda_\ell(i_*)}e^{-\log\left(1+\frac{\delta(\Delta_0-2J)}{4JQ_1}\right)\operatorname{dist}(\mathbf{S}_{\Lambda_{\ell^\tau}(i)}\cap\mathbf{S}_b,\mathbf{M}^{(N)}_{\Lambda,1})} \\
&\le \frac{8J^2Q_1}{\delta\left(1-\frac{2J}{\Delta_0}\right)} + \|R_\Lambda\|\frac{16J^2Q_1^2}{\delta}e^{-\log\left(1+\frac{\delta(\Delta_0-2J)}{4JQ_1}\right)\ell(1-2\ell^{-q})}. \qquad (1.10.55)
\end{aligned}$$

In the preceding argument, Corollary 1.6.5 was applied twice. Notice that for each $b \in \boldsymbol{\partial}\Lambda_\ell(i_*)$ we have that

$$\operatorname{dist}(\mathbf{S}_{\Lambda_{\ell^\tau}(i)}\cap\mathbf{S}_b,\mathbf{M}^{(N)}_{\Lambda,1}) \ge \ell-\ell^\tau-(N-2) \ge \ell-\ell^\tau-L^{\zeta'} \ge \ell-\ell^\tau-\ell^{\zeta'\gamma^2} \ge \ell(1-2\ell^{-q}). \quad (1.10.56)$$

With $q = 1-\max\{\tau,\zeta',\gamma^2\} > 0$. Putting it all together, along with (1.9.15) we have that

$$\begin{aligned}
\|P^-_iR_\Lambda(E)\| &\le e^{\ell^\beta} + 8J^2e^{-m\ell^\tau}\|R_\Lambda(E)\| \qquad (1.10.57) \\
&\quad + e^{\ell^\beta}\left(\frac{8J^2Q_1}{\delta\left(1-\frac{2J}{\Delta_0}\right)} + \frac{16J^2Q_1^2}{\delta}e^{-\log\left(1+\frac{\delta(\Delta_0-2J)}{4JQ_1}\right)\ell(1-2\ell^{-q})}\|R_\Lambda(E)\|\right) \\
&\le \left(1+\frac{8J^2Q_1}{\delta\left(1-\frac{2J}{\Delta_0}\right)}\right)e^{\ell^\beta} + \left(8J^2e^{-\ell^{-\kappa}\ell^\tau} + \frac{16J^2Q_1^2}{\delta}e^{\ell^\beta-\ell^{-\kappa}\ell(1-2\ell^{-q})}\right)\|R_\Lambda(E)\| \\
&\le e^{2\ell^\beta} + Ce^{-\ell^{\tau-\kappa}}\|R_\Lambda(E)\|.
\end{aligned}$$

This holds for sufficiently large ℓ. Therefore

$$\begin{aligned} \|R_\Lambda(E)\| &\leq \sum_{i\in\Lambda_L} \|P_i^- R_\Lambda(E)\| \leq (2L+1)\left(e^{2\ell^\beta} + Ce^{-\ell^{\tau-\kappa}}\|R_\Lambda(E)\|\right) \\ &\leq e^{3\ell^\beta} + \tfrac{1}{2}\|R_\Lambda(E)\| \implies \|R_\Lambda(E)\| \leq 2e^{3\ell^\beta} \leq e^{L^\beta}. \end{aligned} \tag{1.10.58}$$

We have now proved the bound holds when $N \leq L^{\zeta'}$ and when $N > L^{\zeta'}$. $\square$

Proof of Corollary 1.10.2 We assume the conclusions of Theorem 1.10.1. Given a scale $L \geq L_0^\gamma = L_1$, let $k = k(L)$ be a natural number, chosen so that $L_k \leq L < L_{k+1}$. We have that $L_k = L_{k-1}^\gamma \leq L < L_{k-1}^{\gamma^2}$. So $L = L_{k-1}^{\gamma'}$ for some $\gamma' \in [\gamma, \gamma^2)$. Now, given $u, v \in \mathbb{Z}$ with $|u - v| > 2L$, consider the event

$$\mathcal{F}_{u,v,L} = \bigcap_{\substack{u' \in \Xi_{L,L_{k-1}}(u) \\ v' \in \Xi_{L,L_{k-1}}(v)}} \mathcal{R}(\tfrac{m_0}{2}, L_{k-1}, I, u', v'). \tag{1.10.59}$$

From the conclusion of Theorem 1.10.1 and (1.7.14), and the fact that $\xi\gamma' < \xi\gamma^2 < \zeta$,

$$\mathbb{P}\{\mathcal{F}_{u,v,L} \cap \mathcal{Y}_{u,v,L}\} \geq 1 - (2L)^2 e^{-L_{k-1}^\zeta} - \tfrac{1}{4}e^{-L^\zeta} = 1 - (2L)^2 e^{-L^{\zeta/\gamma'}} - \tfrac{1}{4}e^{-L^\zeta} > 1 - e^{-L^\xi}. \tag{1.10.60}$$

Let $\omega \in \mathcal{F}_{u,v,L} \cap \mathcal{Y}_{u,v,L}$. Then $\Lambda_L(u)$ and $\Lambda_L(v)$ are $(1, \mathcal{N})$-reduced, and given $E \in I$ then either $\Lambda_{L_{k-1}}(u')$ is $(m_0/2, E)$-regular for all $u' \in \Xi_{L,L_{k-1}}(u)$, or $\Lambda_{L_{k-1}}(v')$ is $(m_0/2, E)$-regular for all $v' \in \Xi_{L,L_{k-1}}(v)$. Suppose the former, the other case is similar. Notice that the conclusion of Lemma 1.10.3 holds for Λ_L, and that given $i \in \Lambda_L(u)$, (1.10.15) holds for all $j \in \Lambda_L(u)$ with $\Lambda_\ell(j_*) \subset \Lambda_R(i)$, and where $\ell = L_{k-1}$ and $m^{(1)} = m_0/2$. We proceed as in (1.10.21) - (1.10.27) to conclude that $\Lambda_L(u)$ is $(m_0/4, E)$-regular.

Proof of Theorem 1.5.4 for $H^{\geq 4J}$. It follows immediately from Theorem 1.8.2 and Corollary 1.10.2. $\square$

1.11 Eigenfunction localization

The proof of the main theorem in this section will depend on a series lemmas that we will prove in advance. Also we continue to use the convention that if m is a decay rate then $m^{(1)} \geq m(1-C\ell^{-q})$ for some positive explicit constants C and q, and furthermore that $m^{(2)}$, $m^{(3)}$ and so on will denote further modifications of the mass in the same manner, but with possibly different constants.

Lemma 1.11.1. *Suppose Λ_L is (m, I) localizing for H_{Λ_L} where $I \subset I_{1,\delta}$ and $m > L^{-\kappa}$, let $\{(\varphi_\nu, \nu)\}_{\nu \in \sigma(H_{\Lambda_L})}$ be an eigensystem for H_{Λ_L}. Then for all $\nu \in \sigma_I(H_{\Lambda_L})$ we have*

$$\|P_i^- \varphi_\nu\| \|P_j^- \varphi_\nu\| \leq e^{-m^{(1)}|i-j|} \text{ for all } i, j \in \Lambda_L \text{ with } |i-j| \geq L^{\tilde{\tau}}. \tag{1.11.1}$$

It follows that

$$\sum_{\nu \in \sigma(H_{\Lambda_L}) \cap I} \|P_i^- \varphi_\nu\| \|P_j^- \varphi_\nu\| \leq e^{-m^{(2)}|i-j|} \text{ for all } i, j \in \Lambda_L \text{ with } |i-j| \geq L^{\tilde{\tau}}. \tag{1.11.2}$$

Proof. Let $\nu \in \sigma(H_{\Lambda_L}) \cap I$. It follows from Definition 1.5.1 that for all $i \in \Lambda_L$,

$$\|P_i^- \varphi_\nu\| \leq e^{mL^\tau} e^{-m|i-j_\nu|}. \tag{1.11.3}$$

Now if $i, j \in \Lambda_L$ with $|i-j| \geq L^{\tilde{\tau}}$, then

$$\|P_i^- \varphi_\nu\| \|P_j^- \varphi_\nu\| \leq e^{2mL^\tau} e^{-m|i-j_\nu| - m|j-j_\nu|} \leq e^{-m'|i-j|}. \tag{1.11.4}$$

This is exactly (1.11.1). Also (1.11.2) follows from (1.11.1) and (1.7.6). $\square$

Lemma 1.11.2. *Let the interval $\Lambda = \Lambda_L$ be $(1, \mathcal{N})$-reduced and let (ψ, E) be an eigenpair for H_Λ with $E \in I_{1,\delta}$. Let $i \in \Lambda_L$ and assume the interval $\Lambda_\ell = \Lambda_\ell(i_*)$ is (m, E)-regular, where m satisfies* (1.9.15). *Then for sufficiently large L,*

$$\|P_i^- \psi\| \leq e^{-m^{(1)} \max |r_1 - i|, \ell^\tau} \|P_{r_1}^- \psi\| \text{ for some } r_1 \in \Lambda_L. \tag{1.11.5}$$

Proof. Since Λ is $(1, \mathcal{N})$-reduced and $E \in I_{1,\delta}$, we must have $\mathcal{N}_\Lambda \psi = N\psi$ for some $N \leq L^{\zeta'}$. We have

$$\begin{aligned} P_i^- \psi &= P_i^- R_{\Lambda \backslash \partial \Lambda_\ell}(E)(H_{\Lambda \backslash \partial \Lambda_\ell} - E)\psi = -P_i^- R_{\Lambda \backslash \partial \Lambda_\ell}(E) \Gamma_{\partial \Lambda_\ell} \psi \\ &= -P_i^- R_{\Lambda \backslash \partial \Lambda_\ell}(E) P_{\Lambda_\ell}^+ \Gamma_{\Lambda \backslash \partial \Lambda_\ell} \psi - P_i^- R_{\Lambda \backslash \partial \Lambda_\ell}(E) P_{\Lambda_\ell}^- \Gamma_{\partial \Lambda_\ell} \psi = A + B \end{aligned} \tag{1.11.6}$$

To estimate A, we will focus on the case where $i_* = i$ and use (1.2.20), getting

$$A = -\sum_{p=\ell^\tau}^{\ell-1} P_i^- R_{\Lambda \backslash \partial \Lambda_\ell}(E) P_{\Lambda_p(i)}^+ P_{\{i+p+1, i-p-1\}}^- \Gamma_{\Lambda \backslash \partial \Lambda_\ell} \psi \tag{1.11.7}$$

this calculation is similar to the one in (1.9.38). We proceed by using the fact that $\Lambda_\ell(i)$ is (m, E)-regular, analogous to the derivation of (1.9.45)

$$\begin{aligned} \|A\| &\leq \sum_{p=\ell^\tau}^{\ell-1} \|P_i^- R_{\Lambda \backslash \partial \Lambda_\ell}(E) P_{\Lambda_p(i)}^+\| \|P_{\{i+p+1, i-p-1\}}^- \Gamma_{\Lambda \backslash \partial \Lambda_\ell} \psi\| \\ &\leq \sum_{p=\ell^\tau}^{\ell-1} e^{-m^{(1)}(p+1)} \|P_{\{i+p+1, i-p-1\}}^- \Gamma_{\Lambda \backslash \partial \Lambda_\ell} \psi\| \\ &\leq 8J^2 \sum_{p=\ell^\tau}^{\ell-2} e^{-m^{(1)}(p+1)} \|P_{\{i+p+1, i-p-1\}}^- \psi\| + e^{-m^{(1)}\ell} \sum_{s \in \partial \Lambda_\ell(i)} \|P_s^- \psi\| \\ &\leq \sum_{r \in \Lambda_{\ell+1} \backslash \Lambda_{\ell^\tau}} e^{-m^{(2)}|r-i|} \|P_r^- \psi\|. \end{aligned} \tag{1.11.8}$$

If $i \neq i_*$, the estimate holds and the same proof will work with some minor adjustments.

$$\begin{aligned}\|B\| &= P_i^- R_{\Lambda\setminus\partial\Lambda_\ell}(E) P_{\Lambda_\ell}^- \Gamma_{\Lambda\setminus\partial\Lambda_\ell}\psi\| \leq \|P_i^- R_{\Lambda\setminus\partial\Lambda_\ell}(E)\| \|P_{\Lambda_\ell}^- \Gamma_{\Lambda\setminus\partial\Lambda_\ell}\psi\| \\ &\leq \|R'_{\Lambda\setminus\partial\Lambda_\ell}(E)\| \|P_{\Lambda_\ell}^- \Gamma_{\Lambda\setminus\partial\Lambda_\ell}\psi\| \leq e^{\ell^\beta} \sum_{s\in\partial\Lambda_\ell} \|P_{\Lambda_\ell}^- P_s^- \psi\|\end{aligned} \tag{1.11.9}$$

We now estimate $\|P_{\Lambda_\ell}^- P_s^- \psi\|$ by Corollary 1.6.7. For the value of s in Corollary 1.6.7 we use $\frac{3}{4}\ell$ and we use (1.9.48).

$$\sum_{s\in\partial\Lambda_\ell} \|P_{\Lambda_\ell}^- P_s^- \psi\| \leq \frac{8(Q_1-1)\sqrt{Q_1}}{\delta \mathcal{W}_0^{(N)}} \sum_{r\in\Lambda} e^{-\log\left(1+\frac{\delta(\Delta_0-2J)}{4JQ_1}\right)\left(1-\frac{4}{3}\ell^{\tau-1}\right)\max\{|r-i|,\frac{3}{4}\ell\}} \|P_r^-\psi\| \tag{1.11.10}$$

We now combine all of the preceding estimates to get the following

$$\begin{aligned}\|P_i^-\psi\| &\leq \sum_{r\in\Lambda_{\ell+1}\setminus\Lambda_{\ell^\tau}} e^{-m^{(2)}|r-i|}\|P_r^-\psi\| + e^{\ell^\beta} C(\delta,J,N) \sum_{r\in\Lambda} e^{-M\max\{|r-i|,\frac{3}{4}\ell\}}\|P_r^-\psi\| \\ &\leq \sum_{r\in\Lambda} e^{-m^{(3)}\max\{|r-i|,\ell^\tau\}} \leq (2L+1) e^{-m^{(3)}\max\{|r_*-i|,\ell^\tau\}}\|P_{r_*}^-\psi\| \\ &\leq e^{-m^{(4)}\max\{|r_*-i|,\ell^\tau\}}\|P_{r_*}^-\psi\| \text{ for some } r_*\in\Lambda.\end{aligned} \tag{1.11.11}$$

This holds for sufficiently large L and we used the fact that $m > L^{-\kappa}$. ☐

Proof of Theorem 1.5.2. Assume L is large enough that (1.5.19) holds at scale $\ell = L^{\gamma^{-1}}$. Let $\Lambda_L = \Lambda_L(j)$ for some $j\in\mathbb{Z}$. Let $\mathcal{Y}_L = \mathcal{Y}_{\Lambda_L(j),1}$ be the event that Λ_L is $(1,\mathcal{N})$-reduced. Consider the event

$$\mathcal{F}_L = \bigcap_{u,v\in\Xi_{L,\ell}(j),\ |u-v|>2\ell} \mathcal{R}(m,\ell,I_{1,\delta},u,v). \tag{1.11.12}$$

It follows from (1.5.19) that

$$\mathbb{P}\{\mathcal{F}_L\} \geq 1 - 4L^2 e^{-\ell^{\xi_1}} = 1 - 4L^2 e^{-L^{\xi_1\gamma^{-1}}} > 1 - \tfrac{1}{2} e^{-L^{\xi_2}}. \tag{1.11.13}$$

Hence, using (1.7.14) and $0 < \zeta_2 < \zeta$ we have that

$$\mathbb{P}\{\mathcal{F}_L \cap \mathcal{Y}_L\} > 1 - e^{-L^{\xi_2}}. \tag{1.11.14}$$

To finish the proof we need to show that $\mathcal{F}_L \cap \mathcal{Y}_L \subset \mathcal{Q}(\frac{m}{2}, L, I_{1,\delta}, j)$. Fix $\omega \in \mathcal{F}_L \cap \mathcal{Y}_L$, and let (ψ, E) be an eigenpair for H_{Λ_L} with $E \in I$. Since $\omega \in \mathcal{Y}_L$, we have that $\mathcal{N}_{\Lambda_L}\psi = N\psi$ for some $N \leq L^{\zeta'}$. Notice that there must exist a $q \in \Xi_{L,\ell}(j)$ such that the interval $\Lambda_\ell(q)$ is not (m, ℓ)-localizing. Otherwise, from Lemma 1.11.2 we have that $\|P_i^- \psi\| \leq e^{-m^{(1)}\ell^\tau}$ for all $i \in \Lambda_L$, and hence

$$1 = \|\psi\| \leq \sum_{i \in \Lambda_L} \|P_i^- \psi\| \leq (2L+1)e^{-m^{(1)}\ell^\tau} < 1, \tag{1.11.15}$$

a contradiction. Now since $\omega \in \mathcal{F}_L$, it follows that $\Lambda_\ell(u)$ is (m, ℓ)-localizing for all $u \in \Xi_{L,\ell}$ with $|u - q| > 2\ell$. So let $i \in \Lambda_L$ with $|i - q| > L^\tau$. Since $\Lambda_\ell(i_*)$ is (m, ℓ)-localizing, it follows from Lemma 1.11.2 that

$$\|P_i^- \psi\| \leq e^{-m^{(1)} \max\{|r_1 - i|, \ell^\tau\}} \|P_{r_1}^- \psi\| \text{ for some } r_1 \in \Lambda_L. \tag{1.11.16}$$

If $|r_1 - q| \leq 3\ell$, we have

$$\|P_i^- \psi\| \leq e^{-m^{(1)}|r_1 - i|} \leq e^{-m^{(1)}(|i-q| - |r_1 - q|)} \leq e^{-m^{(1)}(|i-q| - 3\ell)} \leq e^{-m^{(2)}|i-q|}, \tag{1.11.17}$$

since $|i - q| > L^\tau$. Now if $|r_1 - q| > 3\ell$, then $|(r_1)_* - q| > 2\ell$ and another application of Lemma 1.11.2 yields

$$\begin{aligned} \|P_i^- \psi\| &\leq e^{-m^{(1)} \max\{|r_1 - i|, \ell^\tau\}} e^{-m^{(1)} \max\{|r_2 - r_1|, \ell^\tau\}} \|P_{r_2} \psi\| && (1.11.18) \\ &\leq e^{-m^{(1)} \max\{|r_2 - i|, 2\ell^\tau\}} \text{ for some } r_2 \in \Lambda_L && (1.11.19) \end{aligned}$$

If $|r_2 - q| \leq 3\ell$ we have (1.11.17) and we are done, otherwise we have that $|(r_2)_* - q| > 2\ell$ and we can apply Lemma 1.11.2 again in the same way. We repeat this argument n times, stopping when either $|r_n - q| \leq 3\ell$ or $n\ell^\tau \geq |i - q|$, obtaining the desired estimate

$$\|P_i^- \psi\| \leq e^{-m^{(2)}|i-q|}. \tag{1.11.20}$$

We have shown that ψ is $(q, m^{(2)})$ localized. Now if we pick L sufficiently large we will have that $m^{(2)} \geq m/2$. Thus $\omega \in \mathcal{Q}(m/2, L, I_{1,\delta}, j)$. Given L sufficiently large, the estimates (1.5.12) and (1.5.13) follow from (1.11.14) and Lemma 1.11.1 respectively. □

1.12 Fixed Particle Numbers

The results up to this point suffice to prove Theorem 1.5.4 and Theorem 1.5.2 for the case where $N \geq 4J$. In multiple places we relied on the bound from Theorem 1.4.5 and the fact that for $N \geq 4J$, $\mathcal{W} \geq 4J^2$. We will show that an analog of Theorem 1.5.4 will hold for the operator $H^{<4J}$. In the low particle number situation we have to take a different approach. In this section we will show that $H^{<4J}$ is, with small modification, unitarily equivalent to the one studied in [51].

From this equivalence we will state the main theorem of this section, show that the hypothesis of the theorem is satisfied, and then finally give a proof of the main theorem.

1.12.1 Configurations in $\mathbb{Z}^N$

Definition 1.12.1. The space of allowed configurations:

$$\mathcal{Z}^{N,J} = \{\mathbf{x} \in \mathbb{Z}^N : x_1 \leq \cdots \leq x_N \text{ and at most } 2J \text{ consecutive coordinates equal.}\} \tag{1.12.1}$$

See [43]. We will treat $\mathcal{Z}^{N,J}$ as a graph by considering it as an induced subgraph of $\mathbb{Z}^N$, in particular this means that two vertices $x, y \in \mathcal{Z}^{N,J}$ are adjacent, denoted by $x \sim y$, if $|x-y|_1 = 1$ in $\mathbb{Z}^N$.

Definition 1.12.2.

$$\ell^2(\mathcal{Z}^{N,J}) = \left\{ f : \mathcal{Z}^{N,J} \to \mathbb{C} \text{ s.t. } \|f\|^2 = \sum_{x \in \mathcal{Z}^{N,J}} |f(x)|^2 < \infty. \right\} \tag{1.12.2}$$

with the obvious inner product. We denote the canonical basis,

$$\psi_x(y) = \delta_{x,y}. \tag{1.12.3}$$

Lemma 1.12.3. *There is a graph isomorphism* $\varphi \colon \mathbf{M}^{(N)} \to \mathcal{Z}^{N,J}$.

Proof. We write the support of $\mathbf{m}$ as a finite set of integers

$$\operatorname{supp}(\mathbf{m}) = \{x_1, x_2, x_3, \dots x_n\} \text{ where } x_i \text{ are increasing and } n \leq N. \tag{1.12.4}$$

Then mapping is given as,

$$\varphi : \mathbf{M}^{(N)} \to \mathcal{Z}^{N,J} \text{ as } \varphi(\mathbf{m}) = (\overbrace{x_1, x_1, \dots}^{\mathbf{m}(x_1)}, \overbrace{x_2, x_2, \dots}^{\mathbf{m}(x_2)}, \dots, \overbrace{x_n, x_n, \dots}^{\mathbf{m}(x_n)}). \tag{1.12.5}$$

Since $\sum_{\mathbb{Z}} \mathbf{m}(x_i) = N$, we see that $\varphi(\mathbf{m}) \in \mathbb{Z}^N$, and since $\mathbf{m} : \mathbb{Z} \to \{0, 1, \dots 2J\}$ we have furthermore that $\varphi(\mathbf{m}) \in \mathcal{Z}^{N,J}$ as desired. The inverse mapping is

$$[\varphi^{-1}(x)](j) = |\{k \in \mathbb{Z} : x_k = j\}|. \tag{1.12.6}$$

See [43] for the proof that φ and φ^{-1} are adjacency preserving. $\square$

Now we associate to φ a unitary operator from $\ell^2(\mathcal{Z}^{N,J})$ to $\ell^2(\mathbf{M}^{(N)})$, U_φ, by extending linearly from the basis vectors, that is

$$U_\varphi(\psi_{\mathbf{m}}) = \psi_{\varphi(\mathbf{m})}. \tag{1.12.7}$$

Fix $N \in \mathbb{N}$. We define a random Schrodinger operator on $\ell^2(\mathcal{Z}^{N,J})$ that is unitarily equivalent to $H^{(N)}$ by using φ.

$$\tilde{H}^{(N)} : \ell^2(\mathcal{Z}^{N,J}) \to \ell^2(\mathcal{Z}^{N,J}) \text{ as } \tilde{H}^{(N)} = U_\varphi H^{(N)} U_\varphi^{-1} \tag{1.12.8}$$

Similarly, with conjugation by U_φ we also define:

$$\tilde{H}^{(N)} = -\frac{1}{2\Delta}\tilde{A}^{(N)} + \tilde{\mathcal{W}}^{(N)} + \lambda \tilde{V}_\omega^{(N)}. \tag{1.12.9}$$

Observation 1.12.4. Since φ is a graph isomorphism, $\tilde{A}^{(N)}$ is a weighted N-dimensional adjacency operator on $\ell^2(\mathcal{Z}^{N,J})$. Furthermore we extend $\tilde{A}^{(N)}$ to a weighted adjacency operator, A, on $\ell^2(\mathbb{Z}^N)$ by assigning weight zero to any edge in $\mathbb{Z}^N$ which connects to a point in $\mathbb{Z}^N \setminus \mathcal{Z}^{N,J}$.

Observation 1.12.5. Consider a random multiplication operator on $\ell^2(\mathbb{Z}^N)$ given by

$$\tilde{V}_\omega^{(N)} f(\mathbf{x}) = \left(\sum_{i=1}^{N} \omega_{x_i}\right) f(\mathbf{x}). \tag{1.12.10}$$

It is straightforward to show that for any $f \in \ell^2(\mathcal{Z}^{N,J})$ we have that

$$\tilde{V}_\omega^{(N)} f = U_\varphi^* V_\omega^{(N)} U_\varphi(f). \tag{1.12.11}$$

Therefore the restriction of $\tilde{V}_\omega^{(N)}$ to $\ell^2(\mathcal{Z}^{N,J})$ is unitarily equivalent to $V_\omega^{(N)}$.

Lemma 1.12.6. *We have*

$$\tilde{\mathcal{W}}^{(N)}(\mathbf{x})f(\mathbf{x}) = \left(2JN - \sum_{1\le i<j\le n} \delta_{|x_j-x_i|,1}\right) f(\mathbf{x}). \tag{1.12.12}$$

Proof. Recalling(1.3.23), we have

$$\tilde{\mathcal{W}}^{(N)}(\mathbf{x}) = \mathcal{W}^{(N)}(\varphi^{-1}\mathbf{x}) = 2JN - \sum_{i\in\mathbb{Z}}[\varphi^{-1}\mathbf{x}](i)[\varphi^{-1}\mathbf{x}](i+1)$$

$$\begin{aligned}\sum_{i\in\mathbb{Z}}[\varphi^{-1}\mathbf{x}](i)[\varphi^{-1}\mathbf{x}](i+1) &= \sum_{i\in\mathbb{Z}}\left(\sum_{j=1}^{N}\delta_{x_j,i}\right)\left(\sum_{k=1}^{N}\delta_{x_k,i+1}\right) = \sum_{i\in\mathbb{Z}}\sum_{j,k\in\{1,\ldots N\}}\delta_{x_j,i}\delta_{x_k,i+1}\\ &= \sum_{j=1}^{N}\sum_{k=1}^{N}\delta_{x_j,x_k-1} = \sum_{1\le j\le k\le N}\delta_{x_k-x_j,1}\\ &= \sum_{1\le j<k\le N}\delta_{x_k-x_j,1} = \sum_{1\le j<k\le N}\delta_{|x_k-x_j|,1}\end{aligned} \tag{1.12.13}$$

The last few steps follow because the points in $\mathcal{Z}^{N,J}$ are ordered.

∎

Now we need to extend the Hamiltonian, $\tilde{H}$ to an operator, K, that acts on all of $\ell^2(\mathbb{Z}^N)$ and is of the form treated in [51]. First we will define a potential on all of $\ell^2(\mathbb{Z}^N)$.

Definition 1.12.7.

$$U(\mathbf{x}) = \sum_{1\le i<j\le N}\tilde{U}(x_j - x_i). \tag{1.12.14}$$

Where

$$\tilde{U}(x) = -\delta_{|x|,1}. \tag{1.12.15}$$

Note that if $\mathbf{x} \in \mathcal{Z}^{N,J}$ then

$$U(\mathbf{x}) = \tilde{\mathcal{W}}^{(N)}(\mathbf{x}) - 2JN. \tag{1.12.16}$$

Definition 1.12.8. We define the main operator on $\ell^2(\mathbb{Z}^n)$

$$K_\omega^{(N)} = -\frac{1}{2\Delta} A + U + \lambda \tilde{V}_\omega^{(N)} = \tilde{H}_\omega^{(N)} - 2JN, \tag{1.12.17}$$

where A is the weighted adjacency operator on $\mathbb{Z}^N$ in Observation 1.12.4, and U is the range 1 interaction, namely $\tilde{\mathcal{W}} - 2JN$, given in Lemma 1.12.6. Lastly $\tilde{V}_\omega^{(N)}$ is the natural extension of the identity in Observation 1.12.5 to a random potential on all of $\mathbb{Z}^N$.

Except for the fact that A is a weighted Laplacian, $K_\omega^{(N)}$ is of the form of operators considered in [51]. However since the weights are non-negative and uniformly bounded, (see (1.3.21) and (1.3.22)), the conclusions from that paper will follow from the same proofs. In particular we obtain all the localization results for $K_\omega^{(N)}$ proved in [51].

Definition 1.12.9. Let $\mathbf{x} = (x_1, x_2, \ldots x_N)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_N)$ be two points in $\mathbb{Z}^N$, the Hausdorff distance $d_H(\mathbf{x}, \mathbf{y})$ is given by

$$d_H(\mathbf{x}, \mathbf{y}) = \max\{\max_i \min_j |x_i - y_j|, \max_j \min_i |x_i - y_j|\}. \tag{1.12.18}$$

Definition 1.12.10. Let $\mathbf{x} \in \mathbb{R}^N$ and let $L > 0$. Define the box of radius L centered at $\mathbf{x}$,

$$B_L(\mathbf{x}) = \{\mathbf{y} \in \mathbb{Z}^N : \|\mathbf{y} - \mathbf{x}\|_\infty \leq L\}. \tag{1.12.19}$$

Definition 1.12.11. For fixed N, given $E \in \mathbb{R}$ and $m > 0$, a box $B = B_L(\mathbf{a})$ is said to be (m, E)-good if

$$m > L^{-\kappa} \tag{1.12.20}$$

$$\operatorname{dist}(E, \sigma(K_B^{(N)})) > e^{-L^\beta} \tag{1.12.21}$$

$$|\langle \delta_{\mathbf{x}}, (K_B^{(N)} - E)^{-1} \delta_{\mathbf{y}} \rangle| \leq e^{-m\|\mathbf{x}-\mathbf{y}\|_\infty} \text{ for all } \mathbf{x} \text{ and } \mathbf{y} \text{ with } \|\mathbf{x}-\mathbf{y}\|_\infty > L^\tau. \tag{1.12.22}$$

Definition 1.12.12.

$$\tilde{\mathcal{A}}(m, L, I, \mathbf{x}) = \{E \in I \implies B_L(\mathbf{x}) \text{ is } (m, E) - \text{good}\}. \tag{1.12.23}$$

$$\tilde{\mathcal{R}}(m, L, I, \mathbf{x}, \mathbf{y}) = \{E \in I \implies B_L(\mathbf{y}) \text{ or } B_L(\mathbf{x}) \text{ is } (m, E) - \text{good.}\} \tag{1.12.24}$$

Now observe in particular that,

$$\mathbb{P}\{\tilde{\mathcal{R}}(m, L, I, \mathbf{x}, \mathbf{y})\} \geq \mathbb{P}\{\tilde{\mathcal{A}}(m, L, I, \mathbf{x})\} \quad \text{for all} \quad \mathbf{x}, \mathbf{y}. \tag{1.12.25}$$

Below is the main theorem from [51] but adapted to our setting. We will prove that the hypothesis of this theorem is satisfied.

Theorem 1.12.13. *Fix $N \geq 1$ and $0 < \zeta < \zeta' < 1$. There exists $\mathcal{L} = \mathcal{L}(N)$, such that if for some $L_0 \geq \mathcal{L}$,*

$$\sup_{\mathbf{x} \in \mathbb{R}^N} \mathbb{P}\{B_{L_0}(\mathbf{x}) \textit{ is } (\tfrac{1}{4}, E) - \textit{nongood }\} \leq e^{-L_0^{\zeta'}} \textit{ for all } E \in \mathbb{R}, \tag{1.12.26}$$

then there exists $L_\zeta = L_\zeta(N, L_0)$ and $\delta_\zeta = \delta_\zeta(N, L_0) > 0$ with the following property: Let $E_1 \in \mathbb{R}$ with $I(E_1) = [E_1 - \delta_\zeta, E_1 + \delta_\zeta]$, then for every $L \geq L_\zeta$ and $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ with $d_H(\mathbf{a}, \mathbf{b}) \geq 2L$, we have

$$\mathbb{P}\{\tilde{\mathcal{R}}(\tfrac{1}{8}, L, I(E_1), \mathbf{a}, \mathbf{b})\} \geq 1 - e^{-L^\zeta}. \tag{1.12.27}$$

Remark 1.12.14. We can get results uniform in $N \in \{1, 2, \ldots 4J - 1\}$.

Definition 1.12.15. Given $E \in \mathbb{R}$ and $m > 0$, a box $B_L(\mathbf{a})$ is said to be (m, E)-good2 if

$$m > 2L^{-\kappa} \tag{1.12.28}$$

$$\operatorname{dist}(E, \sigma(K^{(N)}_{B_L(\mathbf{a})})) > 2e^{-L^\beta} \tag{1.12.29}$$

$$|\langle \delta_{\mathbf{x}}, (K^{(N)}_{B_L(\mathbf{a})} - E)^{-1} \delta_{\mathbf{y}} \rangle| \le e^{-m\|\mathbf{x}-\mathbf{y}\|_\infty}. \text{ for all } \mathbf{x} \text{ and } \mathbf{y} \text{ with } \|\mathbf{x}-\mathbf{y}\|_\infty > L^\tau. \tag{1.12.30}$$

Theorem 1.12.16. *Let $m = \frac{1}{4}$, fix $\lambda_0 > 0$, and let the scale L satisfy*

$$e^{-\frac{1}{6}L^\beta} \le \lambda_0. \tag{1.12.31}$$

Then, if L is sufficiently large, given $\lambda > \lambda_0$ and $\Delta > 2J$ such that

$$e^{L^{\zeta''}} \le \Delta\lambda, \tag{1.12.32}$$

we have

$$\mathbb{P}\{\tilde{\mathcal{R}}(\tfrac{1}{4}, L, I(E, e^{-3L}), \mathbf{x}, \mathbf{y})\} \ge 1 - e^{-L^\zeta}, \tag{1.12.33}$$

for all $E \in \mathbb{R}$ and for all $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^N$.

Lemma 1.12.17. *Fix $N \in \mathbb{N}$ and fix $\lambda_0 > 0$. Then choose $\lambda > \lambda_0$ and $\Delta > 2J$ so that* (1.12.32) *and* (1.12.31) *are both satisfied for a sufficiently large L (depending on N). Let $B = B_L$ be a box of size L, $E \notin \sigma(K^{(N)}_B)$, let $R^{(N)}_B(E) = (K^{(N)}_B - E)^{-1}$. Then for all $E \in \mathbb{R}$, we have*

$$\mathbb{P}\left\{B \text{ is } \left(\tfrac{1}{2}, E\right) - \mathit{good2}\right\} \ge 1 - e^{-L^{\zeta_1}}.$$

Proof. This proof is similar to the proof of Lemma 1.8.3. Given $\eta > 0$, it follows from the

Hölder Continuity of μ, that for all $\mathbf{x} \in \mathbb{Z}^N$ we have

$$\mathbb{P}\left\{|U(\mathbf{x}) + \lambda V_\omega(\mathbf{x}) - E| \le \eta\right\} \le C\left(\tfrac{2\eta}{\lambda}\right)^\alpha, \tag{1.12.34}$$

Consider the event $\mathcal{Q}$:

$$\mathcal{Q} = \left\{\left\|(U + \lambda V_\omega - E)^{-1}\right\| \ge \tfrac{1}{\eta}\right\}, \tag{1.12.35}$$

so that

$$\mathbb{P}\left\{\mathcal{Q}\right\} \le C\left(\tfrac{2\eta}{\lambda}\right)^\alpha (2L+2)^N \le \mathrm{e}^{-L^{\zeta_1}}, \tag{1.12.36}$$

which holds if

$$\eta \le \frac{\lambda}{2}\left(\frac{(2L+2)^{-N}}{C} e^{-L^{\zeta_1}}\right)^{1/\alpha}. \tag{1.12.37}$$

We take

$$\eta = \lambda \mathrm{e}^{-\frac{1}{2}L^{\zeta''}} \quad (\text{recall} \quad \zeta_1 < \zeta' < \zeta'' < \beta), \tag{1.12.38}$$

so (1.12.36) holds for L sufficiently large. When taking $\omega \in \mathcal{Q}^{\mathrm{c}}$, we have,

$$\begin{aligned} R_B^{(N)}(E) &= \left(K_B^{(N)} - E\right)^{-1} = \left(-\tfrac{1}{2\Delta}A_B^{(N)} + U + \lambda V - E\right)^{-1} \\ &= (U + \lambda V - E)^{-1}\left(1 - \tfrac{1}{2\Delta}A_B^{(N)}(U + \lambda V - E)^{-1}\right)^{-1} \\ &= (U + \lambda V - E)^{-1}\left(1 + \sum_{n=1}^{\infty}\left(\tfrac{1}{2\Delta}A_B^{(N)}(U + \lambda V - E)^{-1}\right)^n\right), \end{aligned} \tag{1.12.39}$$

$$\left\|\tfrac{1}{2\Delta}A_B^{(N)}\left(U+\lambda V-E\right)^{-1}\right\| \le \tfrac{1}{2\eta\Delta}\left\|A_B^{(N)}\right\| \le \tfrac{2J}{\eta\Delta}\left\|\mathcal{W}\right\| \le \tfrac{4J^2N}{\eta\Delta} \le \tfrac{1}{2}$$

which we obtain by requiring (1.12.32) with L sufficiently large, depending on N.

$$\langle\delta_{\mathbf{x}},(U+\lambda V-E)^{-1}\left(\tfrac{1}{2\Delta}A_B^{(N)}\left(U+\lambda V-E\right)^{-1}\right)^n\delta_{\mathbf{y}}\rangle = 0 \tag{1.12.40}$$

if $R=\|\mathbf{x}-\mathbf{y}\|_1>n$. Hence

$$\begin{aligned}
&\left|\langle\delta_{\mathbf{x}},(K_B^{(N)}-E)^{-1}\delta_{\mathbf{y}}\rangle\right| \\
&\quad\le \sum_{n\ge R}\left|\langle\delta_{\mathbf{x}},(U+\lambda V-E)^{-1}\left(\tfrac{1}{2\Delta}A_B^{(N)}\left(U+\lambda V-E\right)^{-1}\right)^n\delta_{\mathbf{y}}\rangle\right| \\
&\quad\le \sum_{n\ge R}\tfrac{1}{\eta}\tfrac{1}{2^n} = \tfrac{1}{\lambda}\mathrm{e}^{\frac{1}{2}L^{\zeta''}}\tfrac{2}{2^R} \le \mathrm{e}^{\frac{1}{6}L^{\beta}}\mathrm{e}^{\frac{1}{2}L^{\zeta''}}2\mathrm{e}^{-(\log 2)R} \\
&\quad\le \mathrm{e}^{-(\log 2)R\left(1-\frac{\frac{1}{6}L^{\beta}+\frac{1}{2}L^{\zeta''}+\log 2}{(\log 2)L^{\tau}}\right)} \le \mathrm{e}^{-(\log 2)\left(1-3L^{-\tau+\beta}\right)R} \le \mathrm{e}^{-\frac{1}{2}\|\mathbf{x}-\mathbf{y}\|_1},
\end{aligned} \tag{1.12.41}$$

for L sufficiently large, $R>L^{\tau}$, and by using (1.12.31) to estimate $1/\lambda$. We conclude that for $\omega\in\mathcal{Q}^{\mathrm{c}}$ and $\|\mathbf{x}-\mathbf{y}\|_1>L^{\tau}$ we have

$$|\langle\delta_{\mathbf{x}},(K_B^{(N)}-E)^{-1}\delta_{\mathbf{y}}\rangle| \le \mathrm{e}^{-\frac{1}{2}\|\mathbf{x}-\mathbf{y}\|_1} \le e^{-\frac{1}{2}\|\mathbf{x}-\mathbf{y}\|_\infty}. \tag{1.12.42}$$

We get from (1.12.39) and (1.12.31) that in the event where $\omega\in\mathcal{Q}^c$,

$$\left\|R_B^{(N)}(E)\right\| \le 2\tfrac{1}{\eta} \le \tfrac{2}{\lambda}\mathrm{e}^{\frac{1}{2}L^{\zeta''}} \le \tfrac{2}{\lambda}\mathrm{e}^{\frac{1}{2}L^{\beta}} \le \tfrac{e^{L^{\beta}}}{2}. \tag{1.12.43}$$

Thus, we have proved that $\mathbb{P}\left\{B_L \text{ is } \left(\tfrac{1}{2},E\right)\text{-good2}\right\} \ge 1-\mathrm{e}^{-L^{\zeta_1}}$. □

Lemma 1.12.18. *Fix $N \in \mathbb{N}$ and fix $\lambda_0 > 0$. Then choose $\lambda > \lambda_0$ and $\Delta > 2J$ so that* (1.12.32) *and* (1.12.31) *are both satisfied with some sufficiently large L (depending on N). If the box B_L is $(\frac{1}{2}, E)$-good2 for some E, then the interval B_L is $(\frac{1}{4}, E')$-good for all $E' \in I(E, \theta)$, where*

$$\theta < \min\{\tfrac{1}{2}L^{\tau}e^{-L-2L^{\beta}}, e^{-L^{\beta}}\} < e^{-3L}. \tag{1.12.44}$$

Proof. The proof is identical to the proof of Lemma 1.8.4. □

Proof of Theorem 1.12.16. By Lemma 1.12.18, Lemma 1.12.17 and equation (1.12.25) we have that if L is sufficiently large (depending on N), and our starting parameters satisfy (1.12.31) and (1.12.32) then

$$\mathbb{P}\{\tilde{\mathcal{R}}(\tfrac{1}{4}, L, I(E, e^{-3L}), \mathbf{x}, \mathbf{y})\} \geq \mathbb{P}\{\tilde{\mathcal{A}}(\tfrac{1}{4}, L, I(E, e^{-3L}), \mathbf{x})\} \geq 1 - e^{-L^{\varsigma_1}}.$$

□

In particular notice that this guarantees the existence of an L_0 which satisfies the hypothesis of Theorem 1.12.13.

1.12.2 Main theorems for fixed particle numbers

Observation 1.12.19. Let $\Lambda = \Lambda_L(x_0)$ be an interval, then $\varphi(\mathbf{M}_{\Lambda}^{(N)}) = B_L(\mathbf{x}) \cap \mathcal{Z}^{N,J}$ where $\mathbf{x} = (x_0, \ldots x_0)$. This follows immediately from the definition of φ.

Definition 1.12.20. Notice that in Definition 1.5.3, whether an interval is (m, E)-regular or not depended on the properties of $H^{\geq 4J}$. We replace $H^{\geq 4J}$ with H^N in definition 1.5.3 to give a new definition of (m, E)-regular that is suitable for H^N The definition of $\mathcal{R}(m, L, I, u, v)$ for H^N follows analogously.

Lemma 1.12.21. *Fix $N \in \mathbb{N}$. Let $\mathbf{x} = (x_0, \cdots x_0) \in \mathbb{Z}^N$ and suppose that $B = B_L(\mathbf{x})$ is (m, E)-good, then $\Lambda_L(x_0) = \varphi^{-1}(B_L(\mathbf{x}) \cap \mathcal{Z}^{N,J})$ is $(\frac{m}{2}, E + 2JN)$-regular if L is sufficiently large.*

Proof. We need to show that (1.5.15) and (1.5.16) hold for $\Lambda_L(x_0)$. Notice that since $\mathcal{Z}^{N,J}$ is a reducing subspace $\tilde{A}^{(N)}$ we have that

$$\begin{aligned}\sigma(K_B^{(N)}) &= \sigma(K_{B\cap\mathcal{Z}^{N,J}}^{(N)} \oplus K_{B\cap(\mathcal{Z}^{N,J})^c}^{(N)}) = \sigma(K_{B\cap\mathcal{Z}^{N,J}}^{(N)}) \cup \sigma(K_{B\cap(\mathcal{Z}^{N,J})^c}^{(N)}) \\ &= \sigma(H_\Lambda^{(N)} - 2JN) \cup \sigma(K_{B\cap(\mathcal{Z}^{N,J})^c}^{(N)}) \supset \sigma(H_\Lambda^{(N)} - 2JN).\end{aligned} \tag{1.12.45}$$

Therefore, if $\operatorname{dist}(\sigma(K_B^{(N)}), E) > e^{-L^\beta}$, then $\operatorname{dist}(\sigma(H_\Lambda^{(N)}), E+2JN) > e^{-L^\beta}$, so (1.5.15) holds for Λ.

Now consider the left hand side of (1.5.16), suppose $i \in \Lambda$ and $R > L^\tau$, then

$$\begin{aligned}\|P_i^-(H_{\Lambda_L}^{(N)} - E - 2JN)^{-1}P_{\Lambda_R(i)\cap\Lambda}^+\| &\le \sum_{\substack{\mathbf{m},\mathbf{n}\in\mathbf{M}_{\Lambda_L}^{(N)} \\ d_\Lambda^{(N)}(\mathbf{m},\mathbf{n})>R}} |\langle\delta_\mathbf{m}, (H_{\Lambda_L}^{(N)} - E - 2JN)^{-1}\delta_\mathbf{n}\rangle| \\ &\le \sum_{\substack{\mathbf{m},\mathbf{n}\in\mathbf{M}_{\Lambda_L}^{(N)} \\ d_\Lambda^{(N)}(\mathbf{m},\mathbf{n})>R}} |\langle\delta_{\varphi(\mathbf{m})}, (K_B^{(N)} - E)^{-1}\delta_{\varphi(\mathbf{n})}\rangle| \le \binom{(2L+1)2J}{N}^2 e^{-m(R+1)} \\ &\le \left(\frac{\mathrm{e}(2L+1)2J}{N}\right)^{2N} e^{-m(R+1)} \le e^{-\frac{m}{2}(R+1)}.\end{aligned} \tag{1.12.46}$$

The last inequality holds for L large enough, depending on N. $\square$

Theorem 1.12.22. *(compare with Theorem 1.5.4) Fix $\lambda \geq \lambda_0 > 0$, $\Delta > 2J$, $\delta \in (0,1)$, and $L_0 > 0$ such that* (1.12.26) *holds for all $N \in \{1, \dots, 4J-1\}$. Then there exists $L_\zeta = L_\zeta(L_0)$ and $\delta_\zeta = \delta_\zeta(L_0) > 0$ with the following property: Let $E_1 \in \mathbb{R}$ with $I(E_1) = [E_1 - \delta_\zeta, E_1 + \delta_\zeta]$, then for every $L \geq L_\zeta$ and $a, b \in \mathbb{Z}$ with $|a-b| \geq 2L$, we have*

$$\mathbb{P}\{\mathcal{R}(\tfrac{1}{16}, L, I(E_1), a, b)\} \geq 1 - e^{-L^\zeta} \quad \textit{for all} \quad N \in \{1, \dots, 4J-1\}. \tag{1.12.47}$$

Proof. Let $N \in \{1, \dots, 4J-1\}$. If $\mathbf{a} = (a, a, \dots a)$ and $\mathbf{b} = (b, b, \dots b) \in \mathbb{R}^N$ then clearly $|a-b| = d_H(\mathbf{a}, \mathbf{b}) > 2L$. From Lemma 1.12.21 and Theorem 1.12.13 we have that

$$\mathbb{P}\{\mathcal{R}(\tfrac{1}{16}, L, I(E_1), a, b)\} \geq \mathbb{P}\{\tilde{\mathcal{R}}(\tfrac{1}{8}, L, I(E_1 - 2JN), \mathbf{a}, \mathbf{b})\} \geq 1 - e^{-L^\zeta}. \tag{1.12.48}$$

□

Chapter 2

Embedded Eigenvalues in Defective Periodic Quantum Graphs

2.1 Introduction

Quantum graphs are a common object of interest in the field of mathematical physics. They are a simple model to study solutions of the Schrödinger equation in an molecule like structure. This portion of the thesis is dedicated to a study of periodic quantum graphs and in particular the phenomenon of Fermi-surface reducibility and how reducibility can allow for the construction of defective graphs with embedded eigenvalues in the continuum.

For any d-periodic self-adjoint operator there is an associated function $D : \mathbb{C} \times (\mathbb{C}^*)^d \to \mathbb{R}$, called the dispersion function. In the case of periodic graph operations, the dispersion function is a Laurent polynomial in the variables $z = (z_1, \ldots, z_d) \in (\mathbb{C}^*)^d$ and it has coefficients that are meromorphic functions of $\lambda \in \mathbb{C}$. Since this work is devoted to the study periodic graphs, especially quantum graphs, we will use the terms dispersion polynomial and dispersion function interchangeably. For each $\lambda \in \mathbb{C}$, the set of $z \in (\mathbb{C}^*)^d$ such that $D(\lambda, z) = 0$

is called the Fermi Surface at energy λ; when $|z_1| = \cdots = |z_d| = 1$ the point on the Fermi Surface describes a set of wave vectors that are admissible by the operator at that energy. When the dispersion polynomial can be factored, for each fixed energy, as a product of two or more polynomials in z, each irreducible component contributes a sequence of spectral bands and gaps. For periodic quantum graphs the only eigenvalues are ones corresponding to eigenfunctions with compact support. However, when a quantum graph has a reducible Fermi Surface, it is possible that a local defect creates an embedded eigenvalue with a non-compactly supported eigenfunction [57, 58].

Irreducibility seems to be the default situation because a polynomial in several variables is only factorable into nonconstant polynomials of lower degree when its coefficients lie on a specific algebraic variety. For this reason proofs reducibility rely on constructions that are tailored to result in the desired type of dispersion polynomial. This work describes two such constructions. On the other hand, proofs of irreducibility utilize a wide variety of methods. We will briefly outline some known results on reducibility and irreducibility.

Irreducibility of the Fermi surface is known to occur for the discrete Laplace operator plus a periodic potential in any dimension. This was proved in two and three dimensions in [11],[48, Ch. 4],[12, Theorem 2] for all but finitely many energies, and in [64] for all energies in dimension higher than two. Proofs rely on the algebraic structure arising from the relative simplicity of the discrete Laplacian on a square graph. Recently, by applying techniques from algebraic geometry, irreducibility has been proven for a large class of finite range periodic Schrödinger operators [41]. Irreducibility is also known for the continuous Laplacian in three dimensions plus a periodic potential of the form $q_1(x_1) + q_2(x_2, x_3)$ [13, Sec. 2]. Irreducibility for all but finitely many energies is established for discrete graph Laplacians with positive weights and more general graph operators, where the underlying graph is planar with two vertices per period [62].

All constructions so far of graph operators with reducible Fermi surface involve multiple coupled layers [77, 78]. The simplest are constructed by coupling several identical copies of a discrete graph operator, using Hermitian coupling constants [77, §2]; or by coupling two identical layers of a quantum graph by edges between corresponding vertices, where the potential $q_e(x)$ of the Schrödinger operator $-d^2/dx^2 + q_e(x)$ on each coupling edge e is symmetric about the center of the edge [77, §3]. When the layers are not coupled symmetrically, a compatibility condition for the potentials on the coupling edges, which is sufficient for reducibility, was proved in [78, Theorem 4].

Here is a summary of properties of the two types of multi-layer quantum graphs, introduced in this thesis, that have reducible Fermi surface. All potentials are electric; we will not cover magnetic potentials. We include in this summary the bilayer quantum graphs studied in [78], which we call type 0.

Type-0 bilayer graphs. ([78])

1. The two layers are identical, and otherwise there is no restriction on the individual layers.
2. The layers are coupled by single edges connecting corresponding vertices.
3. The potentials on all connecting edges lie in the same "asymmetry class".

Type-1 multilayer graphs. (Fig. 2.3.2 left; and section 2.3.3)

1. Each layer breaks into a collection of finite pieces when a vertex and its shifts are removed (Fig. 2.3.1).
2. Several layers are connected by a finite graph, at the vertices of separation.

Type-2 multilayer graphs. (Fig. 2.3.2 right; and section 2.3.4)

1. Each layer has the same graph, which is bipartite with one vertex of each color per period.

2. The potentials on corresponding edges in different layers are Dirichlet-isospectral.
3. Several layers are connected along vertices of the same color by finite graphs.

Multi-layer graphene models. (Section 2.4)

An important application of the theory we develop is to two-dimensional periodic graphs with hexagonal, or honeycomb, structure. Graphene is the most familiar of these structures. We will use the word "graphene" to include any periodic quantum graph with the hexagonal structure of graphene, shown in Fig. 2.4.1.

Quantum graphs offer an intermediate model between full partial differential equation models and tight-binding ones, and they have been used to model graphene and other honeycomb structures in single-layer form [25, 56, 9] and multi-layer form [26, 27, 28]. The model is also called a quantum or free-electron network model [25]; see [61, Appendix A] for a historical discussion.

The quantum-graph model of single-layer graphene satisfies the properties of the individual layers of both type 1 and type 2 By applying the techniques of both types, one finds that very generally stacked multi-layer graphene has reducible Fermi surface. This includes AA-, AB-, ABC-, and mixed stacking, as illustrated in the figures of Section 2.4.

The differences between single- and multiple-layer graphene are covered in [68, 1], which offers much physical context. The most important feature of two or more layers, which occurs typically but not always, is a transition from conical singularities of the dispersion relation (linear band structure at Dirac points) for a single layer to nonconical singularities (quadratic band structure) for multiple layers, accompanied by spectral gaps; see [70, 65, 21], for example. Refs. [49, 73, 75] present some interesting work on opening gaps by twisting two layers relative to one another. The work [72] provides a review of physical phenomena of stacked bi-layer graphene.

This thesis contributes to the spectral properties of graph models of multi-layer graphene with electric potentials in two ways: (1) The reducibility of the Fermi surface allows the construction of local defects in multi-layer graphene that would allow bound states within the radiation continuum (cf. [58]); and (2) The theorems demonstrate the range of allowed potentials on the layers and the connecting edges in order to obtain a reducible Fermi surface.

Quantum graphs have been used as models for a variety of physical systems, such as single- and multi-layer graphene, as mentioned above, tubes of graphene or other planar materials [52, 53, 56, 69, 79], and related band-gap structures [74, 61, 8, 29, 15]. We also refer the reader to the works [7, 54], the collection [37], and the monograph [18].

The multi-layer graphs types 0, 1, and 2, described above allow general vertex conditions of Robin type, defined below (see 2.2.2). The condition stipulates that the value of a function at a vertex v is proportional to a flux from the vertex into the adjacent edges, and the proportionality constant is called the Robin parameter α_v. Different values of α can model different types of atoms at the vertices. A quantum graph with the Robin vertex condition emerges as the limit of a Schrödinger operator on a thickened graph, as the thickness tends to zero [39, §2–3]. Taking the limit in different ways results in other types of self-adjoint vertex conditions; see also [38, 22] for discussions of these limits and their physical relevance.

The Robin condition also allows different coupling strengths imparted by the edges (denoted by ε_e in (2.2.2) below). This accommodates, for example, adjusting interlayer couplings, which can be much weaker than the intralayer bonds [60, 66, 63, 10], and quite complex variations of atom interactions [20, 70]. This strength parameter has been implemented in the quantum-graph models of multi-layer graphene in [26, 27, 28].

Bound states in the continuum are impossible for locally defective periodic Schrödinger partial differential operators in $\mathbb{R}^d$ [57, 55], and this makes the prospect of creating multi-layer structures where local defects admit embedded eigenvalues intriguing. In Section 2.5, we show how to construct a local defect in a periodic quantum graph with reducible Fermi surface, that creates a bound state at an energy embedded in the continuous spectrum. We do this for AA- and AB-stacked bilayer graphene. The associated eigenstates are not those of compact support that are peculiar to graph operators, but rather, they decay exponentially with unbounded support. Bound states in the continuum are associated with interesting and useful resonance phenomena. A motivation for the present work is to understand the mechanisms underlying the creation of embedded eigenvalues and resonances in diverse material structures.

2.2 Periodic graphs and the Fermi surface

2.2.1 Quantum graphs and notation

We begin with an overview of the fundamentals of periodic quantum graphs. The notation specific to periodic graphs essentially follows [78, §3.1-3.2]; and the standard text [18] gives a more general exposition of quantum graphs.

Definition 2.2.1. A *weighted metric graph* is a graph Γ, with a vertex set $\mathcal{V}(\Gamma)$, an edge set $\mathcal{E}(\Gamma)$, a metric which assigns a length L_e to each edge, and a set of weights $\epsilon_e \in \mathbb{R}$ associated to each edge. When referring to an edge e connecting vertices v and w, we write $e\{v, w\}$ when the edge is unoriented and $e(v, w)$ when the edge is oriented from v to w.

A simple example of a metric graph is the unit interval, this graph has two vertices and one edge of length 1. A periodic metric graph is a metric graph that is imbued with a shift action from $\mathbb{Z}^d$, denoted by $x \to g + x$ for each $x \in \Gamma$ and $g = (g_1, \ldots g_d) \in \mathbb{Z}^d$. A good example of a periodic metric graph is constructed by considering the way that $\mathbb{Z}$ is a subset of $\mathbb{R}$. For this graph we say that the elements of $\mathbb{Z}$ are the vertices and that all other elements of $\mathbb{R}$

belong to some edge of length 1; the $\mathbb{Z}^1$ action on this graph is simply the map $x \to g + x$.

Definition 2.2.2. For a weighted metric graph Γ we define the Hilbert space, $L^2(\Gamma, \varepsilon)$ by imposing that the following norm be finite.

$$\|f\|^2_{L^2(\Gamma,\varepsilon)} = \sum_{e\in\mathcal{E}(\Gamma)} \int_e \varepsilon_e |f(x)|^2 dx \,. \tag{2.2.1}$$

The weights ε_e for each edge $e\{v, w\}$, control the strengths of a connections between adjacent vertices. In multi-layer graphene, the inter-layer coupling is considered to be much weaker than the intra-layer bonds, and this is modeled by small values of ε_e on those edges that connect vertices in different layers.

Definition 2.2.3. A *quantum graph* is a pair (Γ, A) where Γ is a weighted metric graph, and A is a self adjoint operator on the space $L^2(\Gamma, \varepsilon)$.

Our analysis is dedicated to the case where A is a Schrödinger operator with vertex conditions. For each edge e of the metric graph, we associate to it the operator

$$-\frac{d^2}{dx^2} + q_e(x) \qquad (0 < x < L_e).$$

We create a global operator on Γ by coupling these edge operators with conditions at each vertex v. There are two different kinds of vertex conditions that we will consider, Robin and Dirichlet Vertex conditions.

Definition 2.2.4. A *Robin* vertex condition at v is satisfied by f in $L^2(\Gamma, A)$ if

$$\sum_{e\in\mathcal{E}(v)} \varepsilon_e \, f'_e(v) \;=\; \alpha_v f(v), \tag{2.2.2}$$

and f is continuous at v.

This condition is called the Kirchoff or Neumann condition when $\alpha_v = 0$. The sum is over all edges incident to v, and the prime denotes the inward derivative, in the direction from the vertex into the edge. Thus, if an edge $e\{v,w\}$ is parameterized by $x \in [0, L_e]$ running from v to w, then $f_e'(v) = df_e/dx(0)$ and $f_e'(w) = -df_e/dx(L_e)$. The weight α_v, called a Robin coefficient, can be considered as a singular δ-potential of strength α_v at the vertex (see [39, Eq. 2.6] for example), and thus the condition can also be called a δ vertex condition.

Definition 2.2.5. A (homogeneous) *Dirichlet* vertex condition at v is satisfied by f in $L^2(\Gamma, A)$ if $f(v) = 0$ and f is continuous at v.

Remark 2.2.6. One important thing to consider is the situation where some number edges meet at a vertex with Dirichlet condition. Suppose u solves $Au = \lambda u$ on each one of those edges and satisfies the vertex condition. If the graph is modified so that vertex is removed and each one of the dangling edges gets its own separate vertex with a Dirichlet condition, then u will still solve $Au = \lambda u$ on each dangling edge and it will still satisfy the Dirichlet condition at each one of the new terminal vertices. From this observation we see that imposing Dirichlet vertex conditions has a decoupling effect on the dynamics of a quantum graph.

This is sufficient to determine a self-adjoint operator A in the Hilbert space $L^2(\Gamma, \varepsilon)$. The operator A, being unbounded, has domain $\mathcal{D}(A)$ that is not all of $L^2(\Gamma, \varepsilon)$ but consists of those functions in $L^2(\Gamma, \varepsilon)$ whose restriction to each edge e is in the Sobolev space $H^2(e)$ and that are continuous on Γ and satisfy the Robin condition at each vertex. Aside from Dirichlet and Robin there are other vertex conditions that correspond to self-adjoint operators in $L^2(\Gamma, \varepsilon)$. They are described in [18, Theorem 1.4.4], but we will not cover them in this thesis.

The quantum graph pair (Γ, A) is periodic provided that the edges, potentials q_e, vertex conditions, and edge weights ε_e, are all invariant under the shift group $\mathbb{Z}^d$. The operator A can in fact be applied to any continuous function on Γ that is in H^2 of each edge and satisfies the vertex conditions – they do not necessarily lie in $L^2(\Gamma, \varepsilon)$. The extended domain consisting of these functions will be called $\bar{\mathcal{D}}(A)$.

Definition 2.2.7. We say that $u \in \bar{\mathcal{D}}(A)$ is a *simultaneous eigenfunction* of A and the $\mathbb{Z}^d$ action if the following equations are satisfied:

$$Au = \lambda u \tag{2.2.3}$$

$$u(gx) = z^g u(x). \tag{2.2.4}$$

We used the notation that $z^g = \prod_{i=1}^d z_i^{g_i}$ for each $g = (g_1, \ldots, g_d) \in \mathbb{Z}^d$. We say that λ is the energy and that $z = (z_1, \ldots, z_d) = (e^{ik_1}, \ldots, e^{ik_d})$ is the vector of Floquet multipliers. Such u is called a Floquet (or Floquet-Bloch) mode of A. If the wavevector, or quasi-momentum, $k = (k_1, \ldots, k_d)$ is real, then u is a Bloch wave and the k_j are phase shifts of u across the d period vectors of the structure.

The pair (2.2.3, 2.2.4) is used to obtain a "spectral matrix" $\hat{A}(z, \lambda)$ for A, from which is derived the dispersion function and the Fermi surface. We now describe the elements of its derivation, which is described in more detail, including its relation to the Floquet (Fourier) transform, in [78, §3.1-3.2].

2.2.2 The Combinatorial Reduction

Let us first consider the eigenvalue problem $Au = \lambda u$, this means that u satisfies the ODE $-u'' + q_e(x)u = \lambda u$ on each edge. If we consider $u_e(x)$ the solution on a single edge, we know that u_e can be written as a linear combination of two fundamental solutions to the ODE.

Definition 2.2.8. The *generalized sine* and *generalized cosine* are the fundamental solutions on an edge.

$$-c_e''(x;\lambda) + q_e(x)c_e(x;\lambda) = \lambda c(x;\lambda), \text{ where } c_e(0;\lambda) = 1 \text{ and } c_e'(0;\lambda) = 0 \tag{2.2.5}$$

$$-s_e''(x;\lambda) + q_e(x)s_e(x;\lambda) = \lambda s(x;\lambda), \text{ where } s_e(0;\lambda) = 0 \text{ and } s_e'(0;\lambda) = 1 \tag{2.2.6}$$

Observe that if $Au = \lambda u$ then for each edge, we can write $u(x)$ as a linear combination of the c and s functions: $u_e(x) = u(0)c_e(x;\lambda) + u'(0)s_e(x;\lambda)$. We will use the notation $c_e(L_e;\lambda) = c_e(\lambda)$ and $s_e(L_e;\lambda) = s_e(\lambda)$ for simplicity. From these observations we construct the *transfer matrix*:

$$T_e(\lambda) = \begin{bmatrix} c_e(\lambda) & s_e(\lambda) \\ c_e'(\lambda) & s_e'(\lambda) \end{bmatrix}, \text{ notice } T_e(\lambda) \begin{bmatrix} u_e(0) \\ u_e'(0) \end{bmatrix} = \begin{bmatrix} u_e(L) \\ u_e'(L) \end{bmatrix}. \tag{2.2.7}$$

By rearranging the information in the transfer matrix, we derive the Dirichlet-to-Neumann (DtN) map for this ODE on each edge. By definition, the DtN map $N(\lambda)$ takes the values of u at the vertices of the edge to the inward derivatives of u at the vertices. The DtN map is formed from the entries of $T_e(\lambda)$ by,

$$N_e(\lambda) = \frac{1}{s_e(\lambda)} \begin{bmatrix} -c_e(\lambda) & 1 \\ 1 & -s_e'(\lambda) \end{bmatrix}, \text{ here } N_e(\lambda) \begin{bmatrix} u(v) \\ u(w) \end{bmatrix} = \begin{bmatrix} u'(v) \\ u'(w) \end{bmatrix}. \tag{2.2.8}$$

By applying the DtN map, the vertex condition (2.2.2) for u can then be written in terms of the values of u at the vertices. This results in a discrete, or combinatorial, reduction of the graph operator. Let $\bar{u}$ be the restriction of u to $\mathcal{V}(\Gamma)$, then we can define the combinatorial reduction of the operator $A - \lambda I$:

$$(\mathfrak{A}_\lambda \bar{u})(v) := \sum_{\{v,w\}\in\mathcal{E}(v)} \frac{\bar{u}(w)}{s_e(\lambda)} - \bar{u}(v)\left(\alpha_v + \sum_{\{v,w\}\in\mathcal{E}(v)} \frac{c_e(\lambda)}{s_e(\lambda)}\right) \tag{2.2.9}$$

$$(\mathfrak{A}_\lambda \bar{u})(v) = 0 \text{ for all } v \in \mathcal{V} \iff Au = \lambda u \text{ for all } x \in \Gamma \text{ and } s_e(\lambda) \neq 0. \tag{2.2.10}$$

The combinatorial operator $\mathfrak{A}_\lambda$ will be used to derive more essential machinery and we will make use of $\mathfrak{A}_\lambda$ also in section 2.5, where we construct embedded eigenvalues. Notice of course, that $\mathfrak{A}_\lambda$ is undefined if λ is a root of the function $s_e(\lambda)$ for some edge e. This set of all such λ is denoted by

$$\sigma_D(\Gamma, A) \;=\; \{\lambda : \exists\, e \in \mathcal{E}(\Gamma),\, s_e(\lambda) = 0\}\,. \tag{2.2.11}$$

Observe that with the construction of $\mathfrak{A}_\lambda$ as described, (2.2.9) is only valid for $\lambda \notin \sigma_D(\Gamma, A)$. However, these poles of the DtN map are not an "essential" feature of the model: by following a technique introduced in [59, §IV] we can create a quantum graph that is almost identical but where $\sigma_D(\Gamma, A)$ is altered to include or exclude particular values of λ as we see fit. The Dirichlet spectrum can be moved by adding "dummy vertices" – the original and modified quantum graphs are unitarily equivalent.

Proposition 2.1. Let (Γ, A) be a quantum graph, let $e\{v_1, v_2\}$ be an edge of Γ, and suppose that $s_e(\lambda) = 0$, say that e has length L and v_1 is identified with 0 and v_2 with L. Let $(\dot{\Gamma}, \dot{A})$ be the quantum graph obtained by placing an additional vertex v, with Neumann vertex condition ($\alpha_v = 0$), in the interior of e, say at point $\ell \in [0, L]$. This divides e into two edges $e_1\{v_1, v\}$ and $e_2\{v, v_2\}$, with the potentials on e_1 and e_2 being inherited from q_e on e. Let $s(x;\lambda)$, $s_1(x;\lambda)$, and $s_2(x;\lambda)$ be the generalized sine functions for the edges e, e_1, and e_2. We have that quantum graphs are unitarily equivalent and also that it is always possible to choose ℓ such that $s_1(\ell;\lambda) \neq 0$ and $s_2(L;\lambda) \neq 0$.

Proof. First we show that A and $\dot{A}$ are unitarily equivalent. We will define the map ψ : $\bar{\mathcal{D}}(A) \to \bar{\mathcal{D}}(\dot{A})$. We let $\psi(u(x)) = u(x)$ whenever x is outside of the modified edge, when $x \in e$ and $x < \ell$ let $\psi(u(x)) = u(x)\chi_{[0,\ell]}(x)$, and when $x \geq \ell$ let $\psi(u(x)) = u(x+\ell)\chi_{[0,L-\ell]}(x)$. Since $u \in \bar{\mathcal{D}}(A)$ the restriction of u to e is in $H^2(e)$. Thus the restriction of u to e_1 and e_2 are each in H^2 as well, and since u is also continuous and differentiable at $x = \ell$, $\psi(u)$ will satisfy the Neumann vertex condition at v. Therefore $\psi(u) \in \bar{\mathcal{D}}(\dot{A})$, the inverse is also another inclusion map – we have the unitary equivalence.

Now to show that it is possible to select ℓ such that $s_1(\lambda) \neq 0$ and $s_2(\lambda) \neq 0$. We have that $s_1(\lambda) = s_{e_1}(\ell;\lambda) = s_e(\ell;\lambda)$. We know that sinc $s_e(x)$ is the generalized sine function for the operator $-d^2/dx^2 + q(x) - \lambda$ on the interval $[0, L]$, $s_e(x)$ cannot be identically zero. Therefore there is some ℓ such that $s_e(L) \neq 0$. Now we need to show that $s_2(L)$ is also not zero. Notice that s_2 can be extended from a function on $[\ell, L]$ to a function on $[0, L]$, we say that s_2 solves the following initial value problem.

$$-s_2''(x;\lambda) + q_e(x)s_2(x;\lambda) = \lambda s_2(x;\lambda) \text{ where } s_2(\ell;\lambda) = 0 \text{ and } s_2'(\ell;\lambda) = 1 \tag{2.2.12}$$

We proceed by considering the transfer matrix, and we use the fact that the Wronskian of s and c is identically 1.

$$\begin{bmatrix} c(\ell;\lambda) & s(\ell;\lambda) \\ c'(\ell;\lambda) & s'(\ell;\lambda) \end{bmatrix} \begin{bmatrix} s_2(0;\lambda) \\ s_2'(0;\lambda) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \implies \begin{bmatrix} s_2(0;\lambda) \\ s_2'(0;\lambda) \end{bmatrix} = \begin{bmatrix} s'(\ell;\lambda) & -s(\ell;\lambda) \\ -c'(\ell;\lambda) & c(\ell;\lambda) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{2.2.13}$$

Therefore we can write $s_2(x;\lambda)$ in terms of $s(x;\lambda)$ and $c(x;\lambda)$.

$$s_2(x;\lambda) = -s(\ell;\lambda)c(x;\lambda) + c(\ell;\lambda)s(x:\lambda) \tag{2.2.14}$$

$$\implies s_2(L;\lambda) = -s(\ell;\lambda)c(L;\lambda) + c(\ell;\lambda)s(L;\lambda) = -s(\ell;\lambda)c(L;\lambda) \tag{2.2.15}$$

This is because we assumed that $s(L;\lambda) = 0$. We have that $s_2(L;\lambda) \neq 0$ because $s(\ell;\lambda) \neq 0$

by choice of ℓ and $c(L;\lambda) \neq 0$ because c and s are linearly independent. $\square$

2.2.3 The Floquet Transform and Spectral Matrix

Another crucial tool for our analysis of periodic quantum graphs is the Floquet transform. This is closely related to the Fourier transform but it is used to study objects defined on a periodic structure with a $\mathbb{Z}^d$ (as opppoosed to $\mathbb{R}^d$) symmetry. We will step away from quantum graphs for a moment to discuss the Floquet Transform as it acts generally on a periodic measure space.

Definition 2.2.9. Suppose that (Γ, μ) is a measure space that is periodic with a $\mathbb{Z}^d$ action, $g : x \to g + x$. We say that $W \subset \Gamma$ is a *fundamental domain* if for every $x \in \Gamma$ there is a unique $g \in \mathbb{Z}^d$ and $w \in W$ such that $x = g + w$.

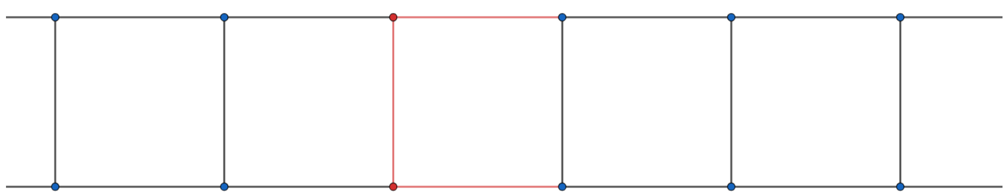

Figure 2.2.1: A ladder quantum graph with fundamental domain in red.

Fundamental domains are not unique; for example a translation of a fundamental domain is also a fundamental domain. Some of our tools require us to select a fundamental domain, although the choice of W is arbitrary and it does not have any impact on the essential properties of a periodic measure space.

Definition 2.2.10. Suppose that $u \in L^2(\Gamma, \mu)$ and that Γ is $\mathbb{Z}^d$ periodic with fundamental domain W. Let $\mathbb{T}^d$ be the unit torus in $\mathbb{C}^d$ and let ν be the unique symmetry respecting measure with $\nu(\mathbb{T}^d) = 1$. Then, the *Floquet transform* of u, is an element of $L^2(W \times \mathbb{T}^d, \mu \times \nu)$

given by:

$$[\mathcal{F}u](w;\vec{k}) = \sum_{(g_1,\dots g_n)\in\mathbb{Z}^n} u(g+w)e^{-i(k_1g_1+k_2g_2\cdots+k_ng_n)}$$
$$:= \sum_g u(g+w)z_1^{-g_1}\cdots z_n^{-g_n} = \sum_g u(g+w)z^{-g} = \hat{u}(w,z).$$

Definition 2.2.11. The *inverse Floquet transform* is given by an integral over the complex unit torus. Let $x = g + w$ uniquely, then we have that

$$u(x) = [\mathcal{F}^{-1}\hat{u}](g+w) = \int_{\mathbb{T}^n} \hat{u}(w,\zeta)\zeta^g d\nu(\zeta). \tag{2.2.16}$$

Notice that the Floquet transform will work on both quantum and discrete graphs, and that the measure μ could be a counting measure, Lebesgue measure, or it could be a measure that is scaled by some aforementioned ε coupling constants. As long as the measure is compatible with the $\mathbb{Z}^d$ action the Floquet transform machinery will work smoothly. Now we consider the situation in which $u \in \mathrm{L}^\infty(\Gamma,\mu)$ is an eigenfunction of the $\mathbb{Z}^d$ action. We have that, for some $z \in \mathbb{T}^d$, $u(g+x) = z^g u(x)$. Although the Floquet transform of u is not classically defined, we can interpret $\hat{u}$ as a distribution:

$$\hat{u}(w,z) = u|_W(w)\delta_z(\zeta). \tag{2.2.17}$$

When we interpret the Floquet transform of a quasiperiodic function in this way, it should be clear that the inverse Floquet transform yields exactly the desired result. Suppose that u is a quasi-periodic function on a periodic quantum graph. We use both the Floquet transform and the combinatorial reduction to write,

$$\hat{\hat{u}}(w,z) = u|_{\mathcal{V}(W)}\delta_z(\zeta). \tag{2.2.18}$$

Since we assumed that a periodic quantum graph has fundamental domains with finitely many vertices, we can interpret $\hat{\hat{u}}$ simply as an element of $\mathbb{C}^{|\mathcal{V}(W)|} \times \mathbb{T}^d$. We will define a linear map from $\mathbb{C}^{|\mathcal{V}(W)|} \to \mathbb{C}^{|\mathcal{V}(W)|}$ that incorporates all of the elements of our discussion so far:

Definition 2.2.12. The "spectral matrix", $\hat{A}$ is a $|\mathcal{V}(W)| \times |\mathcal{V}(W)|$ matrix with entries that depend on z and λ. Let $u \in \mathbb{C}^{|\mathcal{V}(W)|}$ then,

$$\hat{A}(z,\lambda)u := \mathcal{F}\mathfrak{A}_\lambda \mathcal{F}^{-1}(u\delta_z). \tag{2.2.19}$$

Notice that if u is a simultaneous eigenfunction, then the pair (2.2.3, 2.2.4) become equivalent to a homogeneous system of linear equations. The coefficients depend on λ and z and the variables are values of u on the finite set of vertices in a fundamental domain. The spectral matrix for this system, $\hat{A}(z,\lambda)$, necessarily has the following property:

$$\hat{A}(z,\lambda)\hat{\hat{u}} = 0 \qquad \iff \qquad u \text{ satisfies } (2.2.3, 2.2.4). \tag{2.2.20}$$

This suffices to derive the matrix $\hat{A}(z,\lambda)$ abstractly, as well as to demonstrate its essential properties. However we will also detail an algorithmic construction. Let $\mathcal{V}_0$ and $\mathcal{E}_0$ denote the vertices and edges of a fixed fundamental domain W for (Γ, A). The matrix $\hat{A}(z,\lambda)$ is indexed by the vertices $\mathcal{V}_0$ minus those that have the Dirichlet condition. Given an edge $e \in \mathcal{E}_0$, there are vertices $v, w \in \mathcal{V}_0$ and $g \in \mathbb{Z}^d$ such that e connects v and $g + w$. If both v and w have Robin conditions (including Neumann when the Robin parameter is 0) and $v \neq w$, then the following modified DtN matrix goes into the 2×2 submatrix of $\hat{A}(z,\lambda)$ indexed by v and w:

$$\tilde{N}_e(g,z,\lambda) \;=\; \frac{\varepsilon_e}{s_e(\lambda)} \begin{bmatrix} -c_e(\lambda) & z^g \\ z^{-g} & -s'_e(\lambda) \end{bmatrix}, \tag{2.2.21}$$

in which the off-diagonal entries come from the Floquet eigenvalue condition (2.2.4). If $v\!=\!w$, then $\varepsilon_e(z^g + z^{-g} - c_e(\lambda) - s'_e(\lambda))$ goes into the diagonal entry indexed by v. If w has the Dirichlet condition, then $-\varepsilon_e c_e(\lambda)/s_e(\lambda)$ goes in the diagonal entry for v. Then a diagonal matrix with entries $-\alpha_v$ is added.

The dispersion function for (Γ, A) is defined by

$$D_{(\Gamma,A)}(z,\lambda) \;:=\; \det \hat{A}(z,\lambda), \tag{2.2.22}$$

and its zero set is the set of all (z,λ) pairs at which (Γ, A) admits a Floquet mode.

The spectral matrix does depend on the choice of fundamental domain, but of course the Floquet modes of (Γ, A) corresponding to null vectors of $\hat{A}(z,\lambda)$ are independent of this choice. More importantly, the entries of $\hat{A}(z,\lambda)$ and $D_{(\Gamma,A)}(z,\lambda)$ have poles in λ. All of them are Laurent polynomials in z with coefficients that are meromorphic functions of λ. This can be remedied by considering Proposition 2.1. If we periodically add dummy variables in the prescribed way then we can consider $\hat{A}(z,\lambda)$ and $D_{(\Gamma,A)}(z,\lambda)$ to be defined up to meromorphic factors.

Proposition 2.2. Let (Γ, A) be a periodic quantum graph, and let $(\dot{\Gamma}, \dot{A})$ be the quantum graph obtained by applying Proposition 2.1 periodically. We have that,

$$s_1(\lambda)s_2(\lambda)\, D_{(\dot{\Gamma},\dot{A})}(z,\lambda) \;=\; \pm s(\lambda)\, D_{(\Gamma,A)}(z,\lambda). \tag{2.2.23}$$

Proof. The proof is follows by proceeding along the strategy outlined in the proof of Proposition 2.1. First consider the simple case of a quantum graph where the underlying graph E consists of two vertices and the edge $e\{v_1, v_2\}$ between them, identified with the interval $[0, L]$. Let $\dot{E}$ be the graph obtained by applying Proposition 2.1. We place a dummy vertex, v, at the point of e corresponding to $x = \ell \in\ (0, L)$, and ℓ is chosen so that $s_1(\lambda) \neq 0$ and $s_2(\lambda) \neq 0$. Denote the transfer matrices for $-d^2/dx^2 + q(x)$ on $[0, L]$, on $[0, \ell]$, and on $[\ell, L]$

by

$$T(\lambda) = \begin{bmatrix} c(\lambda) & s(\lambda) \\ c'(\lambda) & s'(\lambda) \end{bmatrix}, \quad T_1(\lambda) = \begin{bmatrix} c_1(\lambda) & s_1(\lambda) \\ c_1'(\lambda) & s_1'(\lambda) \end{bmatrix}, \quad T_2(\lambda) = \begin{bmatrix} c_2(\lambda) & s_2(\lambda) \\ c_2'(\lambda) & s_2'(\lambda) \end{bmatrix}. \tag{2.2.24}$$

Considering $(\dot{E}, \dot{A})$ as one period of a d-periodic disconnected graph, its dispersion function is a meromorphic function of λ alone, as its spectral matrix $\hat{\dot{A}}(z,\lambda)$ cannot depend on z. When Robin conditions are imposed at both endpoints, denote this function by $\dot{h}^{\mathrm{RR}}(\lambda) = \det \hat{\dot{A}}(z,\lambda)$,

$$\dot{h}^{\mathrm{RR}}(\lambda) \;=\; \det \begin{bmatrix} -\frac{c_1(\lambda)}{s_1(\lambda)} - \alpha_1 & \frac{1}{s_1(\lambda)} & 0 \\ \frac{1}{s_1(\lambda)} & -\frac{s_1'(\lambda)}{s_1(\lambda)} - \frac{c_2(\lambda)}{s_2(\lambda)} & \frac{1}{s_2(\lambda)} \\ 0 & \frac{1}{s_2(\lambda)} & -\frac{s_2'(\lambda)}{s_2(\lambda)} - \alpha_2 \end{bmatrix}. \tag{2.2.25}$$

When the Dirichlet condition is imposed at one end and a Robin condition is imposed at the other, we have

$$\dot{h}^{\mathrm{DR}}(\lambda) = \det \begin{bmatrix} -\frac{s_1'(\lambda)}{s_1(\lambda)} - \frac{c_2(\lambda)}{s_2(\lambda)} & \frac{1}{s_2(\lambda)} \\ \frac{1}{s_2(\lambda)} & -\frac{s_2'(\lambda)}{s_2(\lambda)} - \alpha_2 \end{bmatrix}, \tag{2.2.26}$$

$$\dot{h}^{\mathrm{RD}}(\lambda) = \det \begin{bmatrix} -\frac{c_1(\lambda)}{s_1(\lambda)} - \alpha_1 & \frac{1}{s_1(\lambda)} \\ \frac{1}{s_1(\lambda)} & -\frac{s_1'(\lambda)}{s_1(\lambda)} - \frac{c_2(\lambda)}{s_2(\lambda)} \end{bmatrix}, \tag{2.2.27}$$

and $\dot{h}^{\mathrm{DD}}(\lambda) = -\frac{s_1'(\lambda)}{s_1(\lambda)} - \frac{c_2(\lambda)}{s_2(\lambda)}$ is the dispersion function when the Dirichlet condition is imposed at both ends. By using the relation $T = T_2 T_1$, we find that

$$\begin{aligned} \dot{h}^{\mathrm{RR}} &= -\frac{c' + \alpha_1 s' + \alpha_2 c + \alpha_1 \alpha_2 s}{s_1 s_2}, \quad \dot{h}^{\mathrm{DR}} = \frac{s' + \alpha_2 s}{s_1 s_2}, \\ \dot{h}^{\mathrm{RD}} &= \frac{c + \alpha_1 s}{s_1 s_2}, \quad \dot{h}^{\mathrm{DD}} = -\frac{s}{s_1 s_2}. \end{aligned} \tag{2.2.28}$$

For the un-dotted quantum graph (E, A), one obtains these same expressions except with the denominator $s_1(\lambda)s_2(\lambda)$ replaced by $s(\lambda)$,

$$\begin{aligned} h^{\mathrm{RR}} &= -\frac{c' + \alpha_1 s' + \alpha_2 c + \alpha_1 \alpha_2 s}{s}, \quad h^{\mathrm{DR}} = \frac{s' + \alpha_2 s}{s}, \\ h^{\mathrm{RD}} &= \frac{c + \alpha_1 s}{s}, \quad h^{\mathrm{DD}} = -\frac{s}{s}. \end{aligned} \tag{2.2.29}$$

Now we consider what happens for a general periodic quantum graph. The first case is where v_1 and v_2 are not in the same $\mathbb{Z}^d$ orbit. We can assume that they both are in the vertex set $\mathcal{V}_0$ of the fundamental domain chosen for constructing $\hat{A}(z, \lambda)$, since $D(z, \lambda)$ is independent of that choice. Denote by $\hat{A}(z, \lambda)$ and $\hat{\dot{A}}(z, \lambda)$ the discrete reductions at energy λ of the quantum graphs (Γ, A) and $(\dot{\Gamma}, \dot{A})$. Index the rows and columns of $\hat{\dot{A}}(z, \lambda)$ so that the first two correspond to v_1 and v_2; then augment it with a 0^{th} column and a 0^{th} row consisting of a 1 in the leading entry and zeroes elsewhere. Call this matrix $\hat{\hat{\dot{A}}}(z, \lambda)$.

The matrix $\hat{\hat{\dot{A}}}(z, \lambda)$ has the block form

$$\left[\begin{array}{c|c} \Sigma + A & B \\ \hline C & D \end{array}\right], \tag{2.2.30}$$

in which

$$\Sigma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -cs^{-1} & s^{-1} \\ 0 & s^{-1} & -s's^{-1} \end{bmatrix}, \tag{2.2.31}$$

A and B have all zeroes in the first row, and A and C have all zeroes in the first column. The variable z does not appear in Σ because v_1 and v_2 are both in the chosen fundamental domain. The matrix $\hat{\hat{A}}(z,\lambda)$ is obtained by replacing Σ by a matrix $\dot{\Sigma}$, where $\dot{\Sigma}$ is obtained from $h^{\mathrm{RR}}(\lambda)$ (2.2.25) with $\alpha_1 = \alpha_2 = 0$ by switching the first two rows and the first two columns (that is, switching the order of the vertices from (v_1, v, v_2) to (v, v_1, v_2)), to obtain

$$\dot{\Sigma} = \begin{bmatrix} -s\, s_1^{-1} s_2^{-1} & s_1^{-1} & s_2^{-1} \\ s_1^{-1} & -c_1 s_1^{-1} & 0 \\ s_2^{-1} & 0 & -s_2' s_2^{-1} \end{bmatrix}, \tag{2.2.32}$$

where the relation $s = s_1 c_2 + s_1' s_2$ is used in the upper left entry.

The 3×3 matrix $K = A - BD^{-1}C$ has all zeroes in its first row and first column. A computation using the relation $T = T_2 T_1$ yields the key relation

$$s_1(\lambda) s_2(\lambda)\ \det(\dot{\Sigma} + K)\ =\ s(\lambda)\ \det(\Sigma + K), \tag{2.2.33}$$

which holds for any matrix K whose first column and and first row vanish. Using this together with

$$\det \hat{\hat{A}} = \det D\ \det(\dot{\Sigma} + K), \qquad \det \hat{A} = \det \hat{\hat{A}} = \det D \det(\Sigma + K) \tag{2.2.34}$$

yields the statement of the theorem.

If $v_2 = gv_1$ for some $g \in \mathbb{Z}^d$, the process above remains the same, except that

$$\Sigma = \frac{1}{s}\begin{bmatrix} 1 & 0 \\ 0 & -c - s' + z^g + z^{-g} \end{bmatrix}, \qquad \dot{\Sigma} = \frac{1}{s_1 s_2}\begin{bmatrix} -s & s_2 + z^g s_1 \\ s_2 + z^{-g} s_1 & -c_1 s_2 - s_2' s_1 \end{bmatrix}, \quad (2.2.35)$$

and K is a 2×2 matrix with its only nonzero entry being the lower right. In this case, one obtains (2.2.33) with an extra minus sign on one side. $\square$

2.2.4 The Fermi surface

As described above, the zero-set of the dispersion function $D_{(\Gamma,A)}(z,\lambda)$ for (Γ, A) is the set of all (z,λ) pairs at which (Γ, A) admits a Floquet mode. This relation $D_{(\Gamma,A)}(z,\lambda) = 0$ in $(\mathbb{C}^*)^d \times \mathbb{C}$ is called the dispersion relation or the Bloch variety of the periodic operator A. By fixing an energy $\lambda \in \mathbb{C}$, one obtains the *Floquet surface*, or Floquet variety, of (Γ, A):

$$\Phi_\lambda = \Phi_{(\Gamma,A),\lambda} = \left\{ z \in (\mathbb{C}^*)^d : D_{(\Gamma,A)}(z,\lambda) = 0 \right\}. \quad (2.2.36)$$

When considered as a set of wavevectors $(k_1, \ldots, k_d) \in \mathbb{C}$ (with $z_j = e^{ik_j}$), it is the *Fermi surface* of (Γ, A). We will just call Φ_λ the "Fermi surface." The spectrum of (Γ, A) consists of all energies λ such that the Fermi surface intersects the d-torus $\mathbb{T}^d = \{z \in \mathbb{C}^d : |z_1| = \cdots = |z_d| = 1\}$,

$$\sigma_{(\Gamma,A)} = \left\{ \lambda \in \mathbb{C} : \Phi_{(\Gamma,A),\lambda} \cap \mathbb{T}^d \neq \emptyset \right\}. \quad (2.2.37)$$

Importantly, when Γ is disconnected, with each connected component being a compact graph, a fundamental domain can be chosen to be one component Γ_0, and thus $\hat{A}(z,\lambda)$ and $D(z,\lambda)$ are independent of z. All of the matrices (2.2.21) have $g = 0$ and reduce to the Dirichlet-to-Neumann maps for the edges. In this case, the spectral matrix, which can be denoted by $\hat{A}(\lambda)$, is the spectral matrix of A confined to the finite graph Γ_0, and the roots of its determinant $D(\lambda)$ are the eigenvalues of this finite quantum graph.

The Fermi surface is an algebraic set in $(\mathbb{C}^*)^d$, and it is ***reducible*** at λ whenever Φ_λ is the union of two algebraic sets. This occurs whenever $D(z, \lambda)$ is factorable into two polynomials, neither of which is a monomial. There is also the situation when a Laurent polynomial $D_1(z, \lambda)^m$ divides $D(z, \lambda)$, with $m > 1$, particularly when $D(z, \lambda) = D_1(z, \lambda)^m$. This makes Φ_λ reducible on account of having a component of multiplicity greater than 1.

2.3 Reducible Fermi Surfaces

2.3.1 A calculus for joining two periodic graphs

In this section we will prove an essential lemma: it is the building block for the analysis of the Fermi surfaces for multi-layer quantum graph operators of type 1. The lemma is similar to the surgery principles for finite quantum graphs in [17], but for periodic quantum graphs. Those surgery procedures describe how the spectrum of a new graph is related to the spectra of old graphs under various modifications and joinings. When writing and manipulating formulas dealing with joining multiple quantum graphs together, it is convenient to use the following abbreviated notation for the dispersion function of a periodic quantum graph:

$$[\Gamma] := D_{(\Gamma,A)}(z, \lambda). \tag{2.3.1}$$

This notation emphasizes the dependence on Γ; it will be used only in the rest of section 2.3, here A, z, and λ will fall into the background and the ways in which the dispersion polynomial depends on the structure of Γ will be studied.

Definition 2.3.1. Let Γ be a d-periodic graph, and let v be a vertex of Γ of degree r. We define Γ^v to be the periodic graph obtained by replacing, for each $g \in \mathbb{Z}^d$, the vertex $g+v$ by r terminal vertices incident to the r edges that are incident to $g+v$ in Γ.

Definition 2.3.2. Let (Γ, A) be a d-periodic quantum graph containing vertex $v \in \mathcal{V}(\Gamma)$. Denote by (Γ^v, A^v) the quantum graph obtained by replacing the vertex condition at each vertex in the orbit $\{g+v : g \in \mathbb{Z}^d\}$ in Γ with the Dirichlet condition. Defining A^v as an operator on Γ^v is valid because of Remark 2.2.6.

Definition 2.1 (Single-vertex join $\Gamma_1(v_1\, v_2)\Gamma_2$)**.** *Let Γ_1 and Γ_2 be d-periodic quantum graphs with Robin parameter α_1 at $v_1 \in \mathcal{V}(\Gamma_1)$ and α_2 at $v_2 \in \mathcal{V}(\Gamma_2)$. The* single-vertex join *of Γ_1 and Γ_2 at the pair (v_1, v_2), denoted by $\Gamma_1(v_1\, v_2)\Gamma_2$, is a quantum graph with vertex set $\mathcal{V}(\Gamma_1) \cup \mathcal{V}(\Gamma_2)/\equiv$, in which $g+v_1 \equiv g+v_2$ for all $g \in \mathbb{Z}^d$ and edge set $\mathcal{E}(\Gamma_1) \cup \mathcal{E}(\Gamma_2)$. A Robin vertex condition with parameter $\alpha_1+\alpha_2$ is imposed at the joined vertices $g+v_1 \equiv g+v_2$, and all other vertex conditions are inherited from Γ_1 and Γ_2. If the Robin parameter at the joined vertex $v_1 \equiv v_2$ is changed to α, the resulting graph is denoted by $\Gamma_1(v_1\, v_2)_\alpha \Gamma_2$.*

Lemma 2.3.3. *Let Γ_1 and Γ_2 be d-periodic quantum graphs with $v_1 \in \mathcal{V}(\Gamma_1)$ and $v_2 \in \mathcal{V}(\Gamma_2)$. Then the dispersion function for $\Gamma_1(v_1, v_2)\Gamma_2$ is*

$$[\Gamma_1(v_1, v_2)\Gamma_2] \;=\; [\Gamma_1]\,[\Gamma_2^{v_2}] + [\Gamma_1^{v_1}]\,[\Gamma_2]\,, \tag{2.3.2}$$

and the dispersion function for $\Gamma_1(v_1\, v_2)_\alpha \Gamma_2$ is

$$[\Gamma_1(v_1\, v_2)_\alpha \Gamma_2] = [\Gamma_1]\,[\Gamma_2^{v_2}] + [\Gamma_1^{v_1}]\,[\Gamma_2] + (\alpha - \alpha_1 - \alpha_2)\,[\Gamma_1^{v_1}]\,[\Gamma_2^{v_2}]\,. \tag{2.3.3}$$

Proof. Let A_1 and A_2 be the operators associated with the quantum graphs Γ_1 and Γ_2, and let $\hat{A}_1(z,\lambda)$ and $\hat{A}_2(z,\lambda)$ be the spectral matrices of these operators. Let $\hat{A}_1^0(z,\lambda)$ and $\hat{A}_2^0(z,E)$ be the spectral matrices of the operators associated with $\Gamma_1^{v_1}$ and $\Gamma_2^{v_2}$. By ordering the vertices of a fundamental domain of Γ_1 such that v_1 is listed last, and ordering the vertices of a fundamental domain of Γ_2 such that v_2 is listed first, one obtains the block decomposition

$$\hat{A}_1 = \begin{bmatrix} \hat{A}_1^0 & a_1 \\ a_1^* & a_1^0 \end{bmatrix}, \qquad \hat{A}_2 = \begin{bmatrix} a_2^0 & a_2^* \\ a_2 & \hat{A}_2^0 \end{bmatrix}, \tag{2.3.4}$$

in which a_1 and a_2 are column vectors and a_1^0 and a_2^0 are scalars.

The matrix $\hat{A}$ of the operator associated with $\Gamma_1(v_1\, v_2)\Gamma_2$ is

$$\hat{A} \;=\; \begin{bmatrix} \hat{A}_1^0 & a_1 & \mathbf{0} \\ a_1^* & a_1^0 + a_2^0 & a_2^* \\ \mathbf{0} & a_2 & \hat{A}_2^0 \end{bmatrix}. \tag{2.3.5}$$

Notice that the entry $a_1^0 + a_2^0$ incorporates the Robin parameter $\alpha_1 + \alpha_2$. The first statement of the theorem can be derived by using properties of block matrix determinants.

$$\det(\hat{A}) \;=\; \det(\hat{A}_1)\det(\hat{A}_2^0) + \det(\hat{A}_1^0)\det(\hat{A}_2).$$

The matrix Floquet transform for $\Gamma_1(v_1\, v_2)_\alpha \Gamma_2$ is obtained by adding $\alpha - \alpha_1 - \alpha_2$ to the term $a_1^0 + a_2^0$ in (2.3.5), and the second statement of the theorem follows. □

2.3.2 Separable periodic graphs

The class of multi-layer graphs that we call type 1 are built from layers that are separable. Each layer has the property that, when a certain vertex is removed periodically, the graph separates into a d-dimensional array of identical finite graphs, as illustrated in Fig. 2.3.1.

Definition 2.3.4. A d-periodic graph Γ is *separable at* $v \in \mathcal{V}(\Gamma)$ if Γ^v is the union of the $\mathbb{Z}^d$ translates of a finite graph, or, equivalently, if Γ^v has compact connected components.

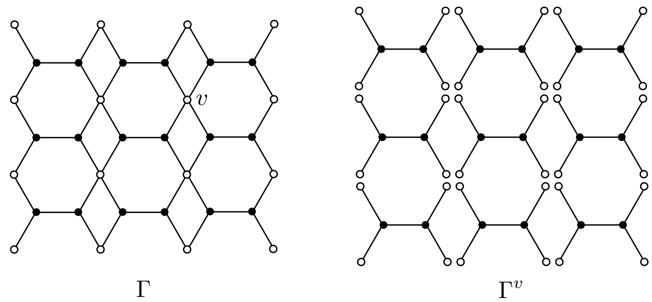

Figure 2.3.1: A periodic graph that is separable at a vertex v; and the corresponding graph Γ^v.

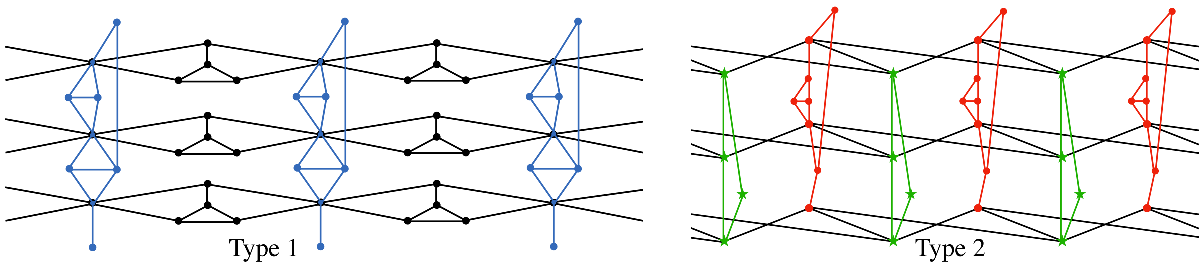

Figure 2.3.2: 1-periodic examples of multi-layer graphs with reducible Fermi surface. Type 1 (left): Each layer is a separable periodic graph whose dispersion function is a polynomial in a fixed Laurent polynomial $\zeta(z, \lambda)$. The layers are connected at corresponding vertices of separation by the periodic translates of a finite (blue) graph. Type 2 (right): Each layer has the same underlying bipartite periodic graph with two vertices per period, and the potentials on corresponding (vertically displaced) edges have the same Dirichlet spectrum. Vertically displaced green vertices are connected by periodic translates of a finite (green) graph; and similarly for the red vertices.

2.3.3 Type 1: Multi-layer graphs with separable layers

This section develops a class of multi-layer graphs whose individual layers are separable and whose Fermi surface is reducible. A 1-periodic illustration is in Fig. 2.3.2(left). An example is AB-stacked graphene, which is discussed in Section 2.4.4.

Definition 2.3.5. A *type-1* quantum graph is built from *layers* (black in Fig. 2.3.2(left)) and *connector graph* (blue in Fig. 2.3.2(left)). Let $\zeta(z, \lambda)$ be a Laurent polynomial in $z = (z_1, \ldots, z_d)$ with coefficients that are meromorphic in λ. The j-th layer ($j = 1, \ldots, n$) is a d-periodic quantum graph (Λ_j, A_j), with a distinguished vertex v_j and such that the dispersion function of (Λ_j, A_j) and the dispersion function of $(\Lambda_j^{v_j}, A_j^{v_j})$ are each polynomial functions of the same composite variable $\zeta(z, \lambda)$ with coefficients that are meromorphic in λ.

The connector graph is a finite quantum graph (Σ, B), together with a list of distinct vertices $w_j \in \mathcal{V}(\Sigma)$ $(j = 1, \ldots, n)$. An n-layer quantum graph (Γ, A) is formed as follows: For each j the vertex v_j is merged, or identified, with w_j. In like manner, for each $g \in \mathbb{Z}^d$, the translated vertices $g + v_1, \ldots, g + v_n$ are coupled by another copy of Σ, called $g + \Sigma$. The resulting periodic graph Γ is called a type-1 multi-layer graph.

The edge set of a type-1 graph consists of the edges of each layer Λ_j and the edges of each translate $g + \Sigma$ of Σ. By denoting the identification of merged vertices by the equivalence relation $\equiv$, the vertex and edge sets of Γ are

$$\mathcal{V}(\Gamma) = \left(\bigcup_{j=1}^{n} \mathcal{V}(\Lambda_j) \cup \bigcup_{g \in \mathbb{Z}^d} \mathcal{V}(g + \Sigma) \right) / \equiv \tag{2.3.6}$$

$$\mathcal{E}(\Gamma) = \bigcup_{j=1}^{n} \mathcal{E}(\Lambda_j) \cup \bigcup_{g \in \mathbb{Z}^d} \mathcal{E}(g + \Sigma). \tag{2.3.7}$$

The Schrödinger operator A on Γ has the same differential-operator expression as the operators A_j and B on the elemental graphs Λ_j and Σ, and the Robin parameter of an equivalence class of merged vertices (now a single vertex of Γ) is assigned the sum of the Robin parameters of all the vertices that were merged.

When we form a type-1 graph we have in mind specifically the situation in which each layer Λ_j is separable at v_j. This is because, in this case, $\Lambda_j^{v_j}$ is a disjoint union of compact graphs, and thus its dispersion function is a meromorphic function $f_j(\lambda)$ and therefore a degree-0 polyomial in $\zeta(z, \lambda)$. However when we form multilayer structures we can allow the layers themselves to be type-1 graphs, in this case the dispersion function of a layer will be a more general polynomial. This is applied to analyze ABC-stacked graphene in Section 2.4.

Observe that it is possible to allow several of the vertices w_j to be equal. In this case, one might as well merge all the layers that are attached to that vertex into a single layer according to the single-vertex join in Definition 2.1, applied several times. According to the calculus of Lemma 2.3.3, the degree of the polynomial p_j for this new layer is the maximum of the degrees of the polynomials of the joined components.

Theorem 2.3. *Let (Γ, A) be an n-layer d-periodic quantum graph of type 1. Its dispersion function $D(z,\lambda)$ is a polynomial in $\zeta(z,\lambda)$ with coefficients that are meromorphic functions of λ,*

$$D(z,\lambda) \;=\; P(\zeta(z,\lambda),\lambda). \tag{2.3.8}$$

Let $p_j(\zeta,\lambda)$ be the dispersion function for the layer (Λ_j, A_j). The degree of P as a polynomial in ζ is

$$\deg P \;=\; \sum_{j=1}^{n} \deg p_j. \tag{2.3.9}$$

Proof. When the number of layers is zero, (Γ, A) is the union $\cup_{g\in\mathbb{Z}^d} g+\Sigma$ of disconnected finite components with the operator B acting on each component. The dispersion function is a meromorphic function of λ, independent of z, and is thus trivially a polynomial in $\zeta(z,\lambda)$ of degree 0, with coefficients that are meromorphic in λ. We proceed by induction. Let the theorem hold with n replaced by $n-1$, with $n \geq 1$.

Let (Γ, A) be the type-1 n-layer quantum graph supposed in the theorem, with layers (Λ_j, A_j) separable at v_j ($1 \leq j \leq n$) and connector graph (Σ, B) with distinct joining vertices $\{w_j\}_{j=1}^n$. If any of the polynomials p_j has degree 0, then Λ_j is a disjoint union of $\mathbb{Z}^d$ translates of a finite graph, and this finite graph might as well be joined with the connector graph Σ. Therefore, we assume that each $\deg p_j \geq 1$ for all $j : 1 \leq j \leq n$.

Denote by $(\tilde\Gamma, \tilde A)$ the type-1 $(n-1)$-layer quantum graph built from the layers $\{(\Lambda_j, A_j)\}_{j=1}^{n-1}$ and the connector objects (Σ, B) and $\{w_j\}_{j=1}^{n-1}$. Note that $(\tilde\Gamma^{w_n}, \tilde A^{w_n})$ is the type-1 (n–1)-layer quantum graph built from the layers $\{(\Lambda_j, A_j)\}_{j=1}^{n-1}$ and the connector objects (Σ^{w_n}, B^{w_n}) and

$\{w_j\}_{j=1}^{n-1}$. Denote by $\tilde{D}(z,\lambda)$ and $\tilde{D}^0(z,\lambda)$ the dispersion functions of $(\tilde{\Gamma},\tilde{A})$ and $(\tilde{\Gamma}^{w_n},\tilde{A}^{w_n})$. By the induction hypothesis, they are polynomials in $\zeta(z,\lambda)$ with coefficients that are meromorphic in λ, and both are of degree $\sum_{j=1}^{n-1}\deg p_j$.

The graph (Γ, A) is the single-vertex join of $(\tilde{\Gamma},\tilde{A})$ and Λ_n,

$$\Gamma \;=\; \tilde{\Gamma}(w_n,v_n)\Lambda_n \tag{2.3.10}$$

and the calculus of Lemma 2.3.3 yields

$$[\Gamma] \;=\; [\tilde{\Gamma}][\Lambda_n^{v_n}] + [\tilde{\Gamma}^{w_n}][\Lambda_n]. \tag{2.3.11}$$

Since Λ_n is separable at v_n, $[\Lambda_n^{v_n}]$ is independent of z, so the degree of the first term on the right-hand side of (2.3.11), as a polynomial in ζ, is $m = \sum_{j=1}^{n-1}\deg p_j$. The degree of the second term as a polynomial in ζ is $m+\deg p_n$. This completes the induction. □

Corollary 2.4. The Fermi surface of a type-1 n-layer d-periodic quantum graph is reducible into $m=\sum_{j=1}^{n}\deg p_j$ components (with possible multiplicities). Each component is of the form

$$\zeta(z,\lambda) \;=\; \mu(\lambda). \tag{2.3.12}$$

For each λ, the m (not necessarily distinct) values of $\mu(\lambda)$ are the roots of $P(\zeta,\lambda)$.

Proof. The polynomial $P(\zeta,\lambda)$ in Theorem 2.3 factors into $m=\sum_{j=1}^{n}\deg p_j$ linear factors as a function of ζ, and each factor corresponds to a component of the Fermi surface of (Γ, A). □

A special case of Theorem 2.3 occurs when there is only one layer. The connector graph Σ is then viewed as a periodic "decoration" of Λ_1. The result is the following corollary. Much more is known about decorated periodic graphs, particularly with regard to opening spectral gaps [76].

Corollary 2.5 (Decorated graphs). Let (Γ, A) be a d-periodic quantum graph that is separable at vertex v, and let (Σ, B) be a finite decorator graph with distinguished vertex $w \in \mathcal{V}(\Sigma)$. Let $\ell(\lambda)$ denote the spectral function of (Γ^v, A^v), and let $h(\lambda)$ and $h^0(\lambda)$ denote the spectral functions of (Σ, B) and (Σ^w, B^w).

Denote by $(\bar{\Gamma}, \bar{A})$ the "decorated graph" obtained by the single-vertex join of (Γ, A) and $\Delta = \cup_{g \in \mathbb{Z}^d} g\Sigma$ at the vertices v and w. If the Fermi surface of (Γ, A) at energy λ is given by

$$D(z, \lambda) \;=\; 0, \tag{2.3.13}$$

then the Fermi surface of $(\bar{\Gamma}, \bar{A})$ at λ is given by

$$D(z, \lambda) \;=\; -\frac{\ell(\lambda) h(\lambda)}{h^0(\lambda)}. \tag{2.3.14}$$

Proof. The theorem says that $[\bar{\Gamma}]$ is a function of $D(\lambda, z)$ that is linear in $D(z, \lambda)$ (take $\zeta(z, \lambda) = D(z, \lambda)$), and we can find the coefficients by applying Lemma 2.3.3. $[\bar{\Gamma}] = [\Gamma][\Delta^w] + [\Gamma^v][\Delta]$, or

$$[\bar{\Gamma}] \;=\; D(z, \lambda) h^0(\lambda) + \ell(\lambda) h(\lambda). \tag{2.3.15}$$

The Fermi surface of $(\bar{\Gamma}, \bar{A})$ is $[\bar{\Gamma}] = 0$, from which follows the result. □

2.3.4 Type 2: Multi-layer graphs with bipartite layers

This section generalizes the construction in [78, §6] from bi-layer to n-layer quantum graphs and from single-edge coupling to coupling by general graphs, as illustrated on the right in

Fig. 2.3.2.

Definition 2.3.6. A *type-2* multilayer quantum graph is built from *layers* with the same graph and two *connector graphs*, (Σ_1, B_1) and (Σ_2, B_2). Each layer is bipartite with exactly one "red" and one "green" vertex in a fundamental domain. The operators associated to each layer must have the same $s(\lambda)$-function on corresponding edges, or, equivalently, parallel potentials must have the same Dirichlet spectrum. The Robin parameters may be different across layers. The connector graphs are both finite graphs with at least n-vertices. The layers are stacked by periodically connecting them copies of the connector graphs. In each fundamental domain the n red vertices are joined by a copy of Σ_1 and the n green vertices are joined by a copy of Σ_2. The Robin vertex parameters are added at each joining vertex.

An example is AA-stacked graphene, which is discussed in Section 2.4.3.

Theorem 2.6 (bipartite layers)**.** *Let (Γ, A) be a multi-layer type-2 periodic quantum graph obtained by coupling n quantum graphs (Λ, A_k), $k = 1, \ldots, n$. The dispersion function is a polynomial, p, of degree n in a composite variable $\zeta(z, \lambda)$ with coefficients that holomorphic in λ. Thus, for each energy λ, the Fermi surface of (Γ, A) has (counting multiplicity) n components. The components are of the form*

$$\zeta(z, \lambda) \;=\; \rho(\lambda), \tag{2.3.16}$$

in which $\rho(\lambda)$ is a root of the polynomial p.

Proof. Given that the quantum graph $(\Lambda, \mathring{A})$, for a given layer, has underlying graph Λ which is bipartite with one red and one green vertex per period, the spectral matrix is a 2×2 matrix

$$\hat{\mathring{A}}(z, \lambda) = \begin{bmatrix} b_1(\lambda) & w(z, \lambda) \\ w(z^{-1}, \lambda) & b_2(\lambda) \end{bmatrix}, \tag{2.3.17}$$

in which $b_i(\lambda)$ are meromorphic functions of λ and $w(z,\lambda)$ is a Laurent polynomial in z with coefficients that are meromorphic in λ. Specifically, $w(z,\lambda)$ is a sum over some finite subset $Z \subset \mathbb{Z}^d$,

$$w(z,\lambda) \;=\; \sum_{\ell \in Z} \frac{\varepsilon_\ell \, z^\ell}{s_\ell(\lambda)}\,, \tag{2.3.18}$$

in which $s_\ell(\lambda)$ is the s-function for the potential $q(x)$ on the edge connecting a green vertex in a given fundamental domain with a red vertex in the domain shifted by $\ell \in \mathbb{Z}^d$, and $z^\ell = z_1^{\ell_1} \cdots z_n^{\ell_n}$ and ε_ℓ is the weight for that edge.

Since (Γ, A) is type-2 we have that $w(z,\lambda)$ are identical over all the layers, however the functions $b_1(\lambda)$ and $b_2(\lambda)$ may vary from layer to layer. The spectral matrix for (Γ, A) is

$$\begin{aligned} \hat{A}(z,\lambda) \;&=\; \begin{bmatrix} \mathbf{b}_1(\lambda) & w(z,\lambda)Q \\ w(z^{-1},\lambda)Q^T & \mathbf{b}_2(\lambda) \end{bmatrix} + \begin{bmatrix} B_1(\lambda) & 0 \\ 0 & B_2(\lambda) \end{bmatrix} \\ &=\; \begin{bmatrix} \tilde{B}_1(\lambda) & w(z,\lambda)Q \\ w(z^{-1},\lambda)Q^T & \tilde{B}_2(\lambda) \end{bmatrix}, \end{aligned} \tag{2.3.19}$$

in which Q is the $m_1 \times m_2$ matrix with the $n \times n$ identity matrix in its upper left, all other entries being zero; $\mathbf{b}_1(\lambda)$ (*resp.* $\mathbf{b}_2(\lambda)$) is a square diagonal matrix of size $n+m_1$ (*resp.*

$n{+}m_2$),

$$\mathbf{b}_1(\lambda) \;=\; \operatorname*{diag}_{j=1\ldots n} b_1^j(\lambda) \oplus 0_{m_1} \;=\; \begin{bmatrix} b_1^1(\lambda) & & & & & \\ & \ddots & & & & \\ & & b_1^n(\lambda) & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix}; \tag{2.3.20}$$

$B_1(\lambda)$ is the $m_1 \times m_1$ spectral matrix of the coupling graph for the red vertices and the $m_2 \times m_2$ matrix $B_2(\lambda)$ is for the green vertices; and $\tilde{B}_j(\lambda){=}\mathbf{b}_j(\lambda) + B_j(\lambda)$.

The dispersion function of (Γ, A) is

$$\begin{aligned} D(z,\lambda) \;=\; \det \hat{A}(z,\lambda) \;&=\; \det\big(\tilde{B}_1(\lambda)\big) \det\Big(\tilde{B}_2(\lambda) - w(z,\lambda)w(z^{-1},\lambda)\, Q^T \tilde{B}_1(\lambda)^{-1} Q\Big) \\ &=\; P\big(w(z,\lambda)w(z^{-1},\lambda),\lambda\big), \end{aligned} \tag{2.3.21}$$

in which $P(\cdot,\lambda)$ is a polynomial of degree n with coefficients that are meromorphic functions of λ. For a single layer, this polynomial is just a linear function of the composite Floquet variable $\zeta(z,\lambda) := w(z,\lambda)w(z^{-1},\lambda)$. $\square$

Remark 2.3.7. An important simplification occurs when the connector graphs are linear graphs. The n successive layers are connected by edges and the matrix Q in expression (2.3.21) becomes the $n{\times}n$ identity matrix I_n. The dispersion function simplifies to

$$D(z,\lambda) \;=\; \det\big(\tilde{B}_1(\lambda)\tilde{B}_2(\lambda) - \zeta(z,\lambda)I_n\big), \tag{2.3.22}$$

and therefore the components of the Fermi surface are

$$\zeta(z,\lambda) \;=\; \rho_j(\lambda), \quad j=1,\dots,n, \tag{2.3.23}$$

where ρ_j are the eigenvalues of the matrix $\tilde{B}_1(\lambda)\tilde{B}_2(\lambda)$. As the connector graphs are linear graphs, their spectral matrices $B_i(\lambda)$ are tridiagonal, with DtN matrices for the connector edges along the principal 2×2 submatrices.

Remark on decorated edges. For the sake of completeness, we mention that the edges in any of the quantum graphs we consider may as well be "decorated edges", as illustrated in Fig. 2.3.3 for a single-layer graphene structure. A decorated edge is a finite graph that has two distinguished terminal vertices that act as the two vertices of the decorated edge. Particularly, in a type-2 multi-layer graph, the single layers, even when decorated, can essentially still be considered as being bipartite. Allowing decorations on an edge can be thought of loosely as allowing a broader class of potentials on the edge. A decorated edge admits a Dirichlet-to-Neumann map that straightforwardly generalizes that of an edge. When forming the spectral matrix $\hat{A}(z,\lambda)$, this DtN map is used, as described in section 2.2.2, and only the two terminal endpoints of the decorated edge enter into the vertex set that indexes the matrix.

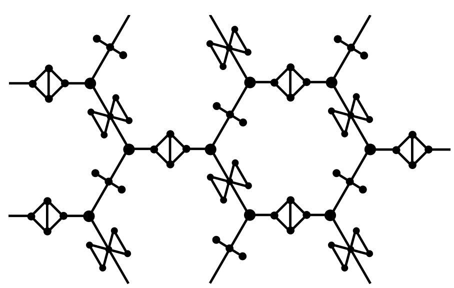

Figure 2.3.3: Quantum-graph graphene model with decorated edges.

2.4 Multilayer Graphene

We apply the theory developed in this work to quantum-graph models of multi-layer graphene structures. By using the theorems for Type-1 and Type-2 quantum graphs, we can show that very general stacking of graphene, where the layers are shifted or rotated, results in a reducible Fermi surface. We also include some brief discussion on the conical singularities at wavevectors $(k_1, k_2) = \pm(2\pi/3, -2\pi/3)$ for single-layer graphene and how stacking multiple layers destroys them.

2.4.1 The single layer

A graph model of graphene is hexagonal and bipartite, having two vertices and three edges of length 1 per fundamental domain. Since Graphene is bipartite, it is also separable at any vertex (see, for example, [56]).

The most general quantum-graph model $(\mathring{\Lambda}, \mathring{A})$ for which the differential operator on the edges is of the form $-d^2/dx^2 + q(x)$ features three potentials, one for each edge in a period, and two Robin parameters α_i, one for each vertex v_i $(i = 1, 2)$ in a period. The potentials will be denoted by $q_i(x)$ $(i = 0, 1, 2)$ as in Fig. 2.4.1 and the corresponding transfer matrices by

$$T_i(\lambda) = \begin{bmatrix} c_i(\lambda) & s_i(\lambda) \\ c_i'(\lambda) & s_i'(\lambda) \end{bmatrix} \qquad (i = 0, 1, 2). \tag{2.4.1}$$

Let ξ_1 and ξ_2, as illustrated in Fig. 2.4.1, be generators of the periodicity in the sense that the action of $(n_1, n_2) \in \mathbb{Z}^2$ on $\mathring{\Gamma}$ shifts the graph along the vector $n_1\xi_1 + n_2\xi_2$ in the plane so that it falls exactly into itself. The components of the vector (z_1, z_2), the Floquet multipliers,

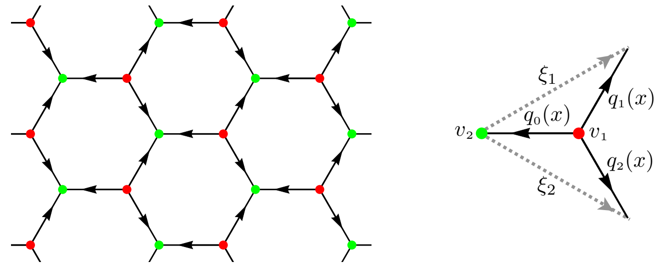

Figure 2.4.1: Single-layer graphene $\mathring{\Gamma}$ and its fundamental domain. The arrows on the edges indicate the direction of the x-interval $[0,1]$ in the parameterization of the edges. The vectors ξ_1 and ξ_2 generate the periodic shifts.

are the eigenvalues of the shifts by ξ_1 and ξ_2 corresponding to a Floquet mode. The spectral matrix (2.3.19) of this quantum graph at energy λ is

$$\hat{\mathring{A}}(z_1, z_2, \lambda) \;=\; \begin{bmatrix} b_1(\lambda) & w(z,\lambda) \\ \\ w(z^{-1},\lambda) & b_2(\lambda) \end{bmatrix}, \tag{2.4.2}$$

Where the entries are the following functions:

$$b_1(\lambda) = -\frac{c_0(\lambda)}{s_0(\lambda)} - \frac{c_1(\lambda)}{s_1(\lambda)} - \frac{c_2(\lambda)}{s_2(\lambda)} - \alpha_1 \tag{2.4.3}$$

$$b_2(\lambda) = -\frac{s_0'(\lambda)}{s_0(\lambda)} - \frac{s_1'(\lambda)}{s_1(\lambda)} - \frac{s_2'(\lambda)}{s_2(\lambda)} - \alpha_2 \tag{2.4.4}$$

$$w(z,\lambda) = \frac{1}{s_0(\lambda)} + \frac{z_1}{s_1(\lambda)} + \frac{z_2}{s_2(\lambda)}\,. \tag{2.4.5}$$

This is the function $w(z,\lambda)$ in (2.3.18). Notice that $w(z,\lambda)$ depends only on the potentials $q_i(x)$ through their Dirichlet spectrum since only the functions $s_i(\lambda)$ appear in the definition of $w(z,\lambda)$. The dispersion function for $(\mathring{\Gamma}, \mathring{A})$ is

$$D(z_1, z_2, \lambda) \;=\; \det \hat{\mathring{A}}(z_1, z_2, \lambda) \;=\; b_1(\lambda)b_2(\lambda) - w(z,\lambda)w(z^{-1},\lambda). \tag{2.4.6}$$

We will use the shorthand $w(z,\lambda)w(z^{-1},\lambda) = \zeta(z,\lambda)$. This notation is chosen intentionally, we will have that $w(z,\lambda)w(z^{-1},\lambda)$ is the composite variable for a variety of multilayer graphene structures. Notice that the dispersion functions of two different single-layer sheets of graphene have the same $\zeta(z,\lambda)$ exactly when corresponding edges are isospectral, because knowing the Dirichlet spectrum of a potential is equivalent to knowing its $s(\lambda)$ function [71, Ch. 2 Theorem 5].

All three edges in a period of a single layer are isospectral exactly when $s_0(\lambda) = s_1(\lambda) = s_2(\lambda)$, and in this case $\zeta(z,\lambda)$ separates as

$$\zeta(z,\lambda) \;=\; s_0(\lambda)^{-2} G(z_1,z_2), \tag{2.4.7}$$

in which

$$G(z_1,z_2) \;=\; (1+z_1+z_2)(1+z_1^{-1}+z_2^{-1}). \tag{2.4.8}$$

The Fermi surface of a single layer at energy λ is given by $D(z_1,z_2,\lambda)=0$, which reduces to

$$s_0(\lambda)^2\, b_1(\lambda) b_2(\lambda) \;=\; G(z_1,z_2). \tag{2.4.9}$$

We will use the symbol $\Delta(\lambda)$ to refer to the left hand side of (2.4.9). Also we call $\Delta(\lambda)$ as the "characteristic function" for this single-layer graphene model.

For $(z_1,z_2) = (e^{ik_1}, e^{ik_2})$ on the torus $\mathbb{T}^2$,

$$\begin{aligned} \tilde{G}(k_1,k_2) \;:=\; G(e^{ik_1},e^{ik_2}) \;&=\; \left|1+e^{ik_1}+e^{ik_2}\right|^2 \\ &=\; 1+8\cos\frac{k_2-k_1}{2}\cos\frac{k_1}{2}\cos\frac{k_2}{2}, \end{aligned} \tag{2.4.10}$$

and this has range $[0,9]$ as a function of real k_1 and k_2, with its minima occuring at $\pm(2\pi/3,-2\pi/3)$ [56, Lemma 3.3]. Thus the bands of this graphene model are the real

λ-intervals over which $\Delta(\lambda)$ lies in $[0, 9]$.

Single-layer quantum-graph graphene sheets and tubes, with a common symmetric potential $q_0(x)$ on all edges, are treated in detail in [56]. In this case, $b_1(\lambda) = b_2(\lambda)$ and the spectrum of the sheet is identical to that of the periodic Hill operator with potential $q_0(x)$ on a period. In contrast to the Hill operator, the dispersion relation exhibits conical singularities, one for each energy λ where $\Delta(\lambda) = 0$. Fig. 2.4.3 shows a graph of $\Delta(\lambda)$.

Notice that $\Delta(\lambda)$ is a non-negative function of real λ that has a minimum value of 0 (Fig. 2.4.3). The quadratic nature of the function at the minima is responsible for the Dirac cones, as explained in [56]. There is a close connection between the dispersion functions of discrete (tight-binding) models and quantum-graph models [56, Remark 3.2]. In the discrete graph model of single-layer graphene with a common interaction strength between atoms, the characteristic function reduces to $\Delta(\lambda) = 9\lambda^2$, and thus there is a Dirac cone at $\lambda = 0$. For multi-layer graphene in the discrete and quantum versions, $\Delta(\lambda)$ becomes a more complicated function of λ. Our analysis treats very general potentials on the three edges of a fundamental domain, and this leads to a more general dispersion function that is not separable into λ-dependent and z-dependent terms because the edges do not in general possess a common $s(\lambda)$ function (see (2.4.6) and (2.4.7)).

2.4.2 Shifting and rotating

We adopt terminology on shifted layers of graphene that is used in the literature. The hexagonal graphene structure is invariant under translation by the sum $\xi_1 + \xi_2$ of the two elementary shift vectors, as illustrated in Fig. 2.4.2. The shift by $(\xi_1 + \xi_2)/3$ (dashed blue) places vertex v_2 onto vertex v_1 and places vertex v_1 onto the center of the hexagon; this will be called the B-shift. The shift by $2(\xi_1 + \xi_2)/3$ (or $-(\xi_1 + \xi_2)/3$, dotted orange) places v_1 onto v_2 and v_2 onto the center of the hexagon; this will be called the C-shift. The unshifted graph is called the A-shift.

By rotating the graphene structure by π about the center of an edge, the potentials reverse direction. This is illustrated on the right of Fig. 2.4.2, in which rotation is about the edge labeled 0. Each labeled oriented edge corresponds to a potential $q_i(x)$, with the parameter x increasing in the direction of the arrow. The labels 0, 1, 2 are preserved under rotation, but their orientations are reversed. Equivalently, rotation effects the change $q_i(x) \mapsto q_i(1-x)$ of the potentials. The rotation also switches the Robin conditions on the two vertices of a period.

Denote a single layer by $(\mathring{\Lambda}, \mathring{A})$ and its 180° rotation by $(\mathring{\Lambda}, \mathring{A}_\pi)$. The potentials $q_i(x)$ and $q_i(1-x)$ have the same Dirichlet spectrum, which coincides with the roots of the function $s_i(\lambda)$. Therefore the function $w(z,\lambda)$ in (2.4.5) is the same for both quantum graphs and their dispersion functions are polynomials in the same composite Floquet variable $\zeta(z,\lambda) = w(z,\lambda)w(z^{-1},\lambda)$.

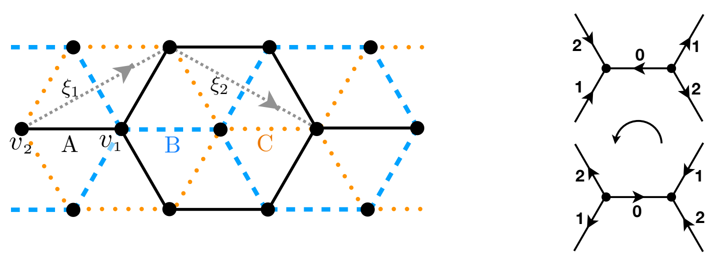

Figure 2.4.2: Left: A- B- and C-shifts of graphene are illustrated in solid black, dashed blue, and dotted orange, as described in the text. Right: Rotating graphene by 180° reverses the orientation of the potentials but preserves their Dirichlet spectra.

2.4.3 AA-stacking and rotation

In AA-stacked graphene, each layer is stacked directly over the previous and each pair of vertically successive vertices is connected by an edge, as in Fig. 2.4.4. As a type-2 n-layer graph, the red vertices in a given period, together with the $n-1$ edges connecting them, form the connector graph (Σ_1, B_1), and the green vertices and the edges connecting them

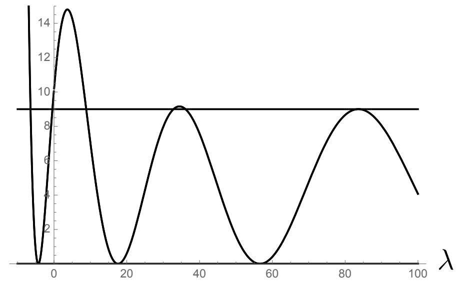

Figure 2.4.3: Graph of the characteristic function $\Delta(\lambda)$ of single-layer graphene, showing the first three spectral bands and the first three gaps. The bands are the λ-intervals for which $\Delta(\lambda) \in [0, 9]$, which is the range of the function $\tilde{G}(k_1, k_2)$. The points where $\Delta(\lambda) = 0$ correspond to conical singularities of the dispersion relation $D(e^{ik_1}, e^{ik_2}, \lambda) = 0$, which occur inside the bands, as discussed in [56].

form (Σ_2, B_2). The hypotheses of Theorem 2.6 allow the potentials $q(x)$ on any pair of vertically aligned edges on two different layers to differ as long as the operators $-d^2/dx^2 + q_e(x)$ possess the same Dirichlet spectrum. The theorem then guarantees that the Fermi surface of the layered structure is reducible with n components.

A particular instance of AA-stacked graphene satisfying the hypotheses of the theorem is constructed from copies of a given single layer and its rotations about the center of an edge. Let a single layer (Λ, A_0) with arbitrary potentials on the three edges of a period and arbitrary Robin parameters on the two vertices be given. Rotation of this graph by 180° about the center of an edge, as described in the previous section and illustrated in Figs. 2.4.2,2.4.4 (right), results in a new layer of graphene (Λ, A_π) with the same underlying graph Λ but with the potentials oriented in the opposite direction and the Robin parameters at the two vertices switched.

Thus Theorem 2.6 applies to an n-layer stack, with each layer being either $(\mathring{\Lambda}, \mathring{A})$ or $(\mathring{\Lambda}, \mathring{A}_\pi)$, in any order, stacked in the AA sense. The Fermi surface of this n-layer graphene has n components. According to section 2.3.4, equation (2.3.22), the relation $D(z, \lambda) = 0$ reduces to n components

$$\mu_i(\lambda) \;=\; G(z_1, z_2) \qquad i = 1, \ldots, n, \tag{2.4.11}$$

in which μ_i are the eigenvalues of the "characteristic matrix"

$$\Delta(\lambda) \;=\; s_0(\lambda)^2\,\tilde{B}_1(\lambda)\tilde{B}_2(\lambda), \tag{2.4.12}$$

which generalizes the characteristic function (2.4.9) by the same name for the single layer.

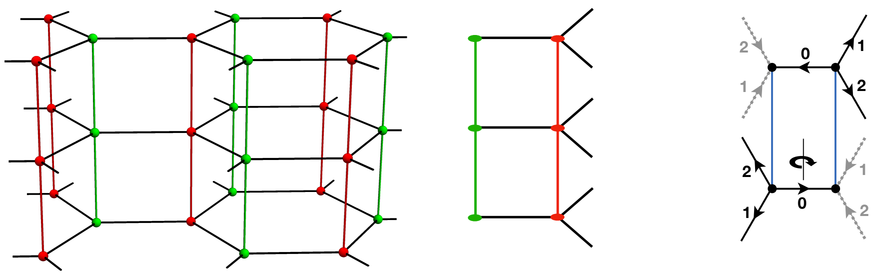

Figure 2.4.4: AA-stacked graphene in three layers and a fundamental domain thereof. If the potentials on corresponding edges on different layers have the same Dirichlet spectrum, then the Fermi surface for the multi-layer graph is reducible. This occurs, in particular, when a layer is rotated by π about the center of an edge.

Examples. (Fig. 2.4.5 and 2.4.6) Let the layers be identical with identical potential $q_0(x)$ on all three edges of a period. We take $q_0(x)$ to be symmetric about $x = 1/2$ so that the DtN map for the edge is independent of the direction. We choose

$$q_0(x) \;=\; -16\,\chi_{[1/3,2/3]}(x) \tag{2.4.13}$$

($\chi_Y(x)$ is the characteristic function of the set $Y \subset \mathbb{R}$) so that the DtN map is explicitly computable and so that the spectrum of the single layer does have gaps (because $q_0(x)$ is not constant; see [56]).

On all of the connector edges, which are all of length 1, we take the potential $q_c(x)$ to be either 0 or $q_0(x)$ or

$$q_c(x) \;=\; -10\,\chi_{[1/2,1]}(x), \tag{2.4.14}$$

which is not symmetric about the center. Fig. 2.4.5 and 2.4.6 show graphs of $\mu_i(\mu)$ for bi-layer and tri-layer graphene. Each eigenvalue contributes a sequence of bands and gaps to the spectrum of the multi-layer graph—the bands for the i^{th} sequence are the λ-intervals for which $\mu_i(\lambda) \in [0,9]$. When the Dirichlet spectral function $s(\lambda)$ on the connecting edges is different from that of the layers, new thin bands are introduced. Conical singularities, or Dirac cones, are characteristic features of single-layer graphene, and in special cases of AA-stacking, they persist. The recent article [28] also observes by computation that a finite number of AA-stacked graphene layers with the same symmetric potential on all edges within each layer and with symmetric connecting potentials, always exhibits Dirac cones. These Dirac cones can also be deduced from the full hexagonal symmetry group, as proved in [16].

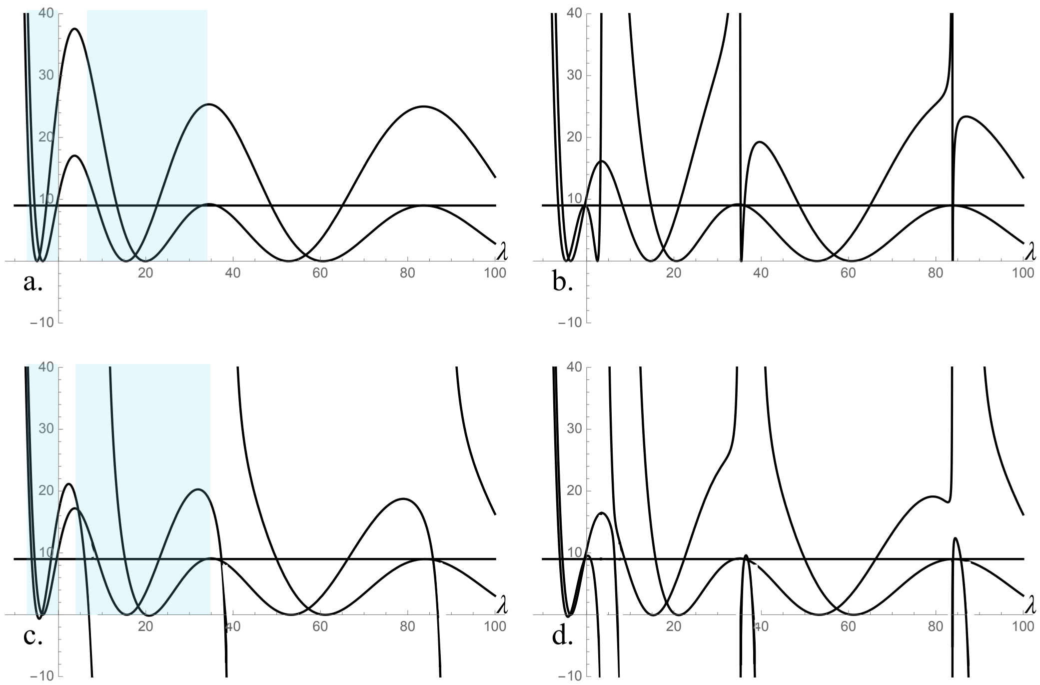

Figure 2.4.5: For double-layer AA-stacked graphene, the two eigenvalues $\mu_i(\lambda)$ $(i = 1, 2)$ of the characteristic matrix $\Delta(\lambda)$ give two sets of spectral bands and gaps. The bands are the λ-intervals for which $\mu_i(\lambda) \in [0, 9]$. **a.** The connecting edges have the same potential as those of the layers (2.4.13). **b.** The potentials (2.4.14) of the two connecting edges are equal to each other but different from that of the layers (2.4.13). This creates additional thin bands (within each of the sets of spectral bands), which have conical singularities of their own. **c.** The potentials of the two connecting edges are different from each other ($q(x) = 0$ and 2.4.13). This destroys conical singularities and introduces additional thin gaps in their place. Additionally, new thin bands are introduced just below the vertical asymptotes. **d.** The potentials of the two connecting edges are different from each other ($q(x) = 0$ and 2.4.14).

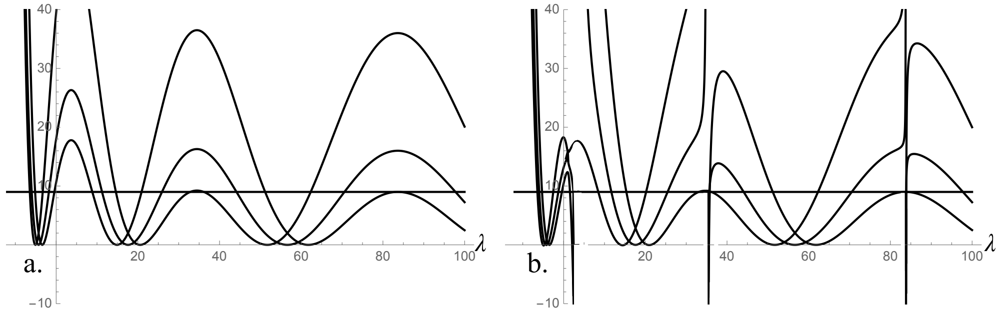

Figure 2.4.6: The three eigenvalues $\mu_i(\lambda)$ ($i = 1, 2, 3$) of the characteristic matrix $\Delta(\lambda)$ for triple-layer AAA-stacked graphene, giving three sets of spectral bands and gaps. The bands are the λ-intervals for which $\mu_i(\lambda) \in [0, 9]$. **a.** The connecting edges have the same potential as those of the layers. **b.** The connecting edges have potential (2.4.13) at vertex v_1 and potential (2.4.14) at vertex v_2; this creates additional thin gaps and destroys Dirac cones (the minima are slightly below 0).

2.4.4 AB-stacking

In AB-stacked graphene, also called Bernal stacking, the layers are in the A-shift or the B-shift and are coupled by a single edge per period. We allow any number of layers with the A- and B-shifts arranged in arbitrary order (such as ABABA, ABBA, *etc.*). Fig. 2.4.7 illustrates three layers with alternating shifts.

In each layer, we allow both $(\mathring{\Lambda}, \mathring{A})$ and $(\mathring{\Lambda}, \mathring{A}_\pi)$ or any potentials $q_i(x)$ ($i = 1, 2, 3$) as long as, for each i, the Dirichlet spectra are invariant across layers. As noted above, this guarantees that the function $\zeta(z, \lambda)$ is independent of the layer since it depends only on the Dirichlet spectral functions $s_i(\lambda)$, which are equivalent to the Dirichlet spectra of the potentials $q_i(x)$. Note that isospectrality (which is explicitly required for type 2) arises for graphene in type-1 stacking. Thus the dispersion function of each layer is of the form (2.4.6) with different $b_i(\lambda)$ but the same $\zeta(z, \lambda)$. In any period of this layered structure, n vertices, one per layer, are aligned along a vertical line, and these are connected by edges. These vertices serve as vertices of separation of the individual layers. Thus Theorem 2.3 on type-1 multi-layer graphs applies.

Computations of AB-stacking in [27], with a common symmetric potential on all edges of each layer, show that three layers result in a Dirac cone (linear point), whereas two layers result in two bands touching quadratically (parabolic point). This raises the question as to whether the parity of the number of layers determines whether the touching point is linear or parabolic and whether symmetry arguments can be used to illuminate this question.

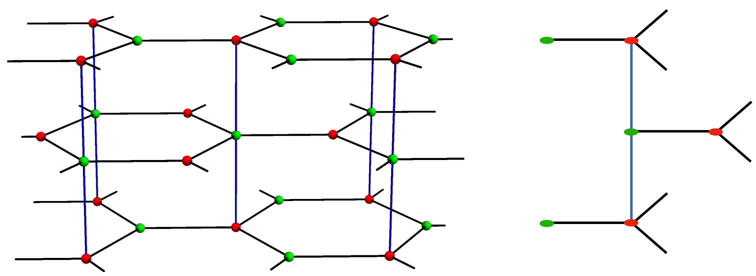

Figure 2.4.7: AB-stacked graphene in three layers (ABA) and a fundamental domain thereof.

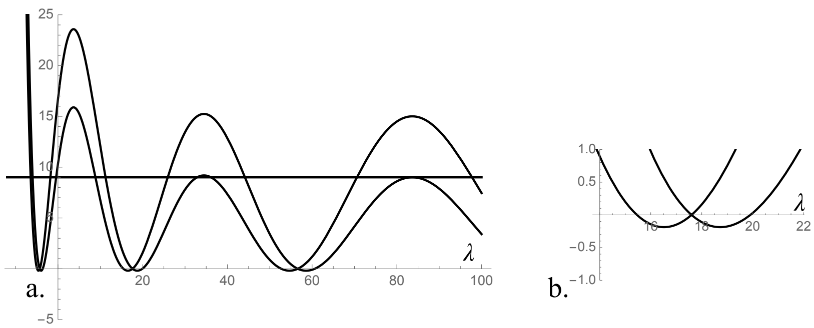

Figure 2.4.8: For double-layer AB-stacked graphene, these are graphs of $s(\lambda)^2\mu_i(\lambda)$ $(i = 1, 2)$, where $\mu_i(\lambda)$ are the two roots of $P(\zeta(z, \lambda), \lambda)$, as in Theorem 2.6. Each root gives a set of spectral bands and gaps. The bands are the λ-intervals for which $\mu_i(\lambda) \in [0, 9]$. **a.** The connecting edge has the same potential as those of the layers. **b.** A close view near $\lambda = 20$ shows that the graphs of $\mu_i(\lambda)$ cross the horizontal axis, and thus Dirac cones are not present.

2.4.5 ABC-stacking

In ABC-stacked graphene, all three shifts are stacked, as illustrated in Fig. 2.4.9. The number of components of the Fermi surface of the ABC-stacked structure is equal to the number of layers. We leave the details of how to use Theorems 2.3 and 2.6 to prove this to the reader. The arguments are similar to those described for the more general mixed stacking below.

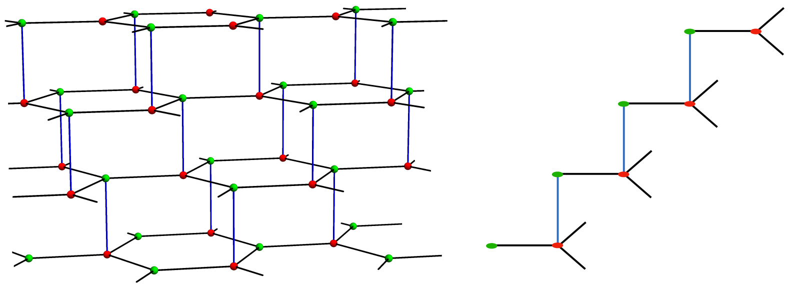

Figure 2.4.9: ABC-stacked graphene in four layers and a fundamental domain thereof.

2.4.6 Mixed stacking

Several graphene sheets can be stacked with arbitrary shifts (A,B,C) to obtain a mixed-stacking multi-layer sheet of graphene with reducible Fermi surface, as long as all the individual layers have dispersion function that is a polynomial in the same composite Floquet variable $\zeta(z,\lambda)$. As discussed above, this occurs when all vertically aligned edges are Dirichlet-isospectral. Fig. 2.4.10 depicts five layers stacked with mixed shifts. The dispersion function is a polynomial in $\zeta(z,\lambda)$ whose degree is the number of single sheets of graphene in the stack. The proof of this uses iterated application of the theorems on type-1 and type-2 multi-layer constructions from section 2.3.

Let Σ_1 and Σ_2 be n-layer and m-layer AA-stacked graphene quantum graphs (type 2). Let the individual layers of both have dispersion functions that are polynomials in a common $\zeta(z,\lambda)$. Let u_1 and u_n be corresponding (vertically aligned) vertices in the first and n-th layers of Σ_1, and let v_1 and v_m be corresponding vertices in the first and m-th layers of Σ_2.

Observe that $\Sigma_1^{u_n}$ has dispersion function that is also a function of the same $\zeta(z,\lambda)$. This is because a single graphene layer is separable at any vertex, and thus $\Sigma_1^{u_n}$ can be viewed as a type-2 $(n-1)$-layer graph with connectors that consist of the edges between the $n-1$ layers plus decorations. The same is true of $\Sigma_2^{v_0}$. Therefore Σ_1 and Σ_2 can be coupled by an edge between u_n and v_0 according to a two-layer type-1 construction, resulting in a quantum graph Γ with dispersion function that is a polynomial in $\zeta(z,\lambda)$.

This construction could just as well be carried out using $\Sigma_2^{v_m}$ in place of Σ_2, resulting in Γ^{v_m}, whose dispersion function is a polynomial in $\zeta(z,\lambda)$. Now yet another AA-stacked multi-layer graphene construction Σ_3 with the same $\zeta(z,\lambda)$ can be attached to Γ, and so on. These arguments need to modified somewhat if any of the AA-stacked sections Σ_i consists of only one layer, as in ABC-stacked graphene.

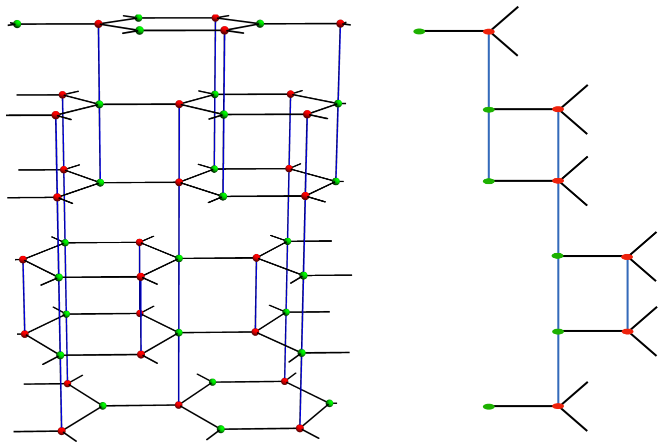

Figure 2.4.10: Five layers of graphene in mixed stacking with a fundamental domain.

2.5 Defects and Embedded eigenvalues

In this section, we show how the reducibility of the Fermi surface can be exploited to build a defective quantum graph with an embedded eigenvalue. We will show the explicit construction for AA stacked graphene, but it would not be difficult to adapt outlined strategy to build an eigenvalue embedding defect for other quantum graph with reducible Fermi surface. The local defect of the periodic quantum graph operator will admit a bound (L^2) state at an energy within the continuous spectrum, and this state will have unbounded support. Such an energy is an embedded eigenvalue of the operator. The condition of unbounded support distinguishes these eigenvalues from the energies of the flat bands, which are peculiar to graph operators, both discrete and quantum. See more discussion of this interesting issue in [58].

A result in [58] gives guidance on how to construct embedded eigenvalues when one does have reducibility. The key is in the inverse Floquet transform, where the dispersion function $D(z,\lambda) = D_1(z,\lambda)D_2(z,\lambda)$ appears in the denominator. This process of construction proceeds in stages and we will show explicitly how the embedding defect is constructed for AA-stacked Graphene.

1. The first step is to choose an energy λ such that exactly one of the factors, say $D_1(z,\lambda)$ but not $D_2(z,\lambda)$, vanishes at some z values on the torus $\mathbb{T}^2 \subset \mathbb{C}^2$.

2. Then we will study an auxiliary forcing problem on the combinatorial graph: $\mathfrak{A}(\lambda)\bar{u} = \phi$. The ϕ is of the form $D_1(z,\lambda)\vec{c}$ for some non-zero vector of constants $\vec{c}$. When we use the inverse Floquet transform to solve for u, the factor $D_1(z,\lambda)$ is canceled in the denominator, so u will be a function with the desired properties.

3. Finally we will to modify the potentials on a finite number of edges so that the same u which solves $\mathfrak{A}(\lambda)\bar{u} = \phi$ will also solve the eigenvalue problem with the modified operator, i.e. $A_V(\lambda)u = \lambda u$.

Consider a single layer of graphene, let the potentials on all of the edges be $q_e(x) = 0$, and let the Robin parameters be identically 0. The spectral matrix for the single layer is

$$\mathring{\hat{A}}(z,\lambda) = \frac{1}{s(\lambda)} \begin{bmatrix} -3c(\lambda) & w(z) \\ w(z^{-1}) & -3c(\lambda) \end{bmatrix}, \tag{2.5.1}$$

in which $s(\lambda) = \sin\sqrt{\lambda}$ and $c(\lambda) = \cos\sqrt{\lambda}$ and $w(z) = 1 + z_1 + z_2$. The spectral matrix (2.3.19) for AA-stacked graphene Γ_{AA} becomes

$$\hat{A}(z,\lambda) = \frac{1}{s(\lambda)} \begin{bmatrix} s(\lambda)\mathring{\hat{A}}(z,\lambda) - c(\lambda)I_2 & I_2 \\ I_2 & s(\lambda)\mathring{\hat{A}}(z,\lambda) - c(\lambda)I_2 \end{bmatrix}, \tag{2.5.2}$$

in which I_2 is the 2×2 identity matrix. The dispersion function $D(z,\lambda)$ is the determinant of this matrix, and since we will choose a λ that is not a root of $s(\lambda)$, we work with

$$\begin{aligned}\tilde{D}(z,\lambda) &= \det\big(s(\lambda)\hat{A}(z,\lambda)\big) = \tilde{D}_1(z,\lambda)\tilde{D}_2(z,\lambda)\\ &= \big(16c(\lambda)^2 - 8c(\lambda) + 1 - \zeta(z)\big)\big(16c(\lambda)^2 + 8c(\lambda) + 1 - \zeta(z)\big),\end{aligned} \tag{2.5.3}$$

in which $\zeta(z) = w(z)w(z^{-1})$. The dispersion relations for the two components of the Fermi surface, as displayed in (2.4.11), are

$$\zeta(z_1,z_2) \;=\; \mu_\pm(\lambda) \;:=\; 16c(\lambda)^2 \pm 8c(\lambda) + 1. \tag{2.5.4}$$

These two components contribute different sets of bands, say $\sigma_\pm(A)$, to the spectrum of the AA-stacked graphene model; σ_- corresponds to the factor $\tilde{D}_1(z,\lambda)$, and σ_+ to the factor $\tilde{D}_2(z,\lambda)$. As described in Section 2.4.3, since the range of $\zeta(e^{ik_1},e^{ik_2})$ is the interval $[0,9]$, the bands are the inverse images of this interval by the functions $\mu_\pm(\lambda)$, whose graphs are shown in Fig. 2.5.1. Precisely, $\sigma_\pm(A) = \{\lambda : 16c(\lambda)^2 \pm 8c(\lambda) + 1 \in [0,9]\}$.

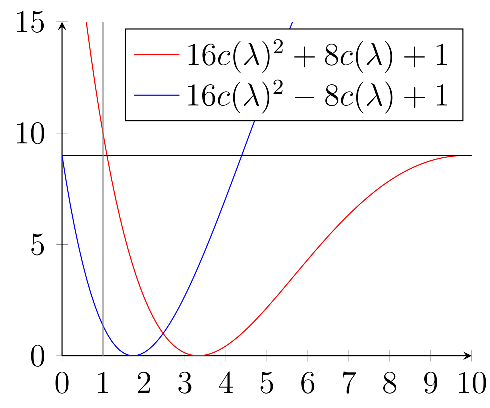

Figure 2.5.1: Two sets of spectral bands for AA stacked, bi-layer graphene. They come from the two components of the Fermi surface, and are the inverse images of the interval $[0,9]$ under the two graphed dispersion functions. The vertical line shows that the energy $\lambda=1$ lies within a band of the relation $\zeta(e^{ik_1},e^{ik_2}) = \mu_-(\lambda)$ but in a gap of the relation $\zeta(e^{ik_1},e^{ik_2}) = \mu_+(\lambda)$.

Notice that $\lambda=1$, for example, lies in $\sigma_-(A)\backslash\sigma_+(A)$. This shows that step of our procedure can be satisfied with $\lambda = 1$, although it is not hard to find other values that would work. Next we need to construct the combinatorial forcing that cancels the desired factor in the inverse Floquet transform. Consider the discrete-graph reduction (sec. 2.2.2) of the full quantum-graph operator on Γ_{AA},

$$\mathfrak{A}(\lambda)u \;=\; \phi, \tag{2.5.5}$$

where ϕ is a forcing function defined on the vertices of Γ_{AA} that is nonzero at only finitely many vertices. Its Floquet transform $\hat{\phi}(z)$ is therefore a Laurent polynomial in $z=(z_1, z_2)$, and its coefficients can be considered to be vectors in $\mathbb{C}^4$ since a fundamental domain W of Γ_{AA} contains four vertices. The solution u is the response (also defined on the vertices of Γ_{AA}) to the forcing function ϕ at energy λ. We obtain $\hat{u}(z)$ algebraically,

$$\hat{u}(z) \;=\; \hat{A}(z,\lambda)^{-1}\hat{\phi}(z) \;=\; \frac{\mathrm{Adj}(\hat{A}(z,\lambda))\hat{\phi}(z)}{D(z,\lambda)}, \tag{2.5.6}$$

here, $\mathrm{Adj}(\hat{A}(z,\lambda))$ is the adjunct of $\hat{A}(z,\lambda)$. Now we fixing $\lambda=1$, and we choose the forcing function in the Fourier domain as:

$$\hat{\phi}(z) \;=\; \tilde{D}_1(z,1)\vec{c} \;:=\; \big(16c(1)^2 - 8c(1) + 1 - \zeta(z_1,z_2)\big)[1,0,0,0]^t\,. \tag{2.5.7}$$

Any choice of $\vec{c} \neq 0$ would suffice but we choose $[1,0,0,0]^t$ because it will simplify many calculations. Since $\hat{\phi}(z)$ is a Laurent polynomial in $z = (z_1, z_2)$, the forcing function ϕ on Γ_{AA} has compact support. We can see exactly what kind of vertex forcing it is by multiplying out the function $\zeta(z_1, z_2)$, we have that

$$\begin{aligned}\hat{\phi}(z) &= (16c(1)^2 - 8c(1) + 1 + (1 + z_1 + z_2)(1 + z_1^{-1} + z_2^{-1}))[1,0,0,0]^t \\ &= (16c(1)^2 - 8c(1) + 4 + z_1 + z_2 + z_1^{-1} + z_2^{-1} + +z_1 z_2^{-1} + z_1^{-1} z_2)[1,0,0,0]^t\end{aligned} \tag{2.5.8}$$

We see that ϕ is a vertex forcing at seven vertices, the same vertex in each of seven adjacent fundamental domains. Say v_1 is the first vertex in the ordering that is determined when we compute the spectral matrix – because of the many symmetries of this version of AA stacked graphene (all potentials and vertex conditions are 0), it does not matter which vertex is chosen to be v_1. We can read from the Floquet transform the exact values of ϕ:

$$\begin{aligned}
\phi(v_1) &= 16c(1)^2 - 8c(1) + 4 \\
\phi((0,1)+v_1) &= \phi((1,0)+v_1) = \phi((0,-1)+v_1) = \phi((-1,0)+v_1) = 1 \\
\phi((-1,1)+v_1) &= \phi((1,-1)+v_1) = 1
\end{aligned}$$

For simplicity say that $\phi(v)$ is supported only on the red vertices in the top layer, and only in seven adjacent fundamental domains. Now we continue from (2.5.6) and we obtain

$$\hat{u}(z) \;=\; \frac{\mathrm{Adj}(\hat{A}(z,1))\,\vec{c}}{16c(1)^2 + 8c(1) + 1 - \zeta(z)}. \tag{2.5.9}$$

The response u has a value at each of the four vertices in each shift of W by any element of $g \in \mathbb{Z}^d$, and so u can be considered to be a function of $g \in \mathbb{Z}^d$ with values in $\mathbb{C}^4$. It is obtained by the inverse Floquet transform,

$$u(g) \;=\; \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} z^g\, \hat{u}(z)\, dz, \tag{2.5.10}$$

in which $\mathbb{T}^2 = \{z \in \mathbb{C}^2 : |z_1| = |z_2| = 1\}$ is the two-dimensional torus in $\mathbb{C}^2$.

Since $\lambda = 1$ is not in $\sigma_2(A)$, the denominator does not vanish on the two-dimensional torus $\mathbb{T}^2 = \{z \in \mathbb{C}^2 : |z_1| = |z_2| = 1\}$ in $\mathbb{C}^2$. Therefore $\hat{u}(z)$ is analytic on $\mathbb{T}^2$, and it follows that $u \in L^2(\mathcal{V}(\Gamma_{AA}))$ (u is in fact exponentially decaying). As long as the denominator $\tilde{D}_2(z,1)$ is not cancelled identically by any of the four components of the vector in the numerator, $\hat{u}(z)$ will not be a Laurent polynomial in z. This means that u has unbounded support, that is, it is nonzero on infinitely many vertices. We compute the values at the four vertices of each shift of W explicitly (since $\vec{c}=[1,0,0,0]^t$):

$$u_1(g) = \frac{s(1)}{(2\pi)^2}\int_{\mathbb{T}^2} z^g \frac{-64c(1)^3+4c(1)\zeta(z)}{16c(1)^2+8c(1)+1-\zeta(z)}dz \tag{2.5.11}$$
$$u_2(g) = \frac{s(1)}{(2\pi)^2}\int_{\mathbb{T}^2} z^g \frac{(1+z_1+z_2)(16c(1)^2-1-\zeta(z))}{16c(1)^2+8c(1)+1-\zeta(z)}dz \tag{2.5.12}$$
$$u_3(g) = \frac{s(1)}{(2\pi)^2}\int_{\mathbb{T}^2} z^g \frac{1-\zeta(z)-16c(1)^2}{16c(1)^2+8c(1)+1-\zeta(z)}dz \tag{2.5.13}$$
$$u_4(g) = \frac{s(1)}{(2\pi)^2}\int_{\mathbb{T}^2} z^g \frac{8c(1)(1+z_1+z_2)}{16c(1)^2+8c(1)+1-\zeta(z)}dz, \tag{2.5.14}$$

and confirm that the denominator is not simultaneously a factor of all four of the numerators. This so far suffices for conclude step 2 of the procedure that was outlined in the beginning of the section. For the final step, we need to demonstrate how we can use the combinatorial forcing to build the desired embedding potential. The following lemma is essential for this step.

Lemma 2.5.1. *Given real numbers α, β, γ, and λ with $\alpha\beta\neq 0$, there exists a real-valued potential $q(x)$ on $[0,1]$ for which $c(\lambda)=\alpha$, $s(\lambda)=\beta$, and $s'(\lambda)=\gamma$.*

Proof. Let $c(x;\lambda)$ be the solution to $-u''+q(x)u=\lambda u$ such that $c(0;\lambda)=1$ and $c'(0;\lambda)=0$; and let $s(x;\lambda)$ be the solution such that $s(0;\lambda)=0$ and $s'(0;\lambda)=1$. Then, by definition, $c(\lambda)=c(1;\lambda)$, $s(\lambda)=s(1;\lambda)$, and $s'(\lambda)=s'(1;\lambda)$, with the prime referring to d/dx. The ODE implies

$$q(x)-\lambda = \frac{c''(x;\lambda)}{c(x;\lambda)}. \tag{2.5.15}$$

We prescribe additionally that $c(x;\lambda)$ is an analytic function of x such that

1. $c(1;\lambda)=\alpha$ and $c'(1;\lambda)=\frac{\alpha\gamma-1}{\beta}$,

2. $c(x;\lambda)$ has finitely many roots in $(0,1)$, and at each root the first derivative is nonzero and the second derivative is zero,

3. $\int_0^1 c(x;\lambda)^{-2}dx = \frac{\beta}{\alpha}$.

Let us show that the potential $q(x)$ obtained from this function $c(x;\lambda)$ is well defined and has the three desired properties.

Because of the second property, $c''(x;\lambda)/c(x;\lambda)$ is analytic at each zero of $c(x;\lambda)$. By construction $c(1;\lambda) = \alpha$. From the Wronskian identity $c(x;\lambda)s'(x;\lambda) - s(x;\lambda)c'(x;\lambda) = 1$, one obtains

$$s(x;\lambda) = c(x;\lambda)\int_0^x \frac{1}{c(x;\lambda)^2}dx, \tag{2.5.16}$$

which, together with properties (1) and (3), yields $s(1;\lambda) = \beta$. The Wronskian identity also gives

$$\alpha s' - \left(\frac{\alpha\gamma - 1}{\beta}\right)\beta = 1, \tag{2.5.17}$$

from which we obtain $s'(1;\lambda) = \gamma$. □

We now show how the solution u to $\mathfrak{A}(1)\bar{u} = \phi$ can be used to create an L^2 eigenstate of a local perturbation of the quantum graph A. Our embedding potential will be supported on seven edges, these edges are all vertical edges which connect two red vertices, and they are in domains $(0,0)$, $(1,0)$, $(0,1)$, $(0,-1)$, $(-1,0)$, $(1,-1)$ and $(-1,1)$. Call these edges $e_1, \ldots e_7$ and say that their terminal vertices are $(v_1, w_1) \ldots (v_7, w_7)$ we know that $\phi(v_1), \ldots \phi(v_7) \neq 0$ and that $\phi(w_1), \ldots \phi(w_7) = 0$. We will construct a perturbation of A by adding potentials $V_i(x)$ the each of the seven edges, e_i. This is equivalent to choosing V so that

$$\mathfrak{A}(1)\bar{u} = \phi \quad \Longleftrightarrow \quad \mathfrak{A}_V(1)\bar{u} = 0. \tag{2.5.18}$$

To do this, we must look at how the matrices $\mathfrak{A}(1)$ and $\mathfrak{A}_V(1)$ are constructed. They are indexed by $\mathcal{V}(\Gamma_{AA})$. The 2×2 submatrix of $\mathfrak{A}(1)$ corresponding to the vertices $\{v, w\}$ of one of the edges $e_1, \dots e_7$ has a contribution from the DtN map (2.2.21) with $g = 0$ and $s_{e_i}(1) = s(1)$ (similarly for c and s'); and the corresponding 2×2 submatrix of $\mathfrak{A}_V(1)$ has a similar contribution, but with $s_{e_i}(1) = s_{V_i}(1)$ (similarly for c and s') coming from the spectral functions for the operator $-d^2/dx^2 + V(x)$. The edges e_i have no vertices in common, so $\mathfrak{A}(1)$ and $\mathfrak{A}_V(1)$ are identical aside from these 7 sub matrices.

Given that $\mathfrak{A}(\lambda)u = \phi$, the task is to find $V(x)$ such that $\mathfrak{A}_V(\lambda)u = \mathfrak{A}(\lambda)u - \phi$. ϕ vanishes except at the vertices v_i, and this equation holds automatically elsewhere. Thus for each i we must solve:

$$\frac{1}{s_{V_i}(1)}\begin{bmatrix} -c_{V_i}(1) & 1 \\ 1 & -s'_{V_i}(1) \end{bmatrix}\begin{bmatrix} u(v_i) \\ u(w_i) \end{bmatrix} = \begin{bmatrix} u'(v_i) - \phi(v_i) \\ u'(w_i) - \phi(w_i) \end{bmatrix} = \begin{bmatrix} u'(v_i) - \phi(v_i) \\ u'(w_i) \end{bmatrix}. \tag{2.5.19}$$

We know that $u'(v_i), u'(w_i)$ can be written in terms of $u(v_i), u(w_i)$ by using the DtN map for an unperturbed edge. We will also invoke Lemma 2.5.1 and choose each V_i such that $s_{V_i}(1) = s(1)$ for each edge i. From this we have that

$$\frac{1}{s(1)}\begin{bmatrix} -c_{V_i}(1) + c(1) & 0 \\ 0 & -s'_{V_i}(1) + s'(1) \end{bmatrix}\begin{bmatrix} u(v_i) \\ u(w_i) \end{bmatrix} = \begin{bmatrix} \phi(v_i) \\ 0 \end{bmatrix} \tag{2.5.20}$$

This system of equations is satisfied by choosing $s'_{V_i}(1) = s'(1)$ and $c_{V_i} = c(1) - \frac{s(1)\phi(v_i)}{u(v_i)}$. If it happens that $u(v_i) = 0$ when v_i is any of the vertices in question then alternate choices of s_V, c_V and s'_V are possible since (2.5.19) is an equation in two variables and three unknowns. Alternatively (2.5.11) can be checked with a computer for all seven choices for g. We know that $u(v_i)$ and $u(w_i)$ cannot both be zero since we chose λ outside of the Dirichlet spectrum.

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