Understanding critical layers in stratified shear flow instabilities: A wave interaction perspective

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Abstract

In this paper we examine the dynamics of unstable critical layers in stratified shear flows. This is done from the framework of the wave interaction theory of shear instabilities, which views instability as arising from the mutual interaction between wave motions that are present in the background shear and density fields. By formulating a simple analytical model of the structure of vorticity within the (continuous) critical layer, we are able to reduce this to an interfacial vortex sheet representation in the far-field that fits naturally into the wave interaction theory. This is applied to describe the physical mechanism of ocean wave generation by wind in an idealised way, though the formulation is applicable to other shear flows in general.

1 Introduction

Instabilities in stratified shear flows are often understood by using "wave interaction theory" (see, for example Holmboe, 1962; Baines and Mitsudera, 1994; Caulfield, 1994; Carpenter et al., 2013). In this theory, shear instabilities result from the interaction of two, otherwise freely propagating waves in the profiles of background velocity and density. In the case of stratified shear flows these waves correspond to internal gravity waves, which propagate on density interfaces where the background density changes abruptly, as well as vorticity waves that propagate on vorticity interfaces where the background vorticity changes abruptly. The formulation is more general, however, and may be extended to include other types of wave motion such as capillary waves (Biancofiore et al., 2015) and magnetohydrodynamical Alfven waves (Heifetz et al., 2015), for example. In wave interaction theory, instability occurs when the waves are able to satisfy two essential conditions: (i) they must have the correct geometry to allow for mutual growth in one another (which is satisfied when their intrinsic propagation is of opposite sign), and (ii) a "phase-locking" must occur such that the waves propagate at the same speed relative to one another. The theory is most successfully applied to piecewise velocity, and step-wise density profiles, so that only a finite number of discrete interacting interfaces must be considered. This has led to physical interpretations of the well known Rayleigh and Fjørtoft Theorems, as well as an explanation of the sometimes destabilising effect of stable stratification. Despite this success, the wave interaction theory cannot be applied in a straight forward way to unstable critical layers that are inherently continuous in nature, since it is formulated in terms of discrete interfaces. In this paper we outline a method of including the critical layer in the framework of the wave interaction theory of shear instability.

The critical layer is a "near-singlularity" that can develop at the location where the background flow speed is equal to the instability phase speed. We use an approximation at the critical layer based on a slowly varying vertical velocity, and a rapidly varying vorticity to formulate a new vortex sheet representation of the critical layer. This analysis of the critical layer provides both a physical understanding of the critical layer structure, as well as an interfacial description. This can then be used within wave interaction theory to understand the role of the critical layer in stratified shear flow instabilities. Specific cases that we examine include the instability responsible for the generation of ocean surface gravity waves when wind blows over water.

The linear stability of stratified shear flows is governed by the Taylor-Goldstein equation

$$w'' - \Big[\frac{U''}{U-c} - \frac{N^2}{(U-c)^2} + k^2\Big]w = 0, \tag{1}$$

where the background horizontal velocity is denoted by $U(z)$, $N^2(z)$ is the squared buoyancy frequency, and the vertical velocity is taken to have the normal mode form $w(z)e^{ik(x-ct)}$, with k the horizontal wavenumber and $c = c_r + ic_i$ the complex wave speed. Unstable solutions have the exponential growth rate of $\sigma \equiv kc_i$.

We can see immediately from the Taylor-Goldstein equation that there is a singularity when $U = c$. However, when we confine our attention to instances where the solutions are unstable, then the singularity is eliminated since U is always real. Nonetheless, the critical layer height, z_c, where $U(z_c) = c_r$, is an important location for the dynamics of unstable flows for small values of c_i. In these cases, a large response is seen in the region surrounding z_c, which is referred to as the critical layer.

In the wave interaction theory it is more useful to express the governing (TG) equation in terms of three fundamental fields: the perturbation vorticity, q, the vertical displacement of material lines (horizontal in an undisturbed background state), η, and w. This can be done through the linearised kinematic condition,

$$\frac{D\tilde{\eta}}{Dt} = \tilde{w} \quad \Rightarrow \quad c\eta = U\eta + \frac{i}{k}w, \tag{2}$$

the vorticity equation,

$$\frac{D\tilde{q}}{Dt} = -U''\tilde{w} + N^2\frac{\partial\tilde{\eta}}{\partial x} \quad \Rightarrow \quad cq = Uq + N^2\eta + \frac{i}{k}U''w, \tag{3}$$

where the tilde quantities refer to the variables before normal modes are assumed (i.e., $\tilde{w}(x,z,t) = \mathrm{Re}\{w(z)e^{ik(x-ct)}\}$), and the primes indicate differentiation with respect to the vertical coordinate z. Finally, the vertical velocity and the vorticity are related by the definition $q \equiv (i/k)(w'' - k^2w)$, which allows us to invert q to find w using the following inversion formula,

$$w(z) = \int_D G(z,s)q(s)ds. \tag{4}$$

Here D is the domain, which we will take to be unbounded, and $G(z, s) = (i/2)e^{-k|z-s|}$ is the Green's function for D. It is important to note that equations (2 – 4) describe all the physics of the Taylor-Goldstein equation, and can be combined mathematically to produce it. We will therefore, frame our investigation of the unstable critical layer in terms of these three fields, w, q, and η.

2 A view inside the critical layer

In order to arrive at an interfacial description of the critical layer, we must first derive the structure of its η, q, and w fields. This can be done with a simple analytical argument, and approximation. What is essential is to realise that η and q are both rapidly varying across the critical layer, while w is slowly varying. This can be seen directly from solutions to the Taylor-Goldstein equation, or from the following argument. Suppose that we have an infinitely rapid variation of vorticity located at the origin, i.e., $q = \delta(z)$. From (4), we can easily calculate the response of the vertical velocity field to this q distribution as $w(z) \propto e^{-k|z|}$. This shows that w varies over a length scale of order k^{-1}, while q and η have no such restriction. It should be noted however, that in reality there will be either molecular or turbulent diffusion acting to smooth any sufficiently rapid variation of q.

Given this knowledge we can now look in detail at the structure of the critical layer. This is done by expanding about z_c in a Taylor series to get $U - c \approx -ic_i + U_c'(z - z_c)$, and $w \approx w_c$, where the subscript denotes quantities evaluated at z_c, and substituting in to the kinematic condition (2), giving

$$\eta(z) \approx \frac{-iw_c}{kU_c'(z - z_c) - i\sigma}. \tag{5}$$

This displacement structure within the critical layer is plotted in figure 1. In the figure, we have non-dimensionalised by the thickness length scale of the critical layer δ_c (to be defined shortly), and the maximum value of η of w_c/σ. It can be seen that the maximum amplitude of η is at z_c, with a decay away from this level that occurs over the length scale δ_c. In addition, there is a phase change of π that occurs across the critical layer. The physical reason for this phase change is due to the relative importance of two different terms in setting the value of η. Far outside the critical layer we have advection by the mean flow dominating the denominator of (5), whereas close to z_c it is dominated by unsteady growth. The ratio of these two terms (advection and growth) defines the critical layer width, $\delta_c \equiv \sigma/k|U_c'|$, over which the phase change occurs. The reason for the phase change is that the advection changes sign across z_c, with the vertical velocity and the displacement in phase directly at z_c (since there is no advection there).

Once the displacement field is obtained, it is a simple matter to find the vorticity. For the sake of simplicity, here we will assume that there is no density stratification within the critical layer, so that we can use the simple relation of $q = -U''\eta$. This shows that the vorticity field is simply a scaled displacement field as shown in figure 1, where we assume that U'' is slowly varying within the critical layer.

3 The critical layer as interface

The wave interaction theory of shear instability models the interaction of discrete interfacial waves to explain and understand the mathematical results of linear stability analysis.

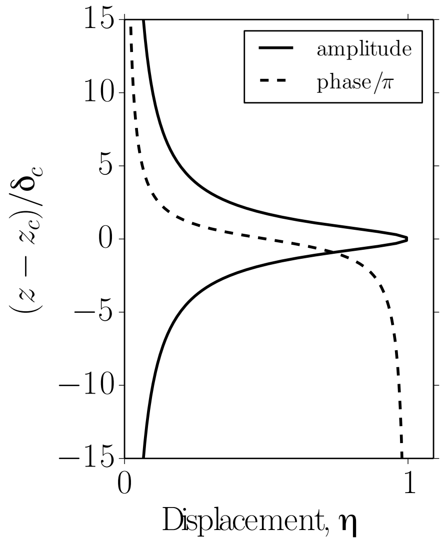

Figure 1: Displacement structure of the unstable critical layer. The vertical axis has been non-dimensionalised by the thickness of the critical layer, δ_c. The vorticity perturbation is also represented by the same structure in the case of a homogeneous fluid.

The waves (usually of the vorticity/counter-propagating Rossby or internal gravity type) are represented by different types of vortex sheets, with the strength of the sheet usually related to the displacement in some way. They interact through the influence of their vertical velocity on the displacement of the interface which can produce growth or decay of the perturbations, or changes in the phase speed (see Carpenter et al., 2013, for a review). We now seek to develop the same interfacial representation of the critical layer. Given the distribution of q found in the last section, we can use the inversion formula in (4) to find the w field that is induced by the critical layer. However, since we are interested in a local description of the critical layer, we do not integrate over the entire domain, but only over a region around the critical layer that is small compared to k^{-1}, but still large compared to δ_c. This reduces the integral to an equivalent vortex sheet representation with a vortex sheet strength of

$$Q_c = \int_{CL} q(z)dz = -2\pi\beta_c w_c \quad \text{and} \quad w(z) = Q_c e^{-k|z-z_c|}, \tag{6}$$

where we have used $\beta_c \equiv U_c''/2kU_c'$. This allows us to model the critical layer as another type of (vortex sheet) interface, and incorporate it in the wave interaction theory. An important difference is that the sheet strength is proportional to the vertical velocity incident on it, rather than on the displacement of the interface.

4 A simple wave interaction model for the critical layer

The results of the last section allow us to construct a simple model of the critical layer dynamics that leads to an understanding of the growth of instabilities in which the critical layer plays a role. As a particular example, we look at the instability of wind blowing over water, which is thought to be responsible for the generation of oceanic wind-waves. This wind-wave instability is the simplest geometry that we are aware of to apply the critical layer model. The reason for this is because of the very non-Boussinesq nature of the system, with the density ratio between air and water of $r \equiv \rho_a/\rho_w = O(10^{-3})$. This has the effect of producing an instability with small growth rate ($\sigma = O(r)$) and a critical layer height that can be well approximated by $U(z_c) \approx c_0$, where $c_0 = (g/k)^{1/2}$ is the deep water gravity wave speed.

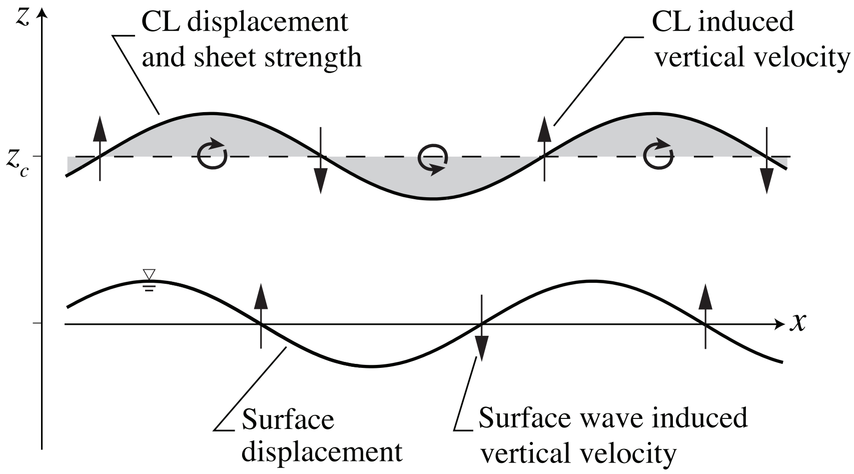

Figure 2: Displacement structure of the unstable critical layer. The vertical axis has been non-dimensionalised by the thickness of the critical layer, δ_c. The vorticity perturbation is also represented by the same structure in the case of a homogeneous fluid.

Choosing the simplest configuration possible, we neglect all vorticity contributions other than the density interface (located at the water surface, $z = 0$) and at the critical layer. The interaction of the two interfaces can then be quantified. We write the vertical velocity as the sum of the two interfacial components

$$w(z) = i/2(Q_s e^{-k|z|} + Q_c e^{-k|z-z_c|}), \tag{7}$$

where Q_s is the vortex sheet strength of the surface wave. Once we substitute for Q_c from above, and solve for w_c in terms of the arbitrary Q_s, we can see that the structure of the wave fields is as shown in the typical case of figure 2. This figure demonstrates the essential instability mechanism of the wind-wave instability; the critical layer takes the incoming vertical velocity of the surface wave and produces a concentrated (in a layer of order δ_c in thickness) vorticity response that gives a vertical velocity component downwards in the troughs of the surface wave and upwards at the crests. This is the essence of the wave interaction interpretation of shear instability. It is also interesting to note that the critical layer is responsible for slowing down the surface wave propagation speed since the critical layer vertical velocity has a component that is downwards in the downwards sloping nodes of the surface displacement. This effect is however, of order r just as with the growth rate.

5 Summary

We have demonstrated a method of incorporating critical layers into the framework of the wave interaction theory of stratified shear instabilities. This has allowed us to develop a wave interaction view of the wind-wave instability that has been lacking up to this point. This interpretation of wind-wave growth follows the classic lines of Lighthill (1962), but in a rather different fashion; we frame our physical interpretation solely in terms of the three fundamental fields of vertical velocity, displacement, and vorticity, rather than appealing to energy, momentum and vortex force. In addition, we view the critical layer as a continuous entity with an internal structure that is well approximated by a simple advection and growth balance of displacement. This physical explanation of the wind-wave instability also brings together the physical description of the instability mechanisms from stratified shear flows and wind-generated gravity waves into a single wave interaction framework. Future work is currently focussed on extending these results to density stratified critical layers.

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