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Authors
Levan, Nhan
Kubrusly, Carlos S

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TIME-SHIFTS GENERALIZED MULTIRESOLUTION ANALYSIS OVER DYADIC-SCALING REDUCING SUBSPACES

NHAN LEVAN AND CARLOS S. KUBRUSLY

Abstract. A Generalized Multiresolution Analysis (GMRA) associated with a wavelet is a sequence of nested subspaces of the function space $\mathcal{L}^2(\mathbb{R})$, with specific properties, and arranged in such a way that each of the subspaces corresponds to a scale 2^m over *all* time-shifts n. These subspaces can be expressed in terms of a generating-wandering subspace — of the dyadic-scaling operator — spanned by orthonormal wavelet-functions — generated from the wavelet. In this article we show that a GMRA can also be expressed in terms of subspaces for each time-shift n over all scales 2^m. This is achieved by means of "elementary" reducing subspaces of the dyadic-scaling operator. Consequently, Time-Shifts GMRA associated with wavelets, as well as "sub-GMRA" associated with "sub-wavelets" will then be introduced.

1. Introduction

An orthonormal wavelet $\psi(\cdot)$, or simply a wavelet, is a unit vector of the function space $\mathcal{L}^2(\mathbb{R})$ which is such that the *wavelet-functions* $\{\psi_{m,n}(\cdot)\}_{(m,n)\in\mathbb{Z}^2}$, generated from $\psi(\cdot)$ by repeated "unit-time-shift" followed by repeated "dyadic-scaling", that is,

$$\psi_{m,n}(\cdot) := \sqrt{2}^{\,m}\psi\big(2^m(\cdot) - n\big) = D^m T^n \psi(\cdot), \quad (m,n) \in \mathbb{Z}^2, \tag{1.1}$$

are pairwise orthogonal

$$\int_{-\infty}^{\infty} \psi_{m,n}(t)\, \overline{\psi}_{m',n'}(t)\, dt = \delta_{m,m'}\, \delta_{n,n'}, \quad (m,m'),(n,n') \in \mathbb{Z}^2, \tag{1.2}$$

and form a $\mathcal{L}^2(\mathbb{R})$-basis [14], see also [3, 12] and the references therein. Here T is the unitary operator *unit-time-shift*:

$$Tf = g, \quad g(t) = f(t-1), \quad t \in \mathbb{R}, \tag{1.3}$$

while D is the unitary operator *dyadic-scaling*:

$$Df = g, \quad g(t) = \sqrt{2}\, f(2t), \quad t \in \mathbb{R}. \tag{1.4}$$

Consequently, $\psi_{m,n}(\cdot)$ can be regarded as $\psi(\cdot)$ for time-shift n and for scale 2^m. We note that there exists no function $\varphi(\cdot) \in \mathcal{L}^2(\mathbb{R})$ such that the functions $\varphi_{n,m}(\cdot) := T^n D^m \varphi(\cdot)$ span the space $\mathcal{L}^2(\mathbb{R})$, [4].

Mathematically speaking, the unitary operators T and D are basically the same in the sense that they are both Hilbert space bilateral shift of countably infinite multiplicity [6], hence they are unitarily equivalent [15]. However, they do not

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commute. This is due to the fact that, from system theoretic viewpoint, T is time-invariant while D is time-varying. Indeed, $DT \neq TD$ but $D^m T^{2^m} = TD^m$ for each $m \in \mathbb{Z}$ so that each T^{2^m} is unitarily equivalent to T via the unitary D^m.

A wavelet can be associated with a Generalized Multiresolution Analysis (GMRA) [2] which is defined as follows.

Definition 1. A sequence of $\mathcal{L}^2(\mathbb{R})$-subspaces $\{\mathcal{V}_m\}_{m\in\mathbb{Z}}$ is a GMRA satisfying the following conditions:

(o) $\mathcal{V}_0$ (called *core subspace* for $\psi(\cdot)$) is T-invariant,
(i) $\mathcal{V}_m \subset \mathcal{V}_{m+1}, \quad m \in \mathbb{Z}$,
(ii) $\bigcap_{m=-\infty}^{\infty} \mathcal{V}_m = \{0\}$,
(iii) $\overline{\bigcup_{m=-\infty}^{\infty}} \mathcal{V}_m = \mathcal{L}^2(\mathbb{R})$,
(iv) $D\mathcal{V}_m = \mathcal{V}_{m+1}, \quad m \in \mathbb{Z}$.

We note that if condition(o) is replaced by

(o') $\mathcal{V}_0$ is spanned by an orthonormal set $\{\phi\big((\cdot) - n\big)\}_{n\in\mathbb{Z}}$,

then the sequence $\{\mathcal{V}_m\}_{m\in\mathbb{Z}}$ is called a Multiresolution Analysis (MRA) [10, 11, 13], while $\phi(\cdot)$ is called *scaling function.* Moreover, the existence of $\phi(\cdot)$ is assumed in condition (o'). The "m-nested" (because of property (i)) subspaces $\mathcal{V}_m$ are *approximation subspaces, for scale* 2^m *and over all time-shifts* n. Also, for a MRA, by properties (o') and (iv), the *scaling orthonormal functions* $\big\{D^m\phi\big((\cdot)-n\big)\big\}_{n\in\mathbb{Z}}$ span the subspaces $\mathcal{V}_m$, for each $m \in \mathbb{Z}$. The GMRA-subspaces $\mathcal{V}_m$ are called "detail-approximation" subspaces, while those of a MRA are "scaling-approximation" subspaces. The core subspace $\mathcal{V}_0$ is actually *incoming subspace* in the Lax-Phillips Scattering Theory [9]. Conditions (o) and (o') apply only to wavelets and are not required for an incoming subspace.

Reducing subspaces play an important role in operator theory. However, for wavelet and signal processing, reducing subspaces have not been that useful! This, perhaps, is due to the fact that an operator is "trapped" by its own reducing subspaces, that is, it can never leave the subspaces. Consequently, signals living in reducing subspaces of an operator cannot be processed by the operator. However, as we shall see in Section 2, there exists a sequence of dyadic-scaling reducing subspaces spanning $\mathcal{L}^2(\mathbb{R})$ and plays a key role in what follows.

We begin in Section 2 by showing relationships between a sequence of orthogonal subspaces — spanning a Hilbert space $\mathcal{H}$, and a $\mathcal{H}$-double-subscripted orthonormal basis. These, in the case of wavelet-functions $\{\psi_{m,n}(\cdot)\}_{(m,n)\in\mathbb{Z}^2}$, result in representations of the space $\mathcal{L}^2(\mathbb{R})$ in terms of orthogonal scale-subspaces — over all time-shifts, as well as orthogonal time-shift-subspaces — over all scales. The former turn out to be wandering subspaces for the dyadic-scaling operator, while the latter are its "elementary" reducing subspaces. These then allow us to introduce the concept of Time-Shifts GMRA on all of $\mathcal{L}^2(\mathbb{R})$, and Sub-GMRA on the elementary reducing subspaces.

2. Main Results

We begin by showing relationships between a Hilbert space Orthogonal Subspaces-Basis and a Double-Subscripted Orthonormal Basis of the Hilbert space.

Let $\mathcal{H}$ be a separable Hilbert space with inner product $\langle\cdot\,,\cdot\rangle$ and norm $\|\cdot\|$. Moreover, $\mathcal{H}$ is spanned by an *Orthogonal Subspaces-Basis* (OSB) $\{\mathcal{B}_k\}_{k\in\mathbb{Z}}$, i.e., a sequence of closed, pairwise orthogonal, subspaces: $\mathcal{B}_k \perp \mathcal{B}'_k$ whenever $k \neq k'$. Therefore

$$\mathcal{H} = \overline{\text{span}}\,\{\mathcal{B}_k,\ k \in \mathbb{Z}\} := \bigvee_{k\in\mathbb{Z}} \mathcal{B}_k = \bigoplus_{k\in\mathbb{Z}} \mathcal{B}_k. \tag{2.1}$$

Note: As usual, we are identifying the closure of the span of closed orthogonal subspaces of a Hilbert space (i.e., the topological sum of closed orthogonal subspaces) with their orthogonal direct sums — these are in fact unitarily equivalent. As a consequence, any $h \in \mathcal{H}$ admits the orthogonal expansion

$$h = \sum_{k\in\mathbb{Z}} b_k, \tag{2.2}$$

where

$$b_k = P_{\mathcal{B}_k} h \quad \text{and} \quad \sum_{k\in\mathbb{Z}} \|b_k\|^2 = \|h\|^2, \tag{2.3}$$

here $P_{\mathcal{B}_k}$ denotes the orthogonal projection onto $\mathcal{B}_k$.

Now suppose that each of the subspaces $\mathcal{B}_k$ is, in turn, spanned by an Orthonormal Basis (ONB) $\{\psi_{k,l}\}_{l\in\mathbb{Z}}$ (i.e., for each $k \in \mathbb{Z}$, $\langle \psi_{k,l}\,, \psi_{k,l'}\rangle = \delta_{l,l'}$ for every $(l, l') \in \mathbb{Z}^2$) so that

$$\mathcal{B}_k = \bigvee_{l\in\mathbb{Z}} \psi_{k,l}, \quad k \in \mathbb{Z}. \tag{2.4}$$

Consequently, (2.2) can be further written as

$$h = \sum_{k\in\mathbb{Z}} \sum_{l\in\mathbb{Z}} \langle h\,, \psi_{k,l}\rangle\, \psi_{k,l}, \quad h \in \mathcal{H}, \tag{2.5}$$

since each b_k can be expanded in terms of the ONB of $\mathcal{B}_k$ as

$$b_k = \sum_{l\in\mathbb{Z}} \langle h\,, \psi_{k,l}\rangle\, \psi_{k,l} = P_{\mathcal{B}_k} h. \tag{2.6}$$

Equation (2.5) implies that the double-subscripted sequence $\{\psi_{k,l}\}_{(k,l)\in\mathbb{Z}^2}$ is orthonormal:

$$\langle \psi_{k,l}\,, \psi_{k',l'}\rangle = \delta_{k,k'} \cdot \delta_{l,l'}, \quad (k, k'), (l, l') \in \mathbb{Z}^2, \tag{2.7}$$

and constitutes an ONB for $\mathcal{H}$. Thus, given an OSB $\{\mathcal{B}_k\}_{k\in\mathbb{Z}}$ we have the "associated" ONB $\{\psi_{k,l}\}_{(k,l)\in\mathbb{Z}^2}$. From which we can also define the closed subspaces

$$\mathcal{H}_l := \bigvee_{k\in\mathbb{Z}} \psi_{k,l}, \quad l \in \mathbb{Z}. \tag{2.8}$$

It is evident from (2.7) that $\mathcal{H}_l \perp \mathcal{H}_{l'}$ whenever $l \neq l'$. Moreover, since $\{\psi_{k,l}\}_{(k,l)\in\mathbb{Z}^2}$ is an ONB for $\mathcal{H}$, the subspaces $\{\mathcal{H}_l\}_{l\in\mathbb{Z}}$ also form an OSB for $\mathcal{H}$. Therefore, in addition to (2.1), we now have a second orthogonal decomposition for $\mathcal{H}$

$$\mathcal{H} = \bigvee_{l\in\mathbb{Z}} \mathcal{H}_l = \bigoplus_{l\in\mathbb{Z}} \mathcal{H}_l. \tag{2.9}$$

The converse is also true, i.e., if $\{\psi_{k,l}\}_{(k,l)\in\mathbb{Z}^2}$ is a $\mathcal{H}$-double-subscripted ONB, then the sequences $\{\mathcal{B}_k\}_{k\in\mathbb{Z}}$ and $\{\mathcal{H}_l\}_{l\in\mathbb{Z}}$, defined by (2.4) and (2.8), respectively, are associated OSB for $\mathcal{H}$.

We summarize the above in the following Lemma.

Lemma 1. *Let* $\{\mathcal{B}_k\}_{k\in\mathbb{Z}}$ *be an orthogonal subspaces-basis for* $\mathcal{H}$. *Suppose that each* $\mathcal{B}_k$ *is spanned by an orthonormal basis* $\{\psi_{k,l}\}_{l\in\mathbb{Z}}$,

$$\mathcal{B}_k := \bigvee_{l\in\mathbb{Z}} \psi_{k,l}, \quad k \in \mathbb{Z}.$$

Then the double-subscripted orthonormal sequence $\{\psi_{k,l}\}_{(k,l)\in\mathbb{Z}^2}$ *is an ONB for* $\mathcal{H}$. *Consequently, the subspaces*

$$\mathcal{H}_l = \bigvee_{k\in\mathbb{Z}} \psi_{k,l}, \quad l \in \mathbb{Z},$$

are pairwise orthogonal and form a second OSB for $\mathcal{H}$. *Conversely, if* $\mathcal{H}$ *admits a double-subscripted orthonormal basis* $\{\psi_{k,l}\}_{(k,l)\in\mathbb{Z}^2}$, *then it also admits an OSB* $\{\mathcal{B}_k\}_{k\in\mathbb{Z}}$, *as well as an OSB* $\{\mathcal{H}_l\}_{l\in\mathbb{Z}}$.

We have from (2.6)

$$P_{\mathcal{B}_k} h = \sum_{l\in\mathbb{Z}} \langle h\,, \psi_{k,l}\rangle\, \psi_{k,l}, \quad h \in \mathcal{H}.$$

Similarly,

$$P_{\mathcal{H}_l} h = \sum_{k\in\mathbb{Z}} \langle h\,, \psi_{k,l}\rangle\, \psi_{k,l}, \tag{2.10}$$

where $P_{\mathcal{H}_l}$ is the projection onto $\mathcal{H}_l$. It then follows that

$$P_{\mathcal{H}_l} P_{\mathcal{B}_k} f = \sum_{l'\in\mathbb{Z}} \langle f\,, \psi_{k,l'}\rangle P_{\mathcal{H}_l} \psi_{k,l'} = \langle f, \psi_{k,l}\rangle\, \psi_{k,l} = P_{\psi_{k,l}} f, \tag{2.11}$$

and

$$P_{\mathcal{B}_k} P_{\mathcal{H}_l} f = \sum_{k'\in\mathbb{Z}} \langle f\,, \psi_{k',l}\rangle P_{\mathcal{B}_k} \psi_{k',l} = \langle f\,, \psi_{k,l}\rangle\, \psi_{k,l} = P_{\psi_{k,l}} f. \tag{2.12}$$

Therefore,

$$P_{\mathcal{B}_k} P_{\mathcal{H}_l} = P_{\mathcal{H}_l} P_{\mathcal{B}_k} = P_{\psi_{k,l}}, \quad (k,l) \in \mathbb{Z}^2. \tag{2.13}$$

This is also evident from the fact that

$$\mathcal{B}_k \cap \mathcal{H}_l = \{\psi_{k,l}\}, \tag{2.14}$$

and the orthogonal complements of $\{\psi_{k,l}\}$ in $\mathcal{B}_k$ and in $\mathcal{H}_l$ are orthogonal.

We now "connect" the above with Generalized Multiresolution Analysis associated with a wavelet $\psi(\cdot)$.

First, consider the function space $\mathcal{L}^2(\mathbb{R})$ with the double-subscripted orthonormal basis — consisted of the wavelet-functions $\{\psi_{m,n}(\cdot)\}_{(m,n)\in\mathbb{Z}^2}$ generated from $\psi(\cdot)$. Then, by Lemma 1, $\mathcal{L}^2(\mathbb{R})$ admits the orthogonal decomposition

$$\mathcal{L}^2(\mathbb{R}) = \bigoplus_{m\in\mathbb{Z}} \mathcal{W}_m, \tag{2.15}$$

where we have renamed $\mathcal{B}_k$ of (2.1) as $\mathcal{W}_m$ so that

$$\mathcal{W}_m := \bigvee_{n\in\mathbb{Z}} \psi_{m,n}(\cdot), \quad m \in \mathbb{Z}, \tag{2.16}$$

as well as the orthogonal decomposition

$$\mathcal{L}^2(\mathbb{R}) = \bigoplus_{n\in\mathbb{Z}} \mathcal{H}_n, \tag{2.17}$$

where $\mathcal{H}_n$ is, as before,

$$\mathcal{H}_n := \bigvee_{m\in\mathbb{Z}} \psi_{m,n}(\cdot), \quad n \in \mathbb{Z}. \tag{2.18}$$

The collection of subspaces $\{\mathcal{W}_m\}_{m\in\mathbb{Z}}$ and $\{\mathcal{H}_n\}_{n\in\mathbb{Z}}$, besides being OSB for $\mathcal{L}^2(\mathbb{R})$, also have special properties due to particular characteristic of the wavelet-functions $\{\psi_{m,n}(\cdot)\}_{(m,n)\in\mathbb{Z}^2}$. First,

$$\mathcal{W}_m = \bigvee_{n\in\mathbb{Z}} D^m T^n \psi(\cdot), \quad m \in \mathbb{Z}. \tag{2.19}$$

Then, since D is unitary, we have [8],

$$\mathcal{W}_m = D^m \bigvee_{n\in\mathbb{Z}} T^n \psi(\cdot) = D^m \mathcal{W}_0, \quad m \in \mathbb{Z}, \tag{2.20}$$

where

$$\mathcal{W}_0 = \bigvee_{n\in\mathbb{Z}} T^n \psi(\cdot). \tag{2.21}$$

Therefore, since $\mathcal{W}_m \perp \mathcal{W}_{m'}$ whenever $m \neq m'$,

$$D^m \mathcal{W}_0 \perp D^{m'} \mathcal{W}_0 \quad \text{whenever} \quad m \neq m'. \tag{2.22}$$

This means that the subspace $\mathcal{W}_0$ is a *wandering* subspace for D [6]. Moreover, since $\{\mathcal{W}_m\}_{m\in\mathbb{Z}}$ spans $\mathcal{L}^2(\mathbb{R})$, $\mathcal{W}_0$ is also a D-wandering-*generating* subspace [6]. Note that $\mathcal{W}_m$ is itself D-wandering. Consequently, (2.16) is a wandering subspaces decomposition of $\mathcal{L}^2(\mathbb{R})$, and we write

$$\mathcal{L}^2(\mathbb{R}) = \bigoplus_{m\in\mathbb{Z}} D^m \mathcal{W}_0 = \bigoplus_{m\in\mathbb{Z}} D^m \bigvee_{n\in\mathbb{Z}} \psi\big((\cdot) - n\big). \tag{2.23}$$

This shows that D is a bilateral shift of infinite multiplicity — since $\mathcal{W}_0$ is infinite dimensional [6]. More is true. Once $\mathcal{W}_0$ is D-wandering, the wavelet $\psi(\cdot)$ is itself a D-wandering vector, since it follows from (2.22) that

$$D^m \{\psi(\cdot)\} \perp D^{m'} \{\psi(\cdot)\} \quad \text{whenever} \quad m \neq m'. \tag{2.24}$$

The decomposition (2.23), with $\mathcal{W}_0$ defined by (2.21), establishes connections between wandering subspaces, wandering vectors, and wavelets. These have been observed by others, beginning with [5], and extensively studied more recently in [4, 1]. See also [7].

We now turn to the decomposition (2.17) with $\mathcal{H}_n$ given by (2.18). It is plain that $\mathcal{H}_n$ are D-reducing. Therefore, (2.17) can be rewritten in terms of the part D_n of D on $\mathcal{H}_n$, that is, $D_n := D|_{\mathcal{H}_n}$, as

$$\mathcal{L}^2(\mathbb{R}) = \bigoplus_{n\in\mathbb{Z}} \mathcal{H}_n = \bigoplus_{n\in\mathbb{Z}} \bigvee_{m\in\mathbb{Z}} D_n^m \{\psi\big((\cdot) - n\big)\}, \tag{2.25}$$

where the unitary operator D_n is a *simple* bilateral shift with generating wandering subspace $\big\{\psi\big((\cdot) - n\big)\big\}$.

By construction, each $\mathcal{W}_m$ is defined for fixed scale 2^m and over all time-shifts n. The action of D on $\mathcal{L}^2(\mathbb{R})$ — according to (2.23) — is to process over all time-shifts n and successively from each scale 2^m to the next scale $2^{(m+1)}$: $D\mathcal{W}_m(\psi) = \mathcal{W}_{m+1}(\psi)$, for any $m \in \mathbb{Z}$. Hence the action of D, in this case, can be characterized as "***scales series processing***". It is evident from (2.25) that, since each $\mathcal{H}_n$ is a D-reducing subspace, the action of D on $\mathcal{H}_n$ is simply that of the simple shift D_n — acting at each time-shift n and over all scales 2^m. The action of D over all of $\mathcal{L}^2(\mathbb{R})$ can now be characterized as "***time-shifts parallel processing***".

We summarize the above as follows.

Theorem 1. *Given a wavelet $\psi(\cdot) \in \mathcal{L}^2(\mathbb{R})$, then any $f(\cdot) \in \mathcal{L}^2(\mathbb{R})$ can be represented by its projections on OSB — spanned by wavelet-functions $\psi_{m,n}(\cdot)$ — either with respect to scales or with respect to time-shifts.*

We refer to [8] for an alternate approach to Theorem 1.

Let $\psi(\cdot)$ be a wavelet. It is a simple matter to check that an associated GMRA is given by

$$\mathcal{V}_m = \bigoplus_{m'=-\infty}^{m-1} \mathcal{W}_{m'}, \quad m \in \mathbb{Z}. \tag{2.26}$$

Moreover,

$$\mathcal{V}_{m+1} = \mathcal{V}_m \oplus \mathcal{W}_m, \quad m \in \mathbb{Z}. \tag{2.27}$$

We now prove the following result.

Lemma 2. *Let $\{\mathcal{V}_m\}_{m\in\mathbb{Z}}$ be the GMRA*

$$\mathcal{V}_m = \bigoplus_{m'=-\infty}^{m-1} \mathcal{W}_{m'}, \quad m \in \mathbb{Z},$$

associated with a wavelet $\psi(\cdot)$. Then each approximation subspace $\mathcal{V}_m$ also admits the orthogonal decomposition

$$\mathcal{V}_m = \bigoplus_{n\in\mathbb{Z}} \mathcal{H}_n^{(m)}, \quad m \in \mathbb{Z}, \tag{2.28}$$

where

$$\mathcal{H}_n^{(m)} := \bigvee_{m'=-\infty}^{m-1} \psi_{m',n}(\cdot) = \bigvee_{m'=-\infty}^{m-1} D^{m'}\psi\big((\cdot)-n\big), \quad (m,n) \in \mathbb{Z}^2, \tag{2.29}$$

which is wavelet-detail-subspace for time-shift n over all scales not greater than 2^{m-1}.

Proof. We have from (2.26) and (2.19)

$$\mathcal{V}_m = \bigoplus_{m'=-\infty}^{m-1} \mathcal{W}_{m'} = \bigoplus_{m'=-\infty}^{m-1} \bigvee_{n\in\mathbb{Z}} D^{m'}\psi\big((\cdot)-n\big), \quad m \in \mathbb{Z}. \tag{2.30}$$

Therefore [7],

$$\mathcal{V}_m = \bigoplus_{n\in\mathbb{Z}} \bigvee_{m'=-\infty}^{m-1} D^{m'}\psi\big((\cdot)-n\big), \quad m \in \mathbb{Z}. \tag{2.31}$$

It then follows that

$$\mathcal{V}_m = \bigoplus_{n\in\mathbb{Z}} \mathcal{H}_n^{(m)}, \tag{2.32}$$

where $\mathcal{H}_n^{(m)}$ is defined by (2.29). □

Remark 1. In order to distinguish between the two representations (2.26) and (2.28) of the approximation subspaces $\mathcal{V}_m$, we adopt the notation $\mathcal{V}^{(m)}$ for the latter

$$\mathcal{V}^{(m)} := \bigoplus_{n\in\mathbb{Z}} \mathcal{H}_n^{(m)}, \quad m \in \mathbb{Z}, \tag{2.33}$$

and refer to $\mathcal{V}^{(m)}$ as *wavelet-time-shifts-approximation subspaces,* while reserve $\mathcal{V}_m$ for the former and call them *wavelet-scales-not-greater-than* 2^{m-1}*-approximation subspaces.*

The above suggests the following Definition.

Definition 2. Let $\psi(\cdot)$ be a wavelet. Then the sequence of subspaces $\mathcal{V}_m$,

$$\mathcal{V}_m := \bigoplus_{m'=-\infty}^{m-1} \mathcal{W}_{m'}, \quad m \in \mathbb{Z},$$

is GMRA associated with $\psi(\cdot)$, while the sequence of subspaces $\mathcal{V}^{(m)}$,

$$\mathcal{V}^{(m)} := \bigoplus_{n\in\mathbb{Z}} \mathcal{H}_n^{(m)}, \quad m \in \mathbb{Z},$$

is Time-Shifts GMRA associated with the wavelet.

It is plain that

$$\mathcal{H}_n^{(m)} \subset \mathcal{H}_n, \quad (m,n) \in \mathbb{Z}^2. \tag{2.34}$$

Moreover, $\mathcal{H}_n^{(m)}$ is an irreducible D^*-invariant subspace, that is, it does not contain any D-reducing subspaces. Also, for each fixed $n \in \mathbb{Z}$,

$$\mathcal{H}_n^{(m)} \subset \mathcal{H}_n^{(m+1)}, \quad m \in \mathbb{Z}, \tag{2.35}$$

$$D\mathcal{H}_n^{(m)} = D_n\mathcal{H}_n^{(m)} = \mathcal{H}_n^{(m+1)}, \quad m \in \mathbb{Z}. \tag{2.36}$$

and

$$\mathcal{H}_n^{(m+1)} = \mathcal{H}_n^{(m)} \oplus \{\psi_{m,n}(\cdot)\}. \tag{2.37}$$

Now, on each Hilbert space $\mathcal{H}_n$ there is a dyadic-scaling operator D_n which, as we have seen, is a simple shift whose wandering-generating subspace, denoted by $\mathcal{W}_{0,n}$, and is given by

$$\mathcal{W}_{0,n} := \bigvee \psi\big((\cdot) - n\big).$$

The unit function $\psi\big((\cdot) - n\big)$ can therefore be defined as *"sub-wavelet"* living in $\mathcal{H}_n$ — since, by (2.18), the orthonormal wavelet-functions spanning $\mathcal{H}_n$ are the functions $\big\{\psi_{m,n}(\cdot) = D^m\psi\big((\cdot)-n\big)\big\}_{m\in\mathbb{Z}}$ — generated from $\psi\big((\cdot)-n\big)$ by repeated dyadic-scaling — alone. This suggests that, as in the case for a wavelet $\psi(\cdot)$, we associate $\psi\big((\cdot) - n\big)$ with a sequence of subspaces $\{\mathcal{V}^{(m,n)}\}_{m\in\mathbb{Z}}$ defined by

$$\mathcal{V}^{(m,n)} := \bigvee_{m'=-\infty}^{m-1} D^{m'}\big\{\psi\big((\cdot) - n\big)\big\} = \mathcal{H}_n^{(m)}, \tag{2.38}$$

according to (2.29). It then follows from this and from (2.34)–(2.37) the next proposition.

Proposition 1. *For each $n \in \mathbb{Z}$, the sequences of subspaces $\{\mathcal{V}^{(m,n)}\}_{m\in\mathbb{Z}}$ of $\mathcal{H}_n$, defined by*

$$\mathcal{V}^{(m,n)} := \mathcal{H}_n^{(m)}, \quad (m,n) \in \mathbb{Z}^2, \tag{2.39}$$

satisfy the following four conditions:

(i) $\mathcal{V}^{(m,n)} \subset \mathcal{V}^{(m+1,n)}, \quad m \in \mathbb{Z},$
(ii) $\bigcap_{m=-\infty}^{\infty} \mathcal{V}^{(m,n)} = \{0\},$
(iii) $\overline{\bigcup_{m=-\infty}^{\infty}} \mathcal{V}^{(m,n)} = \mathcal{H}_n,$
(iv) $D_n \mathcal{V}^{(m,n)} = \mathcal{V}^{(m+1,n)}, \quad m \in \mathbb{Z}.$

It follows from (2.33) that the core subspace $\mathcal{V}^{(0)}$ admits the decomposition

$$\mathcal{V}^{(0)} := \bigoplus_{n\in\mathbb{Z}} \mathcal{H}_n^{(0)}, \quad m \in \mathbb{Z}, \tag{2.40}$$

where

$$\mathcal{H}_n^{(0)} = \bigvee_{m=-\infty}^{-1} D^m \psi\big((\cdot) - n\big). \tag{2.41}$$

Even though $\mathcal{V}^{(0)}$ is T-invariant, the core subspaces $\mathcal{H}_n^{(0)}$ are not, since

$$T\,\mathcal{H}_n^{(0)} = \bigvee_{m=-\infty}^{-1} TD^m \psi\big((\cdot) - n\big), \tag{2.42}$$

$$= \bigvee_{m=-\infty}^{-1} D^m \psi\big((\cdot) - (n + 2^m)\big). \tag{2.43}$$

Therefore, for each $n \in \mathbb{Z}$, the sequence $\{\mathcal{V}^{(m,n)}\}_{m\in\mathbb{Z}}$ satisfies all the conditions of a GMRA — except for condition (o). We therefore refer to each sequence $\{\mathcal{V}^{(m,n)}\}_{m\in\mathbb{Z}}$ as *"Sub-GMRA"* associated with the sub-wavelet $\psi\big((\cdot) - n\big)$ — living on $\mathcal{H}_n$.

We now prove the following theorem.

Theorem 2. *Let $\psi(\cdot) \in \mathcal{L}^2(\mathbb{R})$ be a wavelet. Then each "sub-wavelet" $\psi\big((\cdot) - n\big)$, $n \neq 0$, can be associated with a Sub-GMRA $\{\mathcal{V}^{(m,n)}\}_{m\in\mathbb{Z}}$ — defined on $\mathcal{H}_n$. Consequently, the Time-Shifts GMRA $\{\mathcal{V}^{(m)}\}_{m\in\mathbb{Z}}$ — associated with the wavelet $\psi(\cdot)$ — can be decomposed into Sub-GMRA $\{\mathcal{V}^{(m,n)}\}_{m\in\mathbb{Z}}$ as*

$$\mathcal{V}^{(m)} = \bigoplus_{n'\in\mathbb{Z}} \mathcal{V}^{(m,n')}. \tag{2.44}$$

Moreover,

$$P_{\mathcal{V}^{(m)}} f(\cdot) = \sum_{n\in\mathbb{Z}} P_{\mathcal{V}^{(m,n)}} f(\cdot), \quad f(\cdot) \in \mathcal{L}^2(\mathbb{R}), \quad m \in \mathbb{Z}, \tag{2.45}$$

where $P_{\mathcal{V}^{(m)}} f(\cdot)$ is the wavelet time-shifts approximation of $f(\cdot)$, and $P_{\mathcal{V}^{(m,n)}} f(\cdot)$ is approximation of $f(\cdot)$ for scale 2^{m-1} and for time-shift n.

Proof. Let $\psi(\cdot)$ be a wavelet and let $\{\mathcal{V}^{(m)}\}_{m\in\mathbb{Z}}$ be its associated Time-Shifts GMRA. (2.44) follows from the decomposition (2.33) and Proposition 1. Next, it follows from (2.33) that any $P_{\mathcal{V}^{(m)}} f(\cdot) \in \mathcal{V}^{(m)}$ admits the orthogonal decomposition

$$P_{\mathcal{V}^{(m)}} f(\cdot) = \sum_{n\in\mathbb{Z}} f_n^{(m)}(\cdot),$$

where

$$f_n^{(m)}(\cdot) \in \mathcal{V}^{(m,n)}, \quad \text{and} \quad \sum_{n\in\mathbb{Z}} \|f_n^{(m)}(\cdot)\|^2 < \infty,$$

keeping in mind that $\mathcal{V}^{(m,n)} = \mathcal{H}_n^{(m)}$. Therefore,

$$f_n^{(m)} = P_{\mathcal{V}^{(m,n)}} P_{\mathcal{V}^{(m)}}\{f(\cdot)\} = P_{\mathcal{V}^{(m,n)}}\{f(\cdot)\},$$

since $\mathcal{V}^{(m,n)} \subset \mathcal{V}^{(m)}$, by (2.44). This finishes the proof. □

An immediate consequence of the above is:

Corollary 1. *With respect to a wavelet $\psi(\cdot) \in \mathcal{L}^2(\mathbb{R})$, the space $\mathcal{L}^2(\mathbb{R})$ admits the orthogonal decomposition*

$$\mathcal{L}^2(\mathbb{R}) = \mathcal{V}^{(m)} \oplus \bigoplus_{m'=m}^{\infty} \mathcal{W}_{m'}, \quad m \in \mathbb{Z}, \tag{2.46}$$

and therefore,

$$\mathcal{L}^2(\mathbb{R}) = \bigoplus_{n\in\mathbb{Z}} \mathcal{H}_n^{(m)} \oplus \bigoplus_{n\in\mathbb{Z}} \mathcal{H}_{n,(m)}, \quad m \in \mathbb{Z}, \tag{2.47}$$

where

$$\mathcal{H}_{n,(m)} := \bigvee_{m'=m}^{\infty} D^{m'}\psi\big((\cdot) - n\big), \quad (m,n) \in \mathbb{Z}^2, \tag{2.48}$$

is the wavelet-detail subspace for time-shift n over all scales not smaller than 2^m.

3. Conclusions

We have developed in this article the concept of Time-Shifts Generalized Multiresolution Analysis associated with a wavelet, and the concept of Sub-GMRA associated with a sub-wavelet.

We have shown that if $\{\mathcal{V}^{(m)}\}_{m\in\mathbb{Z}}$ is Time-Shifts GMRA associated with a wavelet $\psi(\cdot)$. Then the set of "m-nested" subspaces $\{\mathcal{H}_n^{(m)}\}_{m\in\mathbb{Z}}$ of $\mathcal{H}_n$, defined by

$$\mathcal{H}_n^{(m)} := \bigvee_{m'=-\infty}^{m-1} \psi_{m',n}(\cdot) = \bigvee_{m'=-\infty}^{m-1} D^{m'}\psi\big((\cdot) - n\big), \quad (m,n) \in \mathbb{Z}^2,$$

constitutes Sub-GMRA for the sub-wavelet $\psi\big((\cdot) - n\big)$ — living in $\mathcal{H}_n$.

The reason for referring to $\psi\big((\cdot) - n\big) \in \mathcal{H}_n$ as *sub-wavelet* is that it behaves like a wavelet on the subspace $\mathcal{H}_n$ — since the orthonormal wavelet-functions $\{\psi_{m,n}(\cdot)\}_{m\in\mathbb{Z}}$ spanning $\mathcal{H}_n$, are generated from $\psi\big((\cdot) - n\big)$ by repeated dyadic-scaling only, instead of by repeated unit-time-shift followed by repeated dyadic-scaling, as in the case of the wavelet-functions $\{\psi_{m,n}(\cdot)\}_{(m,n)\in\mathbb{Z}^2}$.

A scaling function $\phi(\cdot)$ is referred to as father wavelet, while a wavelet $\psi(\cdot)$ is called mother wavelet. We therefore propose the term "children wavelets" for the functions $\psi\big((\cdot) - n\big)$, $0 \neq n \in \mathbb{Z}$.

The central result of our work can be succinctly stated as: "The GMRA associated with each of the (children) wavelets $\psi\big((\cdot)-n\big)$ for $n \neq 0$ constitute the GMRA associated with the (mother) wavelet $\psi(\cdot)$".

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Department of Electrical Engineering, University of California in Los Angeles, Los Angeles, CA 90024-1594, USA
E-mail address: levan@ee.ucla.edu

Catholic University of Rio de Janeiro, 22453-900, Rio de Janeiro, RJ, Brazil
E-mail address: carlos@ele.puc-rio.br