Title

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Abstract

The existence and uniqueness of solutions of the Navier-Stokes equation driven with additive noise in three dimensions is proven, in the presence of a strong uni-directional mean flow with some rotation. The physical relevance of this solution and its relation to the classical solution, whose existence and uniqueness is also proven, is explained. The existence of a unique invariant measure is established and the properties of this measure are described. The invariant measure is used to prove Kolmogorov's scaling in 3-dimensional turbulence including the celebrated $-5/3$ power law for the decay of the power spectrum of a turbulent 3-dimensional flow.

1 Introduction

Kolmogorov's theory of turbulence published in 1941 [14] set the stage for the resolution of one of the oldest problems in modern mathematics, that of the math-

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ematical formulation of the equations for turbulent flow and their statistical solution. However, to provide a rigorous derivation of Kolomgorov's statistical theory of turbulence has proven to be elusive. This has held back improvements of many application of his theory including application to numerical simulation of turbulent flow. A detailed mathematical theory is expected to have major applications to current technology once it is fully developed.

There are two main reasons why the mathematical theory of turbulent flow has been hard to develop. The first is that the Leray's existence theory [17] of solutions to the Navier-Stokes equations has reminded open in three dimensions. Thus the existence and uniqueness of solutions that describe 3-dimensional turbulence has not been established. The second reason is that although there is universal agreement that noise plays an essential role in turbulent flow it has not been clear how to incorporate noise in the Navier-Stokes equations.

In this paper we resolve these two problems and then use the result to develop Kolmogorov's theory of turbulence in three dimensions, including the celebrated $-5/3$ law for the decay of the power spectrum of turbulent flow in three dimensions. We show that with non-zero mean flow, which is always present in turbulent flow on a small scale and can be taken be uni-directional, see [4], on such a scale, see Monin and Yaglom [21, 22], there exist unique weak-solutions of the stochastically driven Navier-Stokes equation. We also have to introduce some rotation in order to take care of components of the three-dimensional flow that are perpendicular to the uni-directional flow. Instabilities are inherent in turbulent flow, see [21, 22], and we show how the small white noise ubiquitous in nature can be exponentially magnified in turbulent flow into large noise that drives the velocity of the fluid. This is the source of additive noise.

In the Lagrangian formulation the flow of a small fluid particle with coordinates $X(t)$ is determined by the equation

$$\frac{dX}{dt} = u(X(t),t) \tag{1}$$

In turbulent flow the path of the fluid particle is going to be influenced by turbulent noise and the resulting trajectory of the fluid particle is going to resemble a random walk. It is reasonable to assume that the velocity u is in fact a random variable and that it satisfies a stochastic equation that can be written as

$$du = \frac{\partial u}{\partial t} dt + df_t \tag{2}$$

Here $\frac{\partial u}{\partial t}$ is the deterministic acceleration of the fluid and df_t is a random force modeling the influence of the random fluctuations in turbulent flow on the velocity.

If we now substitute the right hand side of the deterministic Navier-Stokes in for the time derivative of u in the equation (3) we get the stochastically driven Navier-Stokes equation

$$du = (\nu \Delta u - u \cdot \nabla u - \nabla p)dt + df_t \tag{3}$$

with the incompressibility condition

$$\nabla \cdot u = 0$$

This is the equation that we will analyze in this paper. Once we have solved it for the stochastic velocity $u(x,t)$, u can be substituted into the equation (1) for the random motion of the fluid particle.

Another way of introducing the noise into the Navier-Stokes equation is to try to write down an equation for the random motion of the fluid particle and then use It$\hat{o}$'s formula to get a Navier-Stokes equation with multiplicative noise, see Mikulevicius and Rozovksy [20].

Kolmogorov conjectured that the solutions of the equation (3) approached a statistically stationary state as time increases. In this case an (ensamble) average of the fluid acceleration vanishes $\langle \frac{\partial u}{\partial t} \rangle = 0$. Evidently, $\langle \nu \Delta u - u \cdot \nabla u - \nabla p \rangle = 0$ also and since the viscous term $\langle \nu \Delta u \rangle$ is not believed to be important, the pressure gradient must be balancing the inertial terms in this *inertial range*, described by the statistically stationary state.

To prove Kolmogorov's theory we must model the noise term, and we will make the assumption

$$df_t = \sum_{k \neq 0} h_k^{1/2} d\beta_t^k e_k \tag{4}$$

in this paper. This assumes that in the statistically stationary state the system is driven by noise (fluctuations) that characterize a balance between the noise producing (amplifying) nonlinear terms in (3), this is a common assumption by investigators in this field, see for example [27, 16, 18]. Here the $e_k s$ are basis vectors that can be taken to be Fourier coefficients, they each come with an independent Brownian motion β_t^k and the $h_k^{1/2}$ are decay vectors that depend on the characteristics of the flow. In particular, this assumes that the variance of the noise

$$E(\langle df_t, df_t \rangle)$$

is finite. This form of the noise assumes that the motion of the fluid particles is continuous, an assumption that makes sense on physical grounds. However, it will still need to be fine-tuned to compare with experimental results, see [5]. If the fluid particles moved only under the influence of this noise their velocity would execute an infinite-dimensional Brownian motion.

The existence theory of the three dimensional Navier-Stokes equations is a thorny issue that has kept mathematicians occupied for many years. Onsager [25] pointed out that solutions of the Navier-Stokes equation that possessed Kolmogorov's scaling must be Hölder continuous functions with Hölder's index of continuity $1/3$. This lead many mathematicians to conclude that solutions of the Navier-Stokes equation with smooth initial data had to blow up in finite time. However, the difficulty seem rather to lie with the instabilities inherent in turbulent flow. Problems in turbulence are notoriously unstable and the (nonlinear) ill-posedness of many problems in turbulence results in the magnification of the small (white) noise ubiquitous in nature. The noise grows under the influence of the instabilities and is saturated by nonlinearities in the Navier-Stokes equation into large (colored) noise that drives the system. In Section 2, the magnification of the ambient noise will be made explicit. The color given by the decay of the coefficients $h_k^{1/2}$ is characteristic for the system and to find this decays is now a part of the problem. In other words we will interpret the noise term (4) as a model of the intrinsic noise in turbulence. Below we will model the decay of the coefficients $h_k^{1/2}$ so as to give us the existence of unique rough solutions possessing the Kolmogorov scaling. This view of the problems is analogous to the theory of surface roughening and the evolution of the surface of the earth by erosion, see [8]. Many of the technical details are a generalization of the analogous techniques for one dimensional flow in turbulent rivers, see [6].

In spite of the rotation in the flow the problem solved in this paper is very different from that solved by Babin, Mahalov and Nicolaenko in [1] and [2]. In their papers the rotation plays the main role whereas the uni-directional flow, along the axis of the rotation, is the main actor in this paper. It causes oscillations that permit us to prove the global existence and uniqueness. In this paper the rotation is present for a purely technical reason, to control the velocity components orthogonal to the uniform flow. The two problems are similar in that the initial flow is unstable and the turbulent flow becomes three dimensional. However, in [1] and [2] the three dimensional energy cascade is suppressed and instead there is an inverse cascade similar to two-dimensional flow, whereas in our work the full three-dimensional energy cascade is present and plays a major role in the turbulence production and transfer of energy.

Section 3 contains the definition of the functions spaces we work in and a priori estimates for the stochastic Navier-Stokes equation (3) analogous to those in Leray's existence theory [17]. These a priori estimates play a central role in our existence theory.

It turns out that the stronger the turbulence is, as measured by the Reynold's number, the easier it is to prove the existence of the solutions, not harder as mathematicians have believed until now. The reason is that the turbulent flow is characterized by fast oscillations in the direction of the rotationing flow and the faster these oscillation are the more effective the mixing of the turbulent fluid is. The effective "mixing" of the fluid by the fast oscillations is proven in Section 4. Thus large mean (uni-directional) flow and rotation implies simpler existence theory.

In Section 5, we prove the existence of solutions to the stochastic Navier-Stokes equation (3) that are Hölder continuous functions of Hölder index $1/3$. Then we will use these solutions to prove the existence of a unique invariant measure living on the space of these functions, in Section 7. The existence of this measure allows us to prove Kolmogorov's scaling of the structure functions in turbulence, in Section 7.1. Evidently neither the gradient ∇u of the velocity nor the vorticity $\nabla \times u$ are continuous functions of x so one has to be careful with the sense in which the u solves the equation (3). u must in fact be interpreted to be a weak (mild) solution of the stochastic Navier-Stokes equation. However, with its inititial data specified, it is the *unique* weak solution and it is continuous in time.

2 The Stochastic Initial Value Problem

Consider the Navier-Stokes equation

$$\begin{aligned} w_t + w \cdot \nabla w &= \nu \Delta w - \nabla p \\ w(x,0) &= w^0(x) \end{aligned} \tag{5}$$

with the incompressibility conditions

$$\nabla \cdot w = 0, \tag{6}$$

where ν is the kinematic viscosity. Eliminating the pressure p using (6) gives the equation

$$w_t + w \cdot \nabla w = \nu \Delta w + \nabla \{\Delta^{-1}[trace(\nabla w)^2]\} \tag{7}$$

We want to consider turbulent flow driven by a unidirectional mean flow and to do that we consider the flow to be in a box and impose periodic boundary conditions on the box. Since we are mostly interested in what happens in the direction along the unidirectional flow we take our x_1 axis to be in that direction.

We will assume that the unidirectional flow is fast and select an initial condition of the form

$$w^0(x) = U_o(x_1)\,j_1 \tag{8}$$

where $U_o(x_1)$ is the velocity and j_1 is a unit vector in the x_1 direction. We are ignoring incompressibility to illustrate how the noise gets magnified, incompressibility will be restored below. Clearly this initial condition is not sufficient because the fast flow will be unstable and the white noise ubiquitous in nature will grow into small velocity and pressure oscillations, see for example [3]. To formulate the problem with small noise we look for a solution of the form

$$w(x,t) = U_o(x_1)\,j_1 + u(x,t) \tag{9}$$

where $u(x,t)$ is smaller than U_o but not necessarily small. Then u can be approximated by the solution to the equation (7) linearized about the fast flow $U_o j_1$

$$\begin{aligned} u_t + U_o \partial_{x_1} u &+ U_o' u_1 j_1 + U_0 U_0' j_1 = \nu \Delta u + \nu U_o'' j_1 \\ &+ \nabla\{\Delta^{-1}(U_0'^2 + 2U_0' \partial_x u_1)\} \\ u(x,0) &= u^0(x) \end{aligned} \tag{10}$$

To this equation we must add small forcing by noise ubiquitous in all flow

$$df^0 = \sum_{k \neq 0} c_k^{1/2} d\beta_t^k e_k \tag{11}$$

The $e_k = e^{2\pi i k \cdot x}$ are (three-dimensional) Fourier components and each comes with its own independent Brownian motion β_t^k. None of the coefficients of the vectors $c_k^{1/2} = (c_1^{1/2}, c_2^{1/2}, c_3^{1/2})$ vanish because the small noise is seeded by truly white noise (white both is space and in time). However, df^0 is not white in space because the coefficients $c_k^{1/2}$ must have some decay in k so that the noise term in (10) makes sense. Notice that $c_k^{1/2} \neq h_k^{1/2}$. The former coefficients are tiny whereas the latter can be large.

The difference between laminar and turbulent flow is that the noise is quelled in laminar flow but in turbulent flow it gets magnified by the instabilities and grows.

Now we restore incompressibility and consider the Navier-Stokes equation linearized about the divergence-free initial flow $\mathbf{U} = U_0 j_1 + U'(x_1, -\frac{x_2}{2}, -\frac{x_3}{2})^T$, where T denotes transpose and $\mathbf{U}$ is construed to be the periodic extension of the above formula from $\mathbb{T}^3$ to $\mathbb{R}^3$,

$$\begin{aligned} u_t &+ U_0\partial_{x_1}u + U'\begin{pmatrix} u_1 \\ -\frac{u_2}{2} \\ -\frac{u_3}{2} \end{pmatrix} + U'\begin{pmatrix} x_1 \\ -\frac{x_2}{2} \\ -\frac{x_3}{2} \end{pmatrix}\cdot\nabla u + U'U_0 j_1 \\ &+ (U')^2\begin{pmatrix} x_1 \\ \frac{x_2}{4} \\ \frac{x_3}{4} \end{pmatrix} = \nu\Delta u + \nabla\Delta^{-1}(\frac{3}{2}U'^2 + 2U'(\partial_{x_1}u_1 - \partial_{x_2}u_2 - \partial_{x_3}u_3)) \quad (12) \\ & u(x,0) = 0 \end{aligned}$$

We assume that there is small noise

$$df^0 = \sum_{k\neq 0} c_k^{1/2} d\beta_t^k e_k$$

present in the fluid. Then u satisfies the linear stochastic PDE

$$\begin{aligned} du &= [\nu\Delta u - U_0\partial_{x_1}u - U'\begin{pmatrix} u_1 \\ -\frac{u_2}{2} \\ -\frac{u_3}{2} \end{pmatrix} - U'\begin{pmatrix} x_1 \\ -\frac{x_2}{2} \\ -\frac{x_3}{2} \end{pmatrix}\cdot\nabla u - U'U_0 j_1 \\ &- (U')^2\begin{pmatrix} x_1 \\ \frac{x_2}{4} \\ \frac{x_3}{4} \end{pmatrix}\cdot\nabla u + \nabla\Delta^{-1}(\frac{3}{2}U'^2 + 2U'(\partial_{x_1}u_1 - \partial_{x_2}u_2 - \partial_{x_3}u_3))]dt \\ &+ \sum_{k\neq 0} c_k^{1/2} d\beta_t^k e_k \quad (13) \end{aligned}$$

where the term $\sum_{k\neq 0} c_k^{1/2} d\beta_t^k e_k$ represents stochastic forcing by the small ambient noise.

The solution of this linear equation can be found by use of a Fourier series and it is

$$\begin{aligned} u(x,t) &= \sum_{k\neq 0}\int_0^t e^{-(4\nu\pi^2|k|^2 + 2\pi i U_0 k_1)(t-s)} \times \\ &\left(c_k^{1/2}(1) j_1 e^{-U'(t-s)} + c_k^{1/2}(2) j_2 e^{\frac{U'}{2}(t-s)} + c_k^{1/2}(3) j_3 e^{\frac{U'}{2}(t-s)}\right) d\beta_t^k e_k \\ &+ O(|U'|) \end{aligned}$$

where $c_k^{1/2}(i), i = 1,2,3$ denotes the ith entry of the three vector $c_k^{1/2}$. Now the expectation of $u(x,t)$ vanishes but the variation is

$$
\begin{aligned}
E(|u|_2^2)(t) &= \sum_{k\neq 0}\int_0^t e^{-8\nu\pi^2|k|^2(t-s)}\times \\
(c_k(1)e^{-2U'(t-s)} &+ c_k(2)e^{U'(t-s)}+c_k(3)e^{U'(t-s)})ds \\
&+ O(|U'|^2)
\end{aligned}
\tag{14}
$$

This shows that one one hand the small noise will grow exponentially in time, in the $j_1 e_k$ direction, if

$$U' < 0 \tag{15}$$

and if $|U'| > 8\pi^2\nu|k|$ for some $k \in \mathbb{Z}^3 \setminus \{0\}$, but $|U'|$ is small compared to the exponentially growing term. If on the other hand

$$U' > 0 \tag{16}$$

the small noise will grow exponentially in the $j_2 e_k$ and $j_3 e_k$ directions (in function space), again with $|U'|$ small compared to the exponentially growing term.

The exponential growth of the noise will, however, only continue for a limited time. The growth is quickly saturated by the nonlinear terms in the equation and fluid becomes fully turbulent. We will denote the mean flow in the fully developed turbulent state by U_1 and assume that uniform flow with rotation is of the form

$$\frac{\partial x}{\partial t} = \mathbf{U} = U_1 j_1 - A\sin(\Omega t+\theta_0)j_2 + A\cos(\Omega t+\theta_0)j_3 \tag{17}$$

where the rotation can be extended in a periodic fashion from $\mathbb{T}^3$ to $\mathbb{R}^3$.[1] One can also extend a convection cell pattern from four copies of $\mathbb{T}^3$ to $\mathbb{R}^3$ but we will not use that in this paper. This implies that the deterministic particle motion in the rotating uniform flow is simply

$$x(t) = [U_1 j_1 + \frac{A}{\Omega}\cos(\Omega t+\theta_0)j_2 + \frac{A}{\Omega}\sin(\Omega t+\theta_0)j_3]$$

By the same reasoning as above we can choose the coordinates so that the mean flow component $U_1 j_1$ (17) is in the x_1 direction and this direction is the axis of the rotation.

[1] For physical applications, see [12], cylindrical coordinates are more appropriate but cumbersome.

The equation (17) represents an assumption, which is however not very restrictive. We are assuming the mean flow $U_1 j_1$ and the rotation $-A\sin(\Omega t+\theta_0)j_2 + A\cos(\Omega t+\theta_0)j_3$ are constant as functions of $x \in \mathbb{R}^3$, but this can easily be relaxed. Moreover, on a small scale the rotation can be taken to be constant and it is going to be present as a constant rotation on this scale, in general, for turbulent flow. The second assumption that we will make in equation (20) below is more serious. We are assuming that in the Eulerian representation the acceleration of the fluid is driven by fixed infinite-dimensional noise for which the fluid velocity is continuous. Thus Equation (20) contains a hypothesis about the form of the noise in fully developed turbulence. This form makes sense on physical grounds and it is now a part of the problem to determine the decay that the coefficients $h_k^{1/2}$ can have as $k \to \infty$. Moreover, we will show in this paper that with this hypothesis in Equation (20) one can prove Kolmogorov's scaling in turbulence.

We can now state the problems that we will solve in this paper. First consider the stirred Navier-Stokes equation

$$\begin{aligned} w_t + w\cdot\nabla w &= \nu\Delta w - \nabla\Delta^{-1}\mathrm{trace}(\nabla w)^2 \\ &- A\Omega\cos(\Omega t+\theta_0)j_2 - A\Omega\sin(\Omega t+\theta_0)j_3 \\ w(x,0) &= U_1 j_1 - A\sin(\theta_0)j_2 + A\cos(\theta_0)j_3 \end{aligned} \tag{18}$$

with the incompressibility conditions

$$\nabla\cdot w = 0 \tag{19}$$

The source of the small noise (11) can now be thought of as fluctuations in the stirring rate of the uniform flow in equation (18).

The corresponding stochastic Navier-Stokes equation can be written as

$$\begin{aligned} du = (&\nu\Delta u - U_1\partial_{x_1}u + A\sin(\Omega t+\theta)\partial_{x_2}u - A\cos(\Omega t+\theta)\partial_{x_3}u \\ &-u\cdot\nabla u + \nabla\Delta^{-1}[\mathrm{trace}(\nabla u)^2])dt + \sum_{k\neq 0} h_k^{1/2} d\beta_t^k e_k \end{aligned} \tag{20}$$

where

$$\begin{aligned} \frac{\partial u}{\partial t} + U_1\partial_{x_1}u - A\sin(\Omega t+\theta)\partial_{x_2}u + A\cos(\Omega t+\theta)\partial_{x_3}u + u\cdot\nabla u \\ = \nu\Delta u + \nabla\Delta^{-1}[\mathrm{trace}(\nabla u)^2] \end{aligned} \tag{21}$$

is the driven Navier-Stokes equation (18) for $u = w - U_1 j_1 + A\sin(\Omega t+\theta)j_2 - A\cos(\Omega t+\theta)j_3$. $U_1 j_1$ is the now the constant mean flow of the (fully developed) turbulent fluid and $\sum_{k\neq 0} h_k^{1/2} d\beta_t^k e_k$ models the noise in fully developed turbulent flow. We will take the initial condition to be zero $u(x,0)=0$ for convenience and assume that the incompressibility condition

$$\nabla \cdot u(x,t) = 0$$

is satisfied. However, the problem is just as easily solved with a nontrivial initial condition, see Theorem 5.2. The goal is to prove the existence of a unique solution to (20) but also to determine the smoothest space where these solutions can live because this will determine the decay of the coefficients $h_k^{1/2}$ in the turbulent noise in (20). The noise that we end up with models the intrinsic noise in turbulence and the model is confirmed both by numerical simulations [28], [19] and a direct calculation of the intrinsic noise in turbulence, see [7].

The initial value problems can also be written as an integral equation

$$u(x,t) = u_o(x,t) - \int_0^t K(t-s) * (u \cdot \nabla u - \nabla \Delta^{-1}[\text{trace}(\nabla u)^2])ds \tag{22}$$

where K is the (oscillatory heat) kernal

$$K * f = \sum_{k\neq 0} \int_0^t e^{-(4\pi^2|k|^2 + 2\pi i U_1 k_1)(t-s) - 2\pi i A(k_2,k_3)[\sin(\Omega t+\theta) - \sin(\Omega s+\theta)]} \hat{f}(k,s) ds e_k, \tag{23}$$

$A(k_2,k_3) = A\sqrt{k_2^2+k_3^2}$, $\theta = \tan^{-1}(\frac{k_2}{k_3}) - \theta_0$ and

$$u_o(x,t) = \sum_{k\neq 0} h_k^{1/2} \int_0^t e^{-(4\pi^2|k|^2 + 2\pi i U_1 k_1)(t-s) - 2\pi i A(k_2,k_3)[\sin(\Omega t+\theta) - \sin(\Omega s+\theta)]} d\beta_s^k e_k(x) \tag{24}$$

is a sum of independent oscillatory processes,

$$A_t^k = \int_0^t e^{-(4\pi^2|k|^2 + 2\pi i U_1 k_1)(t-s) - 2\pi i A(k_2,k_3)[\sin(\Omega t+\theta) - \sin(\Omega s+\theta)]} d\beta_s^k \tag{25}$$

with mean zero, see for example [26]. These processes are reminiscent of Ornstein-Uhlenbeck processes and we will call them oscillatory Ornstein-Uhlenbeck-type processes below.

The mean (average) of the solution u_o of the linear equation is zero by the formula (24) and this implies that the solution u of (20) also has mean (average) zero

$$\bar{u}(t) = \int_{\mathbb{T}^3} u(x,t) dx = 0 \tag{26}$$

This also implies that

$$|w|_2^2 = |\mathbf{U}|^2 + |u|_2^2 \tag{27}$$

for $w = \mathbf{U} + u$ and $\mathbf{U} = U_1 j_1 - A\sin(\Omega t + \theta_0) j_2 + A\cos(\Omega t + \theta_0) j_3$ with $|\mathbf{U}| = \sqrt{U_1^2 + A^2}$. We will derive apriori estimates for w in the next section but then apply them to u in subsequent section using (27).

3 The Function Spaces and a Priori Estimates

In this section we will explain the probabilistic setting and prove some a priori estimates.

We let $(\Omega, \mathcal{F}, \mathbb{P})$, Ω is a set (of events) and $\mathcal{F}$ a σ-algebra on Ω, denote a probability space with $\mathbb{P}$ the probability measure of Brownian motion and $\mathcal{F}_t$ a filtration generated by all the Brownian motions β_t^k on $[t, \infty)$. If $f : \Omega \to H$ is a random variable, mapping Ω into a Hilbert space H, for example $H = L^2(\mathbb{T}^3)$, then $L^2(\Omega, \mathcal{F}, \mathbb{P}; H)$ is a Hilbert space with norm:

$$\|f\|_{L^2(\Omega,\mathcal{F},\mathbb{P};H)}^2 = E(|f(\omega)|_2^2) = \int_\Omega |f(\omega)|_2^2 \mathbb{P}(d\omega) = \int_H |x|^2 f_\# \mathbb{P}(dx)$$

where E denotes the expectation with respect to $\mathbb{P}$ and $f_\#\mathbb{P}$ denotes the pull-back of the measure $\mathbb{P}$ to H. A stochastic process f_t in $\mathcal{L}^2 = L^2([0,T]; L^2(\Omega, \mathcal{F}, \mathbb{P}; H))$ has the norm

$$\|f_t\|_{\mathcal{L}^2}^2 = \int_0^T E(|f(t,\omega)|_2^2) dt$$

and f_t has the following properties, see Oksendal [23].

Definition 3.1

1. *$f(t,\omega) : \mathbb{R}^+ \times \Omega \to \mathbb{R}$ is measurable with respect to $\mathcal{B} \times \mathcal{F}$ where $\mathcal{B}$ is the σ-algebra of the Borel sets on $[0,\infty)$, $\omega \in \Omega$,*

2. *$f(t,\omega)$ is adapted to the filtration $\mathcal{F}_t$,*

3. $$E\left(\int_0^T f^2(t,\omega) dt\right) < \infty.$$

We are mostly interested in the Hilbert spaces $H = H^m(\mathbb{T}^3) = W^{(m,2)}$ that are the Sobolev spaces based on L^2 with the Sobolev norm

$$\|u\|_m^2 = |(1-\Delta^2)^{m/2} u|_2^2$$

The corresponding norm on $\mathcal{L}_m^2 = L^2([0,T]; L^2(\Omega, \mathcal{F}, \mathbb{P}; H^m(\mathbb{T}^3)))$ is

$$\|u\|_{\mathcal{L}_m^2} = \left[\int_0^T E(\|u\|_m^2)dt\right]^{1/2}$$

We will abuse notation slightly in this section by writing u instead of w. This is done for future reference and an easier comparison with Leray's classical estimates.

Let $\langle \cdot, \cdot \rangle$ denote the inner product on $L^2(\mathbb{T}^3)$. The following a priori estimates provide the foundation of the probabilistic version of Leray's theory.

Lemma 3.1 *The L^2 norms $|u|_2(\omega,t)$ and $|\nabla u|_2(\omega,t)$ satisfy the identity*

$$d|u|_2^2 + 2\nu|\nabla u|_2^2 dt = 2\sum_{k\neq 0} \langle u, h_k^{1/2} e_k\rangle d\beta_t^k + \sum_{k\neq 0} h_k dt \tag{28}$$

and the bounds

$$|u|_2^2(\omega,t) \leq |u|_2^2(0)e^{-2\nu\lambda_1 t} + 2\sum_{k\neq 0}\int_0^t e^{-2\nu\lambda_1(t-s)}\langle u, h_k^{1/2}e_k\rangle d\beta_s^k + \frac{1-e^{-2\nu\lambda_1 t}}{2\nu\lambda_1}\sum_{k\neq 0} h_k \tag{29}$$

$$\int_0^t |\nabla u|_2^2(\omega,s)ds \leq \frac{1}{2\nu}(|u|_2^2(0) - |\mathbf{U}|^2) + \frac{1}{\nu}\sum_{k\neq 0}\int_0^t \langle u, h_k^{1/2}e_k\rangle d\beta_s^k + \frac{t}{2\nu}\sum_{k\neq 0} h_k \tag{30}$$

where λ_1 is the smallest eigenvalue of $-\Delta$ with vanishing boundary conditions on the box $[0,1]^3$ and $h_k = |h_k^{1/2}|^2$. $\mathbf{U}$ *is the velocity vector from the previous section. The expectations of these norms are also bounded*

$$E(|u|_2^2)(t) \leq E(|u|_2^2(0))e^{-2\nu\lambda_1 t} + \frac{1-e^{-2\nu\lambda_1 t}}{2\nu\lambda_1}\sum_{k\neq 0} h_k \tag{31}$$

$$E\left(\int_0^t |\nabla u|_2^2(s)ds\right) \leq \frac{1}{2\nu}[E(|u|_2^2(0)) - |\mathbf{U}|^2] + \frac{t}{2\nu}\sum_{k\neq 0} h_k \tag{32}$$

Proof: The identity (28) follows from Leray's theory and Ito's Lemma. We apply Ito's Lemma to the L^2 norm of u squared,

$$d\int_{\mathbb{T}^3}|u|^2dx = 2\int_{\mathbb{T}^3}\frac{\partial u}{\partial t}\cdot udxdt + 2\sum_{k\neq 0}\int_{\mathbb{T}^3}u\cdot h_k^{1/2}e_k dxd\beta_t^k + \sum_{k\neq 0}h_k\int_{\mathbb{T}^3}dxdt \quad (33)$$

where $k\in\mathbb{Z}^3$ and $h_k^{1/2}\in\mathbb{R}^3$. Now by use of the Navier-Stokes equation (18)

$$\begin{aligned} d|u|_2^2 &= 2\int_{\mathbb{T}^3}\nu\Delta u\cdot u + (-u\cdot\nabla u + \nabla\Delta^{-1}(trace(\nabla u)^2)\cdot udxdt \\ &\quad + 2\sum_{k\neq 0}\int_{\mathbb{T}^3}u\cdot h_k^{1/2}e_k dxd\beta_t^k + \sum_{k\neq 0}h_k dt \\ &= -2\nu|\nabla u|_2^2 dt + 2\sum_{k\neq 0}\int_{\mathbb{T}^3}u\cdot h_k^{1/2}e_k dxd\beta_t^k + \sum_{k\neq 0}h_k dt \end{aligned}$$

since the divergent-free vector u is orthogonal both to the gradient $\nabla\Delta^{-1}(trace(\nabla u)^2)$ and $u\cdot\nabla u$ by the divergence theorem. Notice that the inner product (average) of u and the stirring force f in equation (18) vanishes, $\langle u,f\rangle = \bar{u}\cdot f = 0$, so f can be omitted in the computation. The first term in the last expression is obtained by integration by parts. This is the identity (28). The inequality (29) is obtained by applying Poincaré's inequality

$$\lambda_1|u|_2^2 \leq |\nabla u|_2^2 \quad (34)$$

where λ_1 is the smallest eigenvalue of $-\Delta$ with vanishing boundary conditions on the cube $[0,1]^3$. [2] By Poincaré's inequality

$$\begin{aligned} d|u|_2^2 + 2\nu\lambda_1|u|_2^2 dt &\leq d|u|_2^2 + 2\nu|\nabla u|_2^2 dt \\ &= 2\sum_{k\neq 0}\langle u, h_k^{1/2}e_k\rangle d\beta_t^k + \sum_{k\neq 0}h_k dt \end{aligned}$$

Solving the inequality gives (29). (30) is obtained by integrating (28)

$$|u|_2^2(t) + 2\nu\int_0^t|\nabla u|_2^2(s)ds = |u|_2^2(0) + 2\sum_{k\neq 0}\int_0^t\langle u, h_k^{1/2}e_k\rangle d\beta_s^k + t\sum_{k\neq 0}h_k$$

[2] We should subtract the mean from u in Poincaré's inequality because of the periodic boundary conditions, but the mean just washes out in the estimates.

and dropping $|u-\mathbf{U}|_2^2(t)>0$, by use of (27).

Finally we take the expectations of (29) and (30) to obtain respectively (31) and (32), using that the function $\langle u, h_k^{1/2} e_k\rangle(\omega,t)$ is adapted to the filtration $\mathcal{F}_t$. **QED**

The following amplification of Leray's a priori estimates will play an important role in the a priori estimates of the solution of the stochastic Navier-Stokes equation below.

Lemma 3.2 *Let $u_{\frac{1}{2B}} = u(x,t+\frac{1}{2B})$ denote the translation of u in time by the number $\frac{1}{2B}$. Then the L^2 norms of the differences $|u-u_{\frac{1}{2B}}|_2(\omega,t)$ and $|\nabla u - \nabla u_{\frac{1}{2B}}|_2(\omega,t)$ satisfy the identity*

$$d|u-u_{\frac{1}{2B}}|_2^2 + 2\nu|\nabla u-\nabla u_{\frac{1}{2B}}|_2^2 dt = 2\sum_{k\neq 0}\langle u-u_{\frac{1}{2B}}, h_k^{1/2}e_k\rangle d(\beta_t^k - \beta_{t+\frac{1}{2B}}^k) \tag{35}$$

and the bounds

$$\begin{aligned} |u-u_{\frac{1}{2B}}|_2^2(\omega,t) &\le |u-u_{\frac{1}{2B}}|_2^2(0)e^{-2\nu\lambda_1 t} \\ &+ 2\sum_{k\neq 0}\int_0^t e^{-2\nu\lambda_1(t-s)}\langle u-u_{\frac{1}{2B}}, h_k^{1/2}e_k\rangle d(\beta_s^k - \beta_{t+\frac{1}{2B}}^k) \end{aligned} \tag{36}$$

$$\begin{aligned} \int_0^t |\nabla u-\nabla u_{\frac{1}{2B}}|_2^2(\omega,s)ds &\le \frac{1}{2\nu}|u-u_{\frac{1}{2B}}|_2^2(0) \\ &+ \frac{1}{\nu}\sum_{k\neq 0}\int_0^t \langle u-u_{\frac{1}{2B}}, h_k^{1/2}e_k\rangle d(\beta_s^k - \beta_{t+\frac{1}{2B}}^k) \end{aligned} \tag{37}$$

where λ_1 is the smallest eigenvalue of $-\Delta$ with vanishing boundary conditions on the box $[0,1]^3$ and $h_k = |h_k^{1/2}|^2$. The expectations of these norms are also bounded

$$E(|u-\nabla u_{\frac{1}{2B}}|_2^2)(t) \le E(|u-\nabla u_{\frac{1}{2B}}|_2^2(0))e^{-2\nu\lambda_1 t} \tag{38}$$

$$E(\int_0^t |\nabla u-\nabla u_{\frac{1}{2B}}|_2^2(s)ds) \le \frac{1}{2\nu}E(|u-\nabla u_{\frac{1}{2B}}|_2^2(0)) \tag{39}$$

by the expectations of the initial data of the differences.

Proof: The Navier-Stokes equation for $u-u_{\frac{1}{2B}}$ is obtained from (18) and the stochastic Navier-Stokes for $u-u_{\frac{1}{2B}}$ is

$$\begin{aligned} d(u-u_{\frac{1}{2B}}) = [\nu\Delta(u-u_{\frac{1}{2B}}) - (u\cdot\nabla u - (u\cdot\nabla u)_{\frac{1}{2B}}) &+ \nabla\Delta^{-1}(\text{trace}(\nabla u)^2 - \text{trace}(\nabla u)^2_{\frac{1}{2B}})]dt \\ &+ \sum_{k\neq 0} h_k^{1/2} e_k(x) d(\beta_t^k - \beta^k_{t+\frac{1}{2B}}) \end{aligned}$$

We apply Ito's Lemma to the L^2 norm of $u-u_{\frac{1}{2B}}$ squared,

$$\begin{aligned} d\int_{\mathbb{T}^3} |u-u_{\frac{1}{2B}}|^2 dx &= 2\int_{\mathbb{T}^3} \frac{\partial(u-u_{\frac{1}{2B}})}{\partial t}\cdot(u-u_{\frac{1}{2B}})dxdt \\ &+ 2\sum_{k\neq 0}\int_{\mathbb{T}^3}(u-u_{\frac{1}{2B}})\cdot h_k^{1/2} e_k dx d(\beta_t^k - \beta^k_{t+\frac{1}{2B}}) \end{aligned} \tag{40}$$

where $k\in\mathbb{Z}^3$ and $h_k^{1/2}\in\mathbb{R}^3$, again the stirring force $f-f_{\frac{1}{2B}}$ washes out. Now by use of the Navier-Stokes equation (18)

$$\begin{aligned} & d|u-u_{\frac{1}{2B}})|_2^2 = 2\int_{\mathbb{T}^3}\nu\Delta(u-u_{\frac{1}{2B}})\cdot(u-u_{\frac{1}{2B}}) + \\ & [(-u\cdot\nabla u + \nabla\Delta^{-1}(\text{trace}(\nabla u)^2) - (-u\cdot\nabla u + \nabla\Delta^{-1}(\text{trace}(\nabla u)^2)_{\frac{1}{2B}})]\cdot(u-u_{\frac{1}{2B}})dxdt \\ + \ & 2\sum_{k\neq 0}\int_{\mathbb{T}^3}(u-u_{\frac{1}{2B}})\cdot h_k^{1/2} e_k dx d(\beta_t^k - \beta^k_{t+\frac{1}{2B}}) = -2\nu|\nabla u - u_{\frac{1}{2B}}|_2^2 dt \\ + \ & 2\sum_{k\neq 0}\int_{\mathbb{T}^3}(u-u_{\frac{1}{2B}})\cdot h_k^{1/2} e_k dx d(\beta_t^k - \beta^k_{t+\frac{1}{2B}}) \end{aligned}$$

It is easily checked that $u_{\frac{1}{2B}}$ is divergence free and since the divergent-free vector $u-u_{\frac{1}{2B}}$ is orthogonal both to the gradients $\nabla\Delta^{-1}(trace(\nabla u)^2)$, $\nabla\Delta^{-1}(trace(\nabla u)^2)_{\frac{1}{2B}}$ and inertial terms $u\cdot\nabla u$, $(u\cdot\nabla u)_{\frac{1}{2B}}$ by the divergence theorem, these terms integrate to zero. The first term in the last expression is obtained by integration by parts. This is the identity (35). The remainder of the proof is similar to the rest of the proof of Lemma 3.1. **QED**

Remark 3.1 Notice that in the notation of the previous section $|w-w_{\frac{1}{2B}}|_2^2 = |u-u_{\frac{1}{2B}}|_2^2$ because the constant velocity **U** cancels out.

4 An Estimate of the Turbulent Solutions

The mechanism of the turbulence production are fast oscillations driving large turbulent noise, that was initially seeded by small white noise. These fast oscillations are generated by the fast constant flow $U = U_1$, where we have dropped the subscript 1, and the flow is rotating with amplitude A and angular velocity Ω. The frequency of these oscillations increases with U and $A\Omega$. The bigger U and A are the more efficient this turbulence production mechanism becomes. Next lemma plays a key role in the proof of the useful estimate of the turbulent solution. It is a version of the Riemann-Lebesgue Lemma which captures the averaging effect (mixing) of the oscillations.

Lemma 4.1 *Let the Fourier transform in time be*

$$\tilde{w} = \int_0^T w(s) e^{-2\pi i (k_1 U + A(k_2,k_3)\Omega)s} ds$$

where $A(k_2,k_3) = A\sqrt{k_2^2 + k_3^2}$ *and* $w = w(k,t)$, $k = (k_1,k_2,k_3)$ *is a vector with three components. If* T *is an even integer multiple of* $\frac{1}{k_1 U + A(k_2,k_3)\Omega}$, *then*

$$\tilde{w} = \widetilde{\eth w} \tag{41}$$

where

$$\eth w = \frac{1}{2}(w(s) - w(s + \frac{1}{2[k_1 U + A(k_2,k_3)\Omega]})) = \frac{1}{2}\int_{s+\frac{1}{2|k_1 U + A(k_2,k_3)\Omega|}}^{s} \frac{\partial w}{\partial r} dr \tag{42}$$

and $\eth w$ *satisfies the estimate*

$$|\eth w| \leq \frac{1}{4|k_1 U + A(k_2,k_3)\Omega|} \text{ess sup}_{[s,s+\frac{1}{2(k_1 U_1 + A(k_2,k_3)\Omega)}]} |\frac{\partial w}{\partial s}| \tag{43}$$

Proof: The proof is similar to the proof of the Riemann-Lebesgue lemma for the Fourier transform in time, let $B(k) = k_1 U + A(k_2,k_3)\Omega$,

$$\begin{aligned}
\tilde{w}(k) &= \int_0^T w(s) e^{-2\pi i B s} ds \\
&= -\int_0^T w(s) e^{-2\pi i B (s - \frac{1}{2B})} ds \\
&= -\int_0^T w(s + \frac{1}{2B}) e^{-2\pi i B s} ds
\end{aligned}$$

where we have used in the last step that w is a periodic function on the interval $[0,T]$. Taking the average of the first and the last expression we get

$$\tilde{w} = \frac{1}{2}\int_0^T (w(s) - w(s+\frac{1}{2B}))e^{-2\pi iBs}ds = \widetilde{\eth w}$$

Now

$$\begin{aligned} |\eth w| &= \frac{1}{2}|(w(s) - w(s+\frac{1}{2B}))| \\ &\leq \frac{1}{2}\int_s^{s+\frac{1}{2B}} |\frac{\partial w}{\partial r}|dr \\ &\leq \frac{1}{4|B|}\text{ess sup}_{[s,s+\frac{1}{2B}]}|\frac{\partial w}{\partial s}| \end{aligned}$$

by the mean-value theorem. **QED**

Corollary 4.1 *If T is not an even integer multiple of $\frac{1}{B(k)} = \frac{1}{k_1U + A(k_2,k_3)\Omega}$, then*

$$\tilde{w} = \widetilde{\eth w} - \frac{1}{2}\int_{-\frac{1}{2B}}^0 w(s+\frac{1}{2B})e^{-2\pi iBs}ds + \frac{1}{2}\int_{T-\frac{1}{2B}}^T w(s+\frac{1}{2B})e^{-2\pi iBs}ds \tag{44}$$

where $\tilde{w}$ satisfies the estimate

$$|\tilde{w}| \leq |\widetilde{\eth w}| + \frac{1}{|B|}\text{ess sup}_{[-\frac{1}{2B},0]\cap[T-\frac{1}{2B},T]}|w(s+\frac{1}{2B})| \tag{45}$$

Proof: The proof is the same as of the Lemma except for the step

$$\tilde{w}(k) = \int_0^T w(s)e^{-2\pi iBs}ds = -\int_0^T w(s)e^{-2\pi iB(s-\frac{1}{2B})}ds =$$
$$-\int_0^T w(s+\tfrac{1}{2B})e^{-2\pi iBs}ds - \int_{-\frac{1}{2B}}^0 w(s+\tfrac{1}{2B})e^{-2\pi iBs}ds + \int_{T-\frac{1}{2B}}^T w(s+\tfrac{1}{2B})e^{-2\pi iBs}ds$$

QED

The lemma allows us to estimate the Fourier transform (in t) of w in terms of the time derivative of w, with a gain of $(k_1U + A(k_2,k_3)\Omega)^{-1}$. Below we will use it in an estimate showing that the limit of $\eth w$ is zero when $|B(k)| = |(k_1U + A(k_2,k_3)\Omega)| \to \infty$.

Lemma 4.2 *The integral*

$$\int_0^t (2\pi|k|)^p e^{-(4\pi^2\nu|k|^2+2\pi i[B(k)(t-s)+g])} ds$$

where $B(k) = k_1 U + A(k_2,k_3)\Omega$*, is bounded by*

$$(2\pi)^p \int_0^t |k|^p e^{-4\pi^2\nu|k|^2(t-s)} ds \leq C\, t^{1-\frac{p}{2}} \tag{46}$$

for $0 \leq p < 2$*, where C is a constant. In particular,*

$$\int_{t-\delta}^t (2\pi|k|)^p e^{-(4\pi^2\nu|k|^2+2\pi i[B(k)(t-s)+g])} ds \leq C\, \delta^{1-\frac{p}{2}}. \tag{47}$$

Proof: We estimate the integral

$$\begin{aligned}\int_0^t |k|^p e^{-4\pi^2\nu|k|^2(t-s)} ds &= \int_0^t |k|^p e^{-4\pi^2\nu|k|^2 r} dr \\ &\leq (\frac{p}{4\pi^2})^{\frac{p}{2}} e^{-p} \int_0^t r^{-\frac{p}{2}} dr = Ct^{1-\frac{p}{2}},\end{aligned}$$

where

$$k = \frac{1}{2\pi}\sqrt{\frac{p}{r}}$$

is the value of k where the integrand achives its maximum. **QED**

Lemma 4.3 *Suppose that for* $k_1 < 0$ *and* $\frac{\sqrt{k_2^2+k_3^2}}{|k_1|} \neq 0$ or ∞, *the constants* U, A *and* Ω *satisfy the non-resonance condition*

$$\left|\frac{U}{A\Omega} + \frac{\sqrt{k_2^2+k_3^2}}{k_1}\right| \geq \frac{C}{|k_1|^r} \tag{48}$$

where C is a constant and $0 < r < 1$; *then for all* $k = (k_1,k_2,k_2) \neq 0$*,*

$$|Uk_1 + A\Omega\sqrt{k_2^2+k_3^2}| \neq 0 \tag{49}$$

and

$$\lim_{|k|\to\infty} |Uk_1 + A\Omega\sqrt{k_2^2+k_3^2}| = \infty. \tag{50}$$

Moreover,

$$|Uk_1 + A\Omega\sqrt{k_2^2+k_3^2}| \geq B = \min(U, A\Omega, CA\Omega). \tag{51}$$

Proof: If $k_1 > 1$, then

$$|Uk_1 + A\Omega\sqrt{k_2^2 + k_3^2}| = U|k_1| + A\Omega\sqrt{k_2^2 + k_3^2} > 0$$

so (49) and (50) hold. If $k_1 < 0$, then by (48)

$$|Uk_1 + A\Omega\sqrt{k_2^2 + k_3^2}| \geq C\,\Omega A|k_1|^{1-r} > 0$$

and

$$\lim_{|k|\to\infty} |Uk_1 + A\Omega\sqrt{k_2^2 + k_3^2}| \geq C\,\Omega A \lim_{|k_1|\to\infty} |k_1|^{1-r} = \infty$$

if $|k_1| \to \infty$. If on the other hand $|k_1| < \infty$ when $|k| \to \infty$ then (50) also holds. When $k_1 = 0$, (49) and (50) are obvious and also if $k_2 = k_3 = 0$.

The lower bound (51) is read of

$$|Uk_1 + A\Omega\sqrt{k_2^2 + k_3^2}|$$

when $k_1 \geq 1$. Then it is either U or $A\Omega$. When $k_1 = 0$ then it is $A\Omega$ and by (48), when $k_1 \leq -1$ it greater than or equal $CA\Omega$. **QED**

The next question to ask is in which space do the turbulent solutions live? This was determined by Onsager in 1945 [24]. He pointed out that if the solutions satisfy the Kolmogorov scaling down to the smallest scales, they must be Hölder continuous function with Hölder exponent 1/3. In three dimensions this means that they live in the Sobolev space $H^{\frac{11}{6}+\varepsilon}$ based on $L^2(\mathbb{T}^3)$.

If $\frac{q}{p}$ is a rational number let $\frac{q}{p}^+$ denote any real number $s > \frac{q}{p}$.

Theorem 4.1 *Let the velocity $U = U_1$ of the mean flow and the product $A\Omega$ of the amplitude A and the frequency Ω of the rotation be sufficiently large, in the uniform rotating flow (17), with U, $A\Omega$ also satisfying the non-resonance conditions (48). Then the solution of the integral equation (22) is uniformely bounded in $\mathcal{L}^2_{\frac{11}{6}^+}$,*

$$\text{ess sup}_{t\in[0,\infty)} E(\|u\|^2_{\frac{11}{6}^+})(t) \leq (1 - C(\frac{1}{B^2} + \delta^{\frac{1}{6}^-}))^{-1} \left[\sum_{k\neq 0} \frac{3(1+(2\pi|k|)^{\frac{11}{3}^+})}{8\pi^2\nu|k|^2} h_k + \frac{C'}{B}\right] \tag{52}$$

where $B = min(|U|, A\Omega, CA\Omega)$ is large, δ small and C, C' are constants.

Corollary 4.2 Onsager's Conjecture *The solutions of the integral equation (22) are Hölder continuous with exponent* $1/3$.

Remark 4.1 The estimate (52) provides the answer to the question we posed in Section 2 how fast the coefficients $h_k^{1/2}$ had to decay in Fourier space. They have to decay sufficiently fast for the expectation of the $H^{\frac{11}{6}^+} = W^{(\frac{11}{6}^+,2)}$ Sobolev norm of the initial function u_o, to be finite. This expectation appear on the right hand side of (52). In other words the $\mathcal{L}^2_{\frac{11}{6}^+}$ norm of the initial function u_o has to be finite.

Proof: We write the integral equation (22) in the form

$$u(x,t) = \sum_{k\neq 0}[h_k^{1/2}A_t^k - \int_0^t e^{-(\{4\pi^2\nu|k|^2+2\pi i[k_1U_1+A(k_2,k_3)\Omega]\}(t-s)+2\pi i g(k,t,s))} \times(\widehat{u\cdot\nabla u} - \nabla\widehat{\Delta^{-1}(\mathrm{tr}(\nabla}u)^2))(k,s)ds]e_k(x)$$

where $e_k = e^{2\pi i k\cdot x}$ are the Fourier components and the A_t^k are the oscillatory Ornstein-Uhlenbeck-type processes (25) and $\mathrm{tr}(\nabla u)^2$ denotes the trace of the matrix $(\nabla u)^2$. The Fourier transform of the term $\nabla\Delta^{-1}(\mathrm{tr}(\nabla u)^2)$ is just $\frac{-ik}{2\pi|k|^2}\mathrm{tr}\widehat{(\nabla u)}^2$ and we will write the integral equation in the form

$$u(x,t) = \sum_{k\neq 0}[h_k^{1/2}A_t^k - \int_0^t e^{-[4\pi^2\nu|k|^2+2\pi i B(k)](t-s)-2\pi i g(k,t,s)} \times (\widehat{u\cdot\nabla u} + \frac{ik}{2\pi|k|^2}(\widehat{\mathrm{tr}(\nabla u)^2}))(k,s)ds]e_k(x) \tag{53}$$

where $B(k) = Uk_1 + A(k_2,k_3)\Omega$, from here on with

$$g(k,t,s) = A(k_2,k_3)[\Omega(t-s) - (\sin(\Omega t+\theta) - \sin(\Omega s+\theta))] \tag{54}$$

We will also assume the trivial non-resonance conditions that A and Ω are sufficiently incommensurate for the rest of the paper.

We split the t integral into the integral from 0 to $t-\delta$, where δ is a small number, and the integral from $t-\delta$ to t. This is done to first avoid the singularities of the spatial derivatives of the heat kernal at $s=t$ and then to deal with these singularities in the latter integral. Now the first estimate is relatively straightforward. The L^2 norm of

$$\sum_{k\neq 0}\int_{t-\delta}^t e^{-\{(4\pi^2\nu|k|^2+2\pi i B)(t-s)+2\pi i g(k,t,s)\}}(-\widehat{u\cdot\nabla u})ds\ e_k$$

is

$$\sum_{k\neq 0}|\int_{t-\delta}^{t} e^{-\{(4\pi^2\nu|k|^2+2\pi iB)(t-s)+2\pi ig(k,t,s)\}}(-\widehat{u\cdot\nabla u})ds|^2 \ \le\ \delta\sum_{k\neq 0}\int_{t-\delta}^{t}|\widehat{u\cdot\nabla u}|_2^2(k)ds$$

$$\le\ \delta\int_{t-\delta}^{t}|u\cdot\nabla u|_2^2 ds\ \le\ \delta\,\text{ess sup}_{[t-\delta,t]}|u|_\infty^2\int_{t-\delta}^{t}|\nabla u|_2^2 ds$$

$$\le\ (\frac{\delta}{\nu}\int_{t-\delta}^{t}\langle u,h_k^{1/2}e_k\rangle d\beta_s^k+\frac{\delta^2}{2\nu}\sum_{k\neq 0}h_k)\ \text{ess sup}_{[t-\delta,t]}\|u\|_{\frac{3}{2}^+}(s) \tag{55}$$

since by the Gagliardo-Nirenberg inequalities

$$|u|_\infty\le C\|u\|_{\frac{3}{2}^+},$$

where δ is independent of U_1 and C is a constant, and by the a priori estimate in Lemma 3.1. Similarly, the L^2 norm of

$$\sum_{k\neq 0}\int_{t-\delta}^{t} e^{-\{(4\pi^2\nu|k|^2+2\pi iB)(t-s)+2\pi ig(k,t,s)\}}(\frac{ik}{2\pi|k|^2}(\widehat{\text{tr}(\nabla u)^2}))ds\ e_k$$

is

$$\sum_{k\neq 0}|\int_{t-\delta}^{t} e^{-\{(4\pi^2\nu|k|^2+2\pi iB)(t-s)+2\pi ig(k,t,s)\}}(\frac{ik}{2\pi|k|^2}(\widehat{\text{tr}(\nabla u)^2}))ds|^2 \tag{56}$$

$$\le\ \delta\sum_{k\neq 0}\int_{t-\delta}^{t}|(\frac{ik}{2\pi|k|^2}(\widehat{\text{tr}(\nabla u)^2}))|_2^2(k)ds$$

$$\le\ \frac{\delta}{2\pi}\int_{t-\delta}^{t}||w||\nabla u||_2^2 ds\ \le\ \frac{\delta}{2\pi}\,\text{ess sup}_{[t-\delta,t]}|w|_\infty^2\int_{t-\delta}^{t}|\nabla u|_2^2 ds$$

$$\le\ (\frac{\delta}{2\pi\nu}\int_{t-\delta}^{t}\langle u,h_k^{1/2}e_k\rangle d\beta_s^k+\frac{\delta^2}{4\pi\nu}\sum_{k\neq 0}h_k)\ \text{ess sup}_{[t-\delta,t]}\|u\|_{\frac{3}{2}^+}^2(s) \tag{57}$$

where $w=\sum_{k\neq 0}\frac{k}{|k|^2}|k\otimes\hat{u}(k,s)|e_k$, $|w|_2=|u|_2$.

The other integrals are estimated by use of Lemma 4.1. The integral

$$\int_0^{t-\delta} e^{-\{(4\pi^2\nu|k|^2+2\pi iB)(t-s)+2\pi ig(k,t,s)\}}\widehat{u\cdot\nabla u}ds$$

can be estimated by Lemma 4.1, when $t-\delta$ is an even integer multiple of $\frac{1}{B}$, we get that

$$\int_0^{t-\delta} e^{-\{(4\pi^2\nu|k|^2+2\pi iB)(t-s)+2\pi ig(k,t,s)\}}\widehat{u\cdot\nabla u}(s)ds =$$

$$\begin{aligned}
&\tfrac{1}{2} \int_0^{t-\delta}[e^{-\{4\pi^2\nu|k|^2(t-s)+2\pi ig(k,t,s)\}}\widehat{u\cdot\nabla u}(s) \\
-\ & e^{-\{(4\pi^2\nu|k|^2+2\pi iB)(t-(s+\frac{1}{2B}))+2\pi ig(k,t,s+\frac{1}{2B})\}}\widehat{u\cdot\nabla u}(s+\frac{1}{2B})]e^{-2\pi iB(t-s)}ds \\
=\ & \frac{1}{2}\int_0^{t-\delta}[(e^{-\{4\pi^2\nu|k|^2(t-s)+2\pi ig(k,t,s)\}} \\
-\ & e^{-\{4\pi^2\nu|k|^2(t-(s+\frac{1}{2B}))+2\pi ig(k,t,s+\frac{1}{2B})\}})\widehat{u\cdot\nabla u}(s)]e^{-2\pi iB(t-s)}ds \\
+\ & \frac{1}{2}\int_0^{t-\delta}\{e^{-(4\pi^2\nu|k|^2(t-(s+\frac{1}{2B}))+2\pi ig(k,t,s+\frac{1}{2B})\}}([\hat{u}(s)-\widehat{u(s+\frac{1}{2B})}]*\widehat{\nabla u(s)} \\
+\ & \widehat{u(s+\frac{1}{2B})}*[\widehat{\nabla u(s)}-\widehat{\nabla u(s+\frac{1}{2B})}])\}e^{-2\pi iB(t-s)}ds
\end{aligned}$$

The first term in the last line above is estimated by Schwartzes inequality

$$\begin{aligned}
& |\int_0^{t-\delta}[(e^{-\{4\pi^2\nu|k|^2(t-s)+2\pi ig(k,t,s)\}} \\
-\ & e^{-\{4\pi^2\nu|k|^2(t-(s+\frac{1}{2B}))+2\pi ig(k,t,s+\frac{1}{2B})\}})\widehat{u\cdot\nabla u}(s)]e^{-2\pi iB(t-s)}ds|^2 \\
\leq\ & \int_0^{t-\delta}|e^{-\{2\pi^2\nu|k|^2(t-s)+2\pi ig(k,t,s)\}} \\
-\ & e^{-\{2\pi^2\nu|k|^2(t-(s+\frac{1}{B}))+2\pi ig(k,t,s+\frac{1}{2B})\}}|^2|u|_2^2(s)\ ds\ \int_0^{t-\delta}e^{-4\pi^2\nu|k|^2(t-s)}|\nabla u|_2^2(s)ds \\
\leq\ & e^{-2\pi^2\nu|k|\delta}\ \int_0^{t-\delta}|e^{-\{2\pi^2\nu|k|^2(t-s)+2\pi ig(k,t,s)\}}-e^{-\{2\pi^2\nu|k|^2(t-(s+\frac{1}{B}))+2\pi ig(k,t,s+\frac{1}{2B})\}}|^2ds \\
\times\ & \int_0^{t-\delta}e^{-2\pi^2\nu|k|^2(t-s)}|\nabla u|_2^2(s)ds\ \text{ess sup}_{s\in[0,t-\delta]}|u|_2^2(s) \\
\leq\ & \frac{Ce^{-2\pi^2\nu|k|\delta}}{B^2}\ \int_0^{t-\delta}e^{-2\pi\nu|k|^2(t-s)}|\nabla u|_2^2(s)ds\ \text{ess sup}_{s\in[0,t-\delta]}|u|_2^2(s)
\end{aligned}$$

by Lemma 4.1. Similarly the second term is estimated by

$$
\begin{aligned}
& |\int_0^{t-\delta} e^{-(4\pi^2\nu|k|^2(t-(s+\frac{1}{2B}))+2\pi i g(k,t,s+\frac{1}{2B}))}([\hat{u}(s)-\widehat{u(s+\frac{1}{2B})}]*\widehat{\nabla u(s)}e^{-2\pi iB(t-s)}ds|^2 \\
\leq\ & e^{-4\pi^2\nu|k|^2(\delta-\frac{1}{2B})}\int_0^{t-\delta}|u(s)-u(s+\frac{1}{2B})|_2^2 ds \int_0^{t-\delta} e^{-4\pi^2\nu|k|^2 s}|\nabla u|_2^2(s)ds
\end{aligned}
$$

using the Cauchy-Schwartz inequality both on the convolution and the time-integral, and the third term is estimated by

$$
\begin{aligned}
& |\int_0^{t-\delta} e^{-(4\pi^2\nu|k|^2(t-(s+\frac{1}{2B}))+2\pi i g(k,t,s+\frac{1}{2B}))}(\widehat{u(s+\frac{1}{2B})}*[\widehat{\nabla u(s)}-\widehat{\nabla u(s+\frac{1}{2B})}]e^{-2\pi iB(t-s)}ds|^2 \\
& \leq \frac{e^{-8\pi^2\nu|k|(\delta-\frac{1}{2B})}}{8\nu\pi^2|k|^2}\int_0^{t-\delta}|\nabla u(s)-\nabla u(s+\frac{1}{2B})|_2^2 ds \text{ ess sup}_{s\in[0,t]}|u|_2^2(s+\frac{1}{2B})
\end{aligned}
$$

Now the terms

$$
H=\int_0^{t-\delta}|u(s)-u(s+\frac{1}{2B})|_2^2 ds \int_0^{t-\delta} e^{-4\pi^2\nu|k|^2 s}|\nabla u|_2^2(s)ds
$$

and

$$
K=\int_0^{t-\delta}|\nabla u(s)-\nabla u(s+\frac{1}{2B})|_2^2 ds \text{ ess sup}_{s\in[0,t]}|u|_2^2(s+\frac{1}{2B})
$$

are estimated by use of Lemma 3.2 and Lemma 4.4. Thus the a priori bounds on the L^2 norms of u and ∇u and their differences in those two lemmas and in Lemmas 3.1 and 4.5 give the inequality

$$
|\int_0^{t-\delta} e^{-\{(4\pi^2\nu|k|^2+2\pi iB)(t-s)+2\pi i g(k,t,s)\}}\widehat{u\cdot\nabla u}(s)ds|^2
$$

$$
\leq Ce^{-4\pi^2\nu|k|^2(\delta-\frac{1}{2B})}\text{ ess sup}_{s\in[0,t]}(\tfrac{C}{B^2}+H+K+d(k))
$$

and an estimate of its right hand side. The terms H and K are estimated in Lemma 4.6 and the expectation of $d(k)$ vanishes.

Now consider the pressure term. By use of Lemma 4.1, we get that

$$
\begin{aligned}
& \int_0^{t-\delta} e^{-\{4\pi^2\nu|k|^2+2\pi iB)(t-s)+2\pi i g(k,t,s)\}}\frac{ik}{2\pi|k|^2}\widehat{\mathrm{tr}(\nabla u)^2}ds \ = \\
& \frac{1}{2}\int_0^{t-\delta}\{e^{-\{4\pi^2\nu|k|^2(t-s)+2\pi i g(k,t,s)\}}\frac{ik}{2\pi|k|^2}\widehat{\mathrm{tr}(\nabla u)^2}(s) \\
- \ & e^{-\{4\pi^2\nu|k|^2(t-(s+\frac{1}{2B}))+2\pi i g(k,t,s+\frac{1}{2B})\}}\frac{ik}{2\pi|k|^2}\widehat{\mathrm{tr}(\nabla u)^2}(s+\frac{1}{2B})\}e^{-2\pi iB(t-s)}ds
\end{aligned}
$$

$$
\begin{aligned}
= \;& \frac{1}{2}\int_0^{t-\delta}\{e^{-\{4\pi^2\nu|k|^2(t-s)+2\pi i g(k,t,s)\}} \\
- \;& e^{-\{4\pi^2\nu|k|^2(t-(s+\frac{1}{2B}))+2\pi i g(k,t,s+\frac{1}{2B})\}}\}\frac{ik}{2\pi|k|^2}\widehat{\mathrm{tr}(\nabla u)^2}(s)ds \\
+ \;& \frac{1}{2}\int_0^{t-\delta} e^{-\{4\pi^2\nu|k|^2(t-(s+\frac{1}{2B}))+2\pi i g(k,t,s+\frac{1}{2B})\}} \\
\times \;& \frac{ik}{2\pi|k|^2}\mathrm{tr}[(\widehat{\nabla u(s)}-\widehat{\nabla u(s+\frac{1}{2B}})) * (\widehat{\nabla u(s)}+\widehat{\nabla u(s+\frac{1}{2B}}))]e^{-2\pi i B(t-s)}ds
\end{aligned}
$$

The first term in the last expression above is estimated as

$$
\begin{aligned}
& |\int_0^{t-\delta}\{e^{-\{4\pi^2\nu|k|^2(t-s)+2\pi i g(k,t,s)\}} \\
- \;& e^{-\{4\pi^2\nu|k|^2(t-(s+\frac{1}{2B}))+2\pi i g(k,t,s+\frac{1}{2B})\}}\}\frac{ik}{2\pi|k|^2}\widehat{\mathrm{tr}(\nabla u)^2}(s)ds|^2 \\
\leq \;& \int_0^{t-\delta} |e^{-\{2\pi^2\nu|k|^2(t-s)+2\pi i g(k,t,s)\}} \\
- \;& e^{-\{2\pi^2\nu|k|^2(t-(s+\frac{1}{B}))+2\pi i g(k,t,s+\frac{1}{2B})\}}|^2|w|_2^2(s)ds \;\int_0^{t-\delta} e^{-4\pi\nu|k|^2(t-s)}|\nabla u|_2^2(s)ds \\
\leq \;& e^{-2\pi^2\nu|k|\delta}\int_0^{t-\delta}|e^{-\{2\pi^2\nu|k|^2(t-s)+2\pi i g(k,t,s)\}} \\
- \;& e^{-\{2\pi^2\nu|k|^2(t-(s+\frac{1}{B}))+2\pi i g(k,t,s+\frac{1}{2B})\}}|^2 ds\int_0^{t-\delta} e^{-2\pi\nu|k|^2(t-s)}|\nabla u|_2^2(s)ds \\
\times \;& \text{ess sup}_{s\in[0,t-\delta]}|u|_2^2(s) \\
\leq \;& \frac{Ce^{-2\pi^2\nu|k|\delta}}{B^2}\int_0^{t-\delta} e^{-2\pi\nu|k|^2(t-s)}|\nabla u|_2^2(s)ds \;\text{ess sup}_{s\in[0,t-\delta]}|u|_2^2(s)
\end{aligned}
$$

where w is the same function as above and by Lemma 4.1. The second term is estimated by

$$
\begin{aligned}
& |\int_0^{t-\delta} e^{-\{4\pi^2\nu|k|^2(t-(s+\frac{1}{2B}))+2\pi i g(k,t,s+\frac{1}{2B})\}} \\
\times \;& \frac{ik}{2\pi|k|^2}\mathrm{tr}[(\widehat{\nabla u(s)}-\widehat{\nabla u(s+\frac{1}{2B}})) * (\widehat{\nabla u(s)}+\widehat{\nabla u(s+\frac{1}{2B}}))]e^{-2\pi i B(t-s)}ds|^2 \\
\leq \;& e^{-4\pi^2\nu|k|(\delta-\frac{1}{2B})}\int_0^{t-\delta}|\nabla u(s)-\nabla u(s+\frac{1}{2B})|_2^2 ds \;\int_0^{t-(\delta-\frac{1}{2B})} e^{-4\pi^2\nu|k|^2 s}|\nabla u|_2^2(s)ds
\end{aligned}
$$

Thus

$$|\int_0^{t-\delta} e^{-\{(4\pi^2\nu|k|^2+2\pi iB)(t-s)+2\pi i g(k,t,s)\}}\frac{ik}{2\pi|k|^2}\widehat{\mathrm{tr}(\nabla u)^2}ds|^2$$

$$\leq Ce^{-4\pi^2\nu|k|^2(\delta-\frac{1}{2B})}\,\text{ess sup}_{s\in[0,t]}(\frac{C}{|B(k)|^2}+L+d(k))$$

where the expectation of $d(k)$ vanishes and the term

$$L=\int_0^{t-\delta}|\nabla u(s)-\nabla u(s+\frac{1}{2B})|_2^2 ds\ \int_0^{t-(\delta-\frac{1}{2B})} e^{-4\pi^2\nu|k|^2 s}|\nabla u|_2^2(s)ds$$

is estimated in Lemma 4.6, again by the a priori bounds on the L^2 norms of u and ∇u and their differences in Lemma 3.1 and Lemma 4.5, and Lemmas 3.2 and 4.4.

When $t-\delta$ is not an even integer multiple of $\frac{1}{B(k)}$ we get the additional terms in Corollary 4.1. However these are estimated exactly as the integrals from $t-\delta$ to t and simply add another term multiplied by δ^2 if we choose $\frac{1}{|B|}=\sup_{k\neq 0}\frac{1}{|B(k)|}<\delta$.

Now we assemble the estimates. Up to terms that vanish when the expectation is taken, the L^2 norm of u is bounded by

$$\begin{aligned}|u|_2^2 &\leq 3\sum_{k\neq 0} h_k|A_t^k|^2\\ &+ 3\sum_{k\neq 0}|\int_0^{t-\delta} e^{-(\{4\pi^2\nu|k|^2+2\pi i[k_1U_1+A(k_2,k_3)\Omega]\}(t-s)+2\pi i g(k,t,s))}\\ &\times (\widehat{u\cdot\nabla u}-\nabla\Delta^{-1}\widehat{(\mathrm{tr}(\nabla u)^2)})(k,s)ds|^2+\delta^2 C\text{ess sup}_{s\in[t-\delta,t]}\|u\|^2_{\frac{11}{6}^+}\end{aligned}\tag{58}$$

$$\begin{aligned}&\leq 3\sum_{k\neq 0} h_k|A_t^k|^2\\ &+ \sum_{k\neq 0} e^{-4\pi^2\nu|k|(\delta-\frac{1}{2B})}[\frac{C'}{|B(k)|^2}+H+K+L](s)+\delta^2 C\text{ess sup}_{s\in[t-\delta,t]}\|u\|^2_{\frac{11}{6}^+}\\ &\leq 3\sum_{k\neq 0} h_k|A_t^k|^2+C(\frac{1}{B^2}+\delta^2)\text{ess sup}_{s\in[0,t]}\|u\|^2_{\frac{11}{6}^+}+\frac{C'}{B}\end{aligned}$$

by Lemma 4.6.

We now act on the integral equation (53) with the operator $\nabla^{(11/6)^+}$, to estimate the derivative $\nabla^{(11/6)^+}u$

$$
\begin{aligned}
\nabla^{(11/6)^+}u(x,t) &= \sum_{k\neq 0}[(2\pi i|k|)^{(11/6)^+}h_k^{1/2}A_t^k \\
&- \int_0^t (2\pi i|k|)^{(11/6)^+}e^{-[4\pi^2\nu|k|^2+2\pi iB(k)](t-s)-2\pi ig(k,t,s)} \\
&\times \ (\widehat{u\cdot\nabla u}+\frac{ik}{2\pi|k|^2}(\widehat{\mathrm{tr}(\nabla u)^2}))(k,s)ds]e_k(x)
\end{aligned} \tag{59}
$$

where $B(k)$ and $g(k,t,s)$ are as in (53). An estimate similar to Equation (58) now gives

$$
\begin{aligned}
|\nabla^{(11/6)^+}u|_2^2 \leq \ & 3\sum_{k\neq 0}(2\pi|k|)^{(11/3)^+}h_k|A_t^k|^2 \\
&+ 3\sum_{k\neq 0}(|\int_0^{t-\delta}|k|^{\frac{11}{6}^+}e^{-(\{4\pi^2\nu|k|^2+2\pi i[k_1U_1+A(k_2,k_3)\Omega]\}(t-s)+2\pi ig(k,t,s))} \\
&\times \ (\widehat{u\cdot\nabla u}-\nabla\Delta^{-1}\widehat{(\mathrm{tr}(\nabla u)^2)})(k,s)ds|^2 \\
&+ \delta^{\frac{1}{6}^-}C\text{ess sup}_{s\in[t-\delta,t]}\|u\|^2_{\frac{11}{6}^+}
\end{aligned} \tag{60}
$$

$$
\begin{aligned}
&\leq 3\sum_{k\neq 0}(2\pi|k|)^{(11/3)^+}h_k|A_t^k|^2 + \text{ess sup}_{s\in[0,t-\delta]}[\frac{C'}{B^2}+H+K+L](s) \\
&\qquad + \delta^{\frac{1}{6}^-}C\text{ ess sup}_{s\in[t-\delta,t]}\|u\|^2_{\frac{11}{6}^+} \qquad (61)\\
&\leq 3\sum_{k\neq 0}(2\pi|k|)^{(11/3)^+}h_k|A_t^k|^2 + C(\frac{1}{B^2}+\delta^{\frac{1}{6}^-})\text{ess sup}_{s\in[0,t]}\|u\|^2_{\frac{11}{6}^+}+\frac{C'}{B}
\end{aligned}
$$

again by Lemma 4.6.

Combining the estimates (58) and (61) we now get that

$$
\|u\|^2_{\frac{11}{6}^+} \leq 3\sum_{k\neq 0}(1+(2\pi|k|)^{\frac{11}{3}^+})h_k|A_t^k|^2 + C(\frac{1}{B^2}+\delta^{\frac{1}{6}^-})\text{ess sup}_{s\in[0,t]}\|u\|^2_{\frac{11}{6}^+}+\frac{C'}{B}
$$

where $\frac{1}{B}$ and δ can be made arbitrarily small. Then taking the expectation we get

$$
(1-C(\frac{1}{B^2}+\delta^{\frac{1}{6}^-}))E(\text{ess sup}_{[0,t]}\|u\|^2_{\frac{11}{6}^+}) \leq 3\sum_{k\neq 0}(1+(2\pi|k|)^{\frac{11}{3}^+})h_kE(|A_t^k|^2)+\frac{C'}{B} \tag{62}
$$

and evaluating the last expectation

$$\sum_{k\neq 0}(1+(2\pi|k|)^{\frac{11}{3}^+})h_k E(|A_t^k|^2)=\sum_{k\neq 0}\frac{(1+(2\pi|k|)^{\frac{11}{3}^+})}{8\pi^2\nu|k|^2}h_k$$

gives the estimate

$$(1-C(\frac{1}{B^2}+\delta^{\frac{1}{6}^-}))E(\text{ess sup}_{[0,t]}\|u\|^2_{\frac{11}{6}^+})\leq 3\sum_{k\neq 0}\frac{(1+(2\pi|k|)^{\frac{11}{3}^+})}{8\pi^2\nu|k|^2}h_k+\frac{C'}{B}$$

By making δ and $\frac{1}{B}$ sufficiently small we conclude that (52) holds for all t. **QED**

We consider the integral equation

$$\begin{aligned} u(x,t)=\sum_{k\neq 0}[h_k^{1/2}A_t^k \quad &- \quad \int_0^t e^{-[4\pi^2\nu|k|^2+2\pi i B(k)](t-s)-2\pi i g(k,t,s)} \\ &\times \quad (\widehat{u\cdot\nabla u}+\frac{ik}{2\pi|k|^2}(\widehat{\text{tr}(\nabla u)^2}))(k,s)ds]e_k(x) \end{aligned}$$

where $B(k)=Uk_1+A(k_2,k_3)\Omega$. The following three lemmas are used in the proof of Theorem 4.1.

Lemma 4.4 *The initial condition* $(u-u_{\frac{1}{2B}})(0)$ *satisfies the estimate*

$$|u-u_{\frac{1}{2B(k)}}|_2^2(0)\leq 2\sum_{j\neq 0}|A^j_{\frac{1}{2B(k)}}|^2+\frac{C}{|B(k)|^2}\text{ess sup}_{t\in[0,\frac{1}{2B}]}\|u\|^2_{\frac{11}{6}^+} \tag{63}$$

Proof: We use the integral equation

$$\begin{aligned} u \quad - \quad & u_{\frac{1}{2B}}=\sum_{k\neq 0}[h_k^{1/2}(A_t^k-A^k_{t+\frac{1}{2B}}) \\ - \quad & (\int_0^t e^{-[4\pi^2\nu|k|^2+2\pi i B(k)](t-s)-2\pi i g(k,t,s)}\times(\widehat{u\cdot\nabla u}+\frac{ik}{2\pi|k|^2}(\widehat{\text{tr}(\nabla u)^2}))(k,s)ds \\ - \quad & \int_0^{t+\frac{1}{2B}} e^{-[4\pi^2\nu|k|^2+2\pi i B(k)](t+\frac{1}{2B}-s)-2\pi i g(k,t+\frac{1}{2B},s)} \\ \times \quad & (\widehat{u\cdot\nabla u}+\frac{ik}{2\pi|k|^2}(\widehat{\text{tr}(\nabla u)^2}))(k,s)ds]e_k(x) \end{aligned}$$

where $B(k) = Uk_1 + A(k_2,k_3)\Omega$. At $t = 0$,

$$|u - u_{\frac{1}{2B}}|^2(0) = |u_{\frac{1}{2B}}|^2(0) = 2\sum_{j\neq 0} h_j |A^j_{\frac{1}{2B}}|^2 + \frac{C}{|B(k)|^2}\text{ess sup}_{t\in[0,\frac{1}{2B}]}\|u\|^2_{\frac{11}{6}+}$$

by similar estimates as above. **QED**

Lemma 4.5 *The identity (28) in Lemma 3.1 can be modified for $a > 0$*

$$d(e^{\nu at}|u|_2^2) + 2\nu e^{\nu at}|\nabla u|_2^2 dt = \nu a e^{\nu at}|u|_2^2 dt + 2e^{\nu at}\sum_{k\neq 0}\langle u, h_k^{1/2} e_k\rangle d\beta_t^k + e^{\nu at}\sum_{k\neq 0} h_k dt \tag{64}$$

and produces the estimates

$$\begin{aligned} |u|_2^2(t) &\leq |u|_2^2(0)(e^{-\nu at} + \frac{ae^{-2\nu\lambda_1 t}}{(a-2\lambda_1)}) + 2\sum_{k\neq 0}\int_0^t e^{-\nu a(t-s)}\langle u, h_k^{1/2} e_k\rangle d\beta_s^k \\ &+ \quad 2\sum_{k\neq 0}\int_0^t e^{-\nu a(t-s)}\int_0^s e^{-2\nu\lambda_1(s-r)}\langle u, h_k^{1/2} e_k\rangle d\beta_r^k ds + \frac{1}{\nu}(\frac{1}{a} + \frac{1}{2\lambda_1})\sum_{k\neq 0} h_k \end{aligned} \tag{65}$$

and

$$\begin{aligned} \int_0^t e^{-\nu a(t-s)}|\nabla u|_2^2(s)ds &\leq \frac{1}{2\nu}(|u|_2^2(0) - |\mathbf{U}|^2)(e^{-\nu at} + \frac{ae^{-2\nu\lambda_1 t}}{(a-2\lambda_1)}) \\ &+ \frac{1}{\nu}\sum_{k\neq 0}\int_0^t e^{-\nu a(t-s)}\langle u, h_k^{1/2} e_k\rangle d\beta_s^k \\ + \frac{1}{\nu}\sum_{k\neq 0}\int_0^t e^{-\nu a(t-s)}\int_0^s e^{-2\nu\lambda_1(s-r)}&\langle u, h_k^{1/2} e_k\rangle d\beta_r^k ds + \frac{1}{2\nu^2}(\frac{1}{a} + \frac{1}{2\lambda_1})\sum_{k\neq 0} h_k \end{aligned} \tag{66}$$

where λ_1 is the smallest eigenvalue of $-\Delta$ with vanishing boundary conditions on the box $[0,1]^3$ and $h_k = |h_k^{1/2}|^2$.

Proof: We multiply the identity (28) in Lemma 3.1 by $e^{\nu at}$ to get (64). Then integration gives the equality

$$\begin{aligned} |u|_2^2(t) \quad &+ \quad 2\nu\int_0^t e^{-\nu a(t-s)}|\nabla u|_2^2(s)ds = |u|_2^2(0)e^{-\nu at} + \nu a\int_0^t e^{-\nu a(t-s)}|u|_2^2(s)ds \\ &+ \quad 2\sum_{k\neq 0}\int_0^t e^{-\nu a(t-s)}\langle u, h_k^{1/2} e_k\rangle d\beta_s^k + \frac{(1-e^{-\nu a(t-s)})}{\nu a}\sum_{k\neq 0} h_k \end{aligned}$$

Now substituting the estimate (29), from Lemma 3.1, for $|u|_2^2$ on the right hand side gives the two inequalities (65) and (66) as in Lemma 3.1. **QED**

Lemma 4.6 *The functions H,K,L in the proof of Theorem 4.1 satisfy the estimate*

$$E(H+K+L) \leq \frac{C}{|B(k)|^2} E(\text{ess sup}_{t\in[0,\frac{1}{2B}]} \|u\|^2_{\frac{11}{6}+}) + \frac{C'}{B} \tag{67}$$

with $B = min(U, A\Omega, CA\Omega)$.

The proof of the lemma is a straight-forward computation that involves long formulas for H,K and L and is placed in the Appendix.

Remark 4.2 Corollary 4.2 is the resolution of a famous question in turbulence: *Is turbulence always caused by the blow-up of the velocity u ?* The answer according to Theorem 5.1 is *no*; the solutions are not singular. However, they are not smooth either, contrary to the belief, stemming from Leray's theory [17], that if solutions are not singular then they are smooth. By Corollary 4.2 the solutions are Hölder continuous with exponent $1/3$ in three dimensions. This confirms a conjecture made by Onsager [25] in 1945. In particular the gradient ∇u and vorticity $\nabla \times u$ are not continuous in general.

Remark 4.3 U and $A\Omega$ do not have to be made very large for the estimate (52) to be satisfied, because $B(k) \to \infty$ as $|k| \to \infty$. How big U and $A\Omega$ have to be for (52) to hold is probably best answered by a numerical simulation.

We can now prove that $\text{ess sup}_{t\in[0,\infty)} \|u(t)\|^2_{\frac{11}{6}+}$ is bounded with probability close to one.

Lemma 4.7 *For all $\varepsilon > 0$ there exists an R such that,*

$$\mathbb{P}(\text{ess sup}_{t\in[0,\infty)} \|u(t)\|^2_{\frac{11}{6}+} < R) > 1 - \varepsilon \tag{68}$$

Proof: By Chebychev's inequality and the estimate (52) we get that

$$\mathbb{P}(\text{ess sup}_{t\in[0,\infty)} \|u(t)\|^2_{\frac{11}{6}+} \geq R) < \frac{C}{R} < \varepsilon$$

for R sufficiently large. **QED**

5 Existence of Turbulent Solutions

In this section we prove the existence of the turbulent solutions of the initial value problem (20). The following theorem states the existence of turbulent solutions in three dimensions. First we write the initial value problem (20) as the integral equation (69),

$$u(x,t) = u_o(x,t) - \int_0^t K(t-s) * [u \cdot \nabla u - \nabla \Delta^{-1} \mathrm{tr}(\nabla u)^2] ds \tag{69}$$

Here K is the oscillatory heat kernal (23) and

$$u_o(x,t) = \sum_{k \neq 0} h_k^{1/2} A_t^k e_k(x)$$

the A_t^ks being the oscillatory Ornstein-Uhlenbeck-type processes from Equation (24).

Theorem 5.1 *If the uniform flow U and product of the amplitude and frequency $A\Omega$, of the rotation, are sufficiently large, $B = min(|U|, A\Omega, CA\Omega)$, δ is small and the non-resonance conditions (48) are satisfied, so that the a priori bound (52) holds, then the integral equation (69) has unique global solution $u(x,t)$ in the space $C([0,\infty); L^2(\Omega, \mathcal{F}, \mathbb{P}; H^{\frac{11}{6}^+}))$, u is adapted to the filtration generated by the stochastic process*

$$u_o(x,t) = \sum_{k \neq 0} h_k^{1/2} A_t^k e_k$$

and

$$E\left(\int_0^t \|u\|^2_{\frac{11}{6}^+} ds\right) \leq \left(1 - C\left(\frac{1}{B^2} + \delta^{\frac{1}{6}^-}\right)\right)^{-1} \left[\sum_{k \neq 0} \frac{3(1 + (2\pi|k|)^{\frac{11}{3}^+})}{8\pi^2 \nu |k|^2} h_k + \frac{C'}{B} \right] t \tag{70}$$

Proof: We start as in the proof of Theorem 4.1 and first prove that the integral equation (69) maps a bounded subset of $L^\infty([0,t]; H^{\frac{11}{6}^+})$ into itself. Consider

$$\begin{aligned} |u|_2^2 \leq 2|u_o|_2^2 \quad &+ \quad 2 \sum_{k \neq 0} \left(\left| \int_0^t e^{-(\{4\pi^2 \nu |k|^2 + 2\pi i [k_1 U_1 + A(k_2,k_3)\Omega]\}(t-s) + 2\pi i g(k,t,s))} \right. \right. \\ &\times \quad \left. \left. (\widehat{u \cdot \nabla u} - \nabla \Delta^{-1} \widehat{(\mathrm{tr}(\nabla u)^2)})(k,s) ds \right|^2 \right) \\ &\leq \quad 2|u_o|_2^2 + Ct^2 \text{ess } \sup_{s \in [0,t]} \|\nabla u\|^2_{\frac{11}{6}^+} \end{aligned}$$

by Lemma 4.2, Sobolev's inequality and the Gagliardo-Nirenberg inequalities, where g is given by (54). Similarly

$$\begin{aligned}|\nabla^{\frac{11}{6}^+} u|_2^2 \leq\ 2|\nabla^{\frac{11}{6}^+} u_o|_2^2 \ &+\ 2\sum_{k\neq 0}(\int_0^t (2\pi|k|)^{\frac{11}{6}^+} e^{-[4\nu\pi^2|k|^2+2\pi iB(k)](t-s)-2\pi ig(k,t,s)} \\ &\times\ (\widehat{u\cdot\nabla u}+\widehat{u\cdot\nabla u}+\frac{ik}{2\pi|k|}\mathrm{tr}\widehat{(\nabla u)^2})(k,s)ds|^2) \\ &\leq\ 2|\nabla^{\frac{11}{6}^+} u_o|_2^2 + C\, t^{\frac{1}{6}^-}\ \text{ess sup}_{[0,t]}\|u\|^2_{\frac{11}{6}^+}\end{aligned}$$

by Lemma 4.2, Sobolev's inequality and the Gagliardo-Nirenberg inequalities. Adding those two inequalities we get the inequality

$$\text{ess sup}_{[0,t]}\|u\|^2_{\frac{11}{6}^+}(\omega) \leq\ 2\ \text{ess sup}_{[0,t]}\|u_o\|^2_{\frac{11}{6}^+}(\omega) + C\, t^{\frac{1}{6}^-}\ \text{ess sup}_{[0,t]}\|u\|^2_{\frac{11}{6}^+}(\omega) \tag{71}$$

Now suppose that

$$\text{ess sup}_{[0,t]}\|u_o\|^2_{\frac{11}{6}^+} \leq \frac{K}{2}$$

where K is a constant. Then

$$\begin{aligned}\text{ess sup}_{[0,t]}\|u\|^2_{\frac{11}{6}^+} &\leq\ 2\ \text{ess sup}_{[0,t]}\|u_o\|^2_{\frac{11}{6}^+} + C\, t^{\frac{1}{6}^-}\ \text{ess sup}_{[0,t]}\|u\|^4_{\frac{11}{6}^+} \\ &\leq K + Ct^{\frac{1}{6}^-} 2K < 2K\end{aligned} \tag{72}$$

by induction, for t sufficiently small, and the integral equation (69) maps the bounded set $\{u|\text{ess sup}_{[0,t]}\|u\|^2_{\frac{11}{6}^+} < 2K\}$ in $L^\infty([0,t];H^{\frac{11}{6}^+})$ into itself, for every $\omega\in\Omega$.

We let $w = u - v$ and $\alpha = u + v$ where u and v are two solutions of the integral equation. We start by writing

$$\begin{aligned}w(x,t) = -\sum_{k\neq 0}[\int_0^t e^{-[4\nu\pi^2|k|^2+2\pi iB(k)](t-s)-2\pi ig(k,t,s)}(\widehat{w\cdot\nabla\alpha}+\widehat{\alpha\cdot\nabla w} \\ +\frac{ik}{2\pi|k|^2}\mathrm{tr}(\widehat{\nabla w)(\nabla}\alpha))(k,s)ds]e_k(x)\end{aligned}$$

where $e_k = e^{2\pi ik\cdot x}$, $B(k) = Uk_1 + A(k_2,k_3)\Omega$ and g is given by the formula (54).

Then by Lemma 4.2

$$|w|_2^2 \le \sum_{k\neq 0} |\int_0^t e^{-[4\nu\pi^2|k|^2+2\pi iB(k)](t-s)-2\pi ig(k,t,s)}(\widehat{w\cdot\nabla\alpha}+\widehat{\alpha\cdot\nabla w}$$

$$+\frac{ik}{2\pi|k|^2}\widehat{\mathrm{tr}(\nabla w)(\nabla\alpha)})(k,s)ds|^2$$

$$\le C\,t^2 \text{ ess sup}_{[0,t]}(|w|_\infty^2|\nabla\alpha|_2^2+|\alpha|_\infty^2|\nabla w|_2^2+|\nabla\alpha|_4^2|\nabla w|_4^2)$$

$$\le C\,t^2 \text{ ess sup}_{[0,t]}\|\alpha\|^2_{\frac{11}{6}^+}\|w\|^2_{\frac{11}{6}^+}$$

Similarly

$$|\nabla^{\frac{11}{6}^+}w|_2^2 \le \sum_{k\neq 0} |\int_0^t (2\pi|k|)^{\frac{11}{6}^+} e^{-[4\nu\pi^2|k|^2+2\pi iB(k)](t-s)-2\pi ig(k,t,s)}(\widehat{w\cdot\nabla\alpha}+\widehat{\alpha\cdot\nabla w}$$

$$+\frac{ik}{2\pi|k|}\widehat{\mathrm{tr}(\nabla w)(\nabla\alpha)})(k,s)ds|^2$$

$$\le C\,t^{\frac{1}{6}^-} \text{ ess sup}_{[0,t]}(|w|_\infty^2|\nabla\alpha|_2^2+|\alpha|_\infty^2|\nabla w|_2^2+|\nabla\alpha|_2^2|\nabla w|_2^2)$$

$$\le C\,t^{\frac{1}{6}^-} \text{ ess sup}_{[0,t]}\|\alpha\|^2_{\frac{11}{6}^+}\|w\|^2_{\frac{11}{6}^+}$$

by Lemma 4.2, Sobolev's inequality and the Gagliardo-Nirenberg inequalities. Combining those two estimates we get that

$$\text{ess sup}_{[0,t]}\|w^{n+1}\|^2_{\frac{11}{6}^+} \le C\,t^{\frac{1}{6}^-} \text{ ess sup}_{[0,t]}\|\alpha^n\|^2_{\frac{11}{6}^+}\text{ess sup}_{[0,t]}\|w^n\|^2_{\frac{11}{6}^+} \tag{73}$$

for the iteration based on the integral equation (69), and t small. Now the $\|\alpha^n\|^2_{\frac{11}{6}^+}$s are bounded by a constant, independent of n, by (72), for every $\omega\in\Omega$. Thus

$$\text{ess sup}_{[0,t]}\|w^{n+1}\|^2_{\frac{11}{6}^+}(\omega) \le C\,t^{\frac{1}{6}^-} \text{ ess sup}_{[0,t]}\|w^n\|^2_{\frac{11}{6}^+}(\omega) \tag{74}$$

By an application of the contraction mapping principle we get that there exists a random variable τ taking its values for almost every ω in the interval $(0,t]$, with t small, such that the integral equation (69) defines a contraction on $C([0,\tau];H^{\frac{11}{6}^+})$. This proves the local existence of unique solutions to (69).

However, we do not yet have the existence of unique solution in $C([0,\tau);L^2(\Omega,\mathcal{F},\mathbb{P};H^{\frac{11}{6}^+}))$ where we need it to be, in order to apply the a priori estimate (52). To obtain this consider the sequence of Picard iterates of (69)

$$u^{n+1} = \sum_{k=0}^{n}(u^{k+1} - u^k) + u^0$$

This sequence satisfies the estimate

$$\|u^{n+1}\|_{\frac{11}{6}^+} = \frac{1-\theta^n}{1-\theta}\|u^1 - u^0\|_{\frac{11}{6}^+} + \|u^0\|_{\frac{11}{6}^+}$$

where $\theta = \sqrt{C}\tau^{1/12} < 1$. The difference of two iterates satisfies

$$\|u^m - u^n\|_{\frac{11}{6}^+} \le \frac{\theta^n(1-\theta^{m-n})}{1-\theta}\|u^1 - u^0\|_{\frac{11}{6}^+}$$

and the expectation satisfies

$$E(\|u^m - u^n\|^2_{\frac{11}{6}^+}) \le \frac{\theta^{2n}(1-\theta^{m-n})^2}{(1-\theta)^2}4K$$

Thus the sequence of iterates is Cauchy in $\mathcal{L}^2_{\frac{11}{6}^+} = L^2(\Omega,\mathcal{F},\mathbb{P};H^{\frac{11}{6}^+})$ and converges to a unique solution of (69) in $\mathcal{L}^2_{\frac{11}{6}^+}$. A standard argument, see [23], now shows that the limits $u \in C([0,\tau];H^{\frac{11}{6}^+})$ and $u \in C([0,\tau];\mathcal{L}^2_{\frac{11}{6}^+})$ agree for almost every $\omega \in \Omega$.

The global existence uses the bound (52) in Theorem 4.1. Namely, since the norm in $\mathcal{L}^2_{\frac{11}{6}^+} = L^2(\Omega,\mathcal{F},\mathbb{P};H^{\frac{11}{6}^+})$ is bounded a priori for all t, the interval of local existence can be extended to the whole positive t axis $\mathbb{R}^+$. **QED**

We now add the initial condition $u(x,0) = u^0(x)$, with mean zero, to the integral equation (69).

Theorem 5.2 *If the uniform flow U and the product of the amplitude $A\Omega$ and frequency of the rotation, $B = min(|U|, A\Omega, CA\Omega)$, are sufficiently large, δ small, and the non-resonance conditions (48) are satisfied, so that the a priori bound (52) holds, then the integral equation*

$$u(x,t) = K(t) * u^0(x) + u_o(x,t) - \int_0^t K(t-s) * (u \cdot \nabla u - \nabla\Delta^{-1}(\nabla u)^2)\, ds, \quad (75)$$

where K is the oscillating kernal in (23), and the initial data satisfies the bound

$$E(\|u^0(x)\|^2_{\frac{11}{6}^+}) \leq (1 - C(\frac{1}{B^2} + \delta^{\frac{1}{6}^-}))^{-1} \left[\sum_{k \neq 0} \frac{(1 + (2\pi|k|)^{\frac{11}{3}^+})}{8\pi^2 \nu |k|^2} h_k \right] \tag{76}$$

has unique global solution $u(x,t)$ in the space $C([0,\infty); L^2(\Omega, \mathcal{F}, \mathbb{P}; H^{\frac{11}{6}^+}))$, u is adapted to the filtration generated by the stochastic process

$$u_o(x,t) = \sum_{k \neq 0} h_k^{1/2} A_t^k e_k$$

and

$$E(\int_0^t \|u\|^2_{\frac{11}{6}^+} ds) \leq (1 - C(\frac{1}{B^2} + \delta^{\frac{1}{6}^-}))^{-1} \left[\sum_{k \neq 0} \frac{(1 + (2\pi|k|)^{\frac{11}{3}^+})}{\pi^2 \nu |k|^2} h_k + \frac{C'}{B} \right] t. \tag{77}$$

Proof: The proof of the theorem is exactly the same as the proof of Theorem 5.1 once the a priori bound (52) is established. Consider the inequality (62)

$$\|u\|^2_{\frac{11}{6}^+} \leq 3 \sum_{k \neq 0} (1 + (2\pi|k|)^{\frac{11}{3}^+}) h_k |A_t^k|^2 + C(\frac{1}{B^2} + \delta^{\frac{1}{6}^-}) \text{ess sup}_{s \in [0,t]} \|u\|^2_{\frac{11}{6}^+}$$

The same estimate becomes

$$\|u\|^2_{\frac{11}{6}^+} \leq 4\|u^0\|^2_{\frac{11}{6}^+} + 4 \sum_{k \neq 0} (1 + (2\pi|k|)^{\frac{11}{3}^+}) h_k |A_t^k|^2 + C(\frac{1}{B^2} + \delta^{\frac{1}{6}^-}) \text{ess sup}_{s \in [0,t]} \|u\|^2_{\frac{11}{6}^+}$$

with the initial data inserted. Now the same argument as in the proof of Theorem 4.1 gives the a priori bound,

$$\begin{aligned} E(\text{ess sup}_{s \in [0,t]} \|u\|^2_{\frac{11}{6}^+}) &\leq \\ (1 - C(\frac{1}{B^2} + \delta^{\frac{1}{6}^-}))^{-1} [4E(\|u^0(x)\|^2_{\frac{11}{6}^+}) &+ 4[\sum_{k \neq 0} \frac{(1 + (2\pi|k|)^{\frac{11}{3}^+})}{8\pi^2 \nu |k|^2} h_k] + \frac{C'}{B}] \\ &\leq (1 - C(\frac{1}{B^2} + \delta^{\frac{1}{6}^-}))^{-1} \left[\sum_{k \neq 0} \frac{(1 + (2\pi|k|)^{\frac{11}{3}^+})}{\pi^2 \nu |k|^2} h_k + \frac{C'}{B} \right] \end{aligned}$$

QED

Corollary 5.1 *For any initial data $u^0 \in \dot{L}^2(\mathbb{T}^3)$, the L^2 space with mean zero, satisfying (76), and any $t_0 > 0$, there exists a mean flow U, an amplitude and angular velocity $A\Omega$, and δ small, such that (75) has a unique solution in $C([t_0,\infty);L^2(\Omega,\mathcal{F},\mathbb{P};H^{\frac{11}{6}^+}))$.*

Proof: For $t > 0$, $K(t) * u^0(x)$ is smooth. Now apply Theorem 5.2 **QED**

Next we prove a Gronwall estimate that will be used in later sections.

Lemma 5.1 *Let u be a solution of (69) with an initial function $u_o(x,t) = \sum_{k\neq 0} h_k^{1/2} A_t^k e_k$ and initial condition $u^0(x)$ and y a solution of*

$$y_t + \mathbf{U}\cdot\nabla y = \nu\Delta y - y\cdot\nabla y + \nabla\Delta^{-1}\mathrm{tr}(\nabla y)^2 + f \qquad (78)$$

with initial condition $y^0(x)$, then

$$\begin{aligned} \|u-y\|^2_{\frac{11}{6}^+}(t) &\leq [3\|u^0-y^0\|^2_{\frac{11}{6}^+} + 3\|\sum_{k\neq 0} h_k^{1/2} A_t^k e_k - K*f\|^2_{\frac{11}{6}^+} \\ &+ \delta^2 C_1 \mathrm{ess\ sup}_{s\in[t-\delta,t]}(\|u\|^2_{\frac{11}{6}^+} + \|y\|^2_{\frac{11}{6}^+})] e^{C_2\int_0^{t-\delta}(1+\|u\|^2_{\frac{11}{6}^+}+\|y\|^2_{\frac{11}{6}^+})ds} \qquad (79) \end{aligned}$$

where C_1 and C_2 are constants and δ can be made arbitrarily small. The A_t^ks are the oscillatory Ornstein-Uhlenbeck-type processes (25) and K is the oscillatory kernal in (23).

Proof: We subtract the integral equation for y from that of u

$$\begin{aligned} u &= u^0 + \sum_{k\neq 0} h_k^{1/2} A_t^k e_k + K*(-u\cdot\nabla u + \nabla\Delta^{-1}\mathrm{tr}(\nabla u)^2) \\ y &= y^0 + K*f + K*(-y\cdot\nabla y + \nabla\Delta^{-1}\mathrm{tr}(\nabla y)^2) \end{aligned}$$

Thus

$$\begin{aligned} \|u-y\|^2_{\frac{11}{6}^+}(t) &\leq [3\|u^0-y^0\|^2_{\frac{11}{6}^+} + 3\|\sum_{k\neq 0} h_k^{1/2} A_t^k e_k - K*f\|^2_{\frac{11}{6}^+} \\ &+ 3\|K*(-w\nabla u - y\nabla w + \nabla\Delta^{-1}\mathrm{tr}\nabla\alpha\cdot\nabla w)\|^2_{\frac{11}{6}^+}] \end{aligned}$$

where $w = u - y$ and $\alpha = u + y$. Now the same estimates as in Theorem 4.1 give

$$\begin{aligned}
\|u-y\|^2_{\frac{11}{6}+}(t) &\leq 3\|u^0 - y^0\|^2_{\frac{11}{6}+} + 3\|\sum_{k\neq 0} h_k^{1/2} A_t^k e_k - K * f\|^2_{\frac{11}{6}+} \\
&+ C_1 \delta^2 \text{ess sup}_{s\in[t-\delta,t]}(\|u\|^2_{\frac{11}{6}+} + \|y\|^2_{\frac{11}{6}+}) \\
&+ C_2 \int_0^{t-\delta} (1 + \|u\|^2_{\frac{11}{6}+} + \|y\|^2_{\frac{11}{6}+})(\|u-y\|^2_{\frac{11}{6}+}) ds
\end{aligned}$$

Then Grönwall's inequality gives (79). **QED**

6 The Existence of the Classical (Laminar) Solution

Now suppose that we remove the noise from the equation (20) but add a nontrivial initial condition. Then we are solving the deterministic problem initial value problem (21)

$$\begin{aligned}
\frac{\partial u}{\partial t} + U_1 \partial_{x_1} u - A\sin(\Omega t + \theta)\partial_{x_2} u &+ A\cos(\Omega t + \theta)\partial_{x_3} u + u \cdot \nabla u \\
&= \nu \Delta u + \nabla \Delta^{-1}[\text{trace}(\nabla u)^2] \qquad (80) \\
u(x,0) &= u^0(x)
\end{aligned}$$

This is the driven Navier-Stokes equation (18) for $u = w - U_1 j_1 + A\sin(\Omega t + \theta) j_2 - A\cos(\Omega t + \theta) j_3$, but now with the initial data $u^0(x)$. We write the initial value problem as an integral equation

$$u(x,t) = K(t) * u^0(x) - \int_0^t K(t-s) * (u \cdot \nabla u - \nabla \Delta^{-1} \text{tr}(\nabla u)^2)\, ds, \qquad (81)$$

where K is the oscillating kernel (23). Now a similar proof as that of Theorem 4.1 give us the following a priori estimate.

Theorem 6.1 *Let the velocity $U = U_1$ of the mean flow and the product of the amplitude $A\Omega$ and frequency of the rotation, $B = min(|U|, A\Omega, CA\Omega)$, be sufficiently large, in the uniform rotating flow (17), and δ small, with U, $A\Omega$ also satisfying the non-resonance conditions (48). Then if the initial condition $u^0(x)$ in the Picard iteration of the integral equation (81) lies in $H^1(\mathbb{T}^3)$, has mean zero, and satisfies*

$$\|K * u^0\|^2_n(t) < \frac{1}{2}|\mathbf{U}|,$$

for $\mathbf{U} = U_1 j_1 - A\sin(\Omega t+\theta) j_2 + A\cos(\Omega t+\theta) j_3$*, then the solution of the integral equation (22) is uniformely bounded in* $H^n(\mathbb{T}^3)$*, for all* $\frac{3}{2}^+ \leq n < 2$*. In particular,*

$$\text{ess sup}_{t\in[0,\infty)}|u|_\infty(t) < C_1(1 - C_2(\frac{1}{B^2} + \delta^{(2-n)^-}))^{-1}|\mathbf{U}|, \qquad (82)$$

where C_1 *and* C_2 *are constants.*

The proof of this theorem is similar to that of Theorem 4.1. The bound on the initial data guarantees that the integral representation is valid or that the initial data does not cancel the uniform (rotating) flow. $K * u^0$ is smooth for $t > 0$ so we can operate on it with ∇^n for any $n > 0$ not only $n \leq \frac{11}{6}^+$ as was the case with the initial function u_o in Theorem 4.1. The lower limit $n \geq \frac{3}{2}^+$ comes form Gagliardo-Nirenberg inequalities, this is the smallest Sobolev norm bounding $|u|_\infty$. The bound is computed as in Theorom 4.1.

We can prove the existence of a smooth classical solution with the a priori estimate (82).

Theorem 6.2 *If*

$$\|K * u^0\|_n^2(t) < \frac{1}{2}|\mathbf{U}| \qquad (83)$$

for all $\frac{3}{2}^+ \leq n < 2$ *and* $t > 0$*, where the uniform flow U and the amplitude times frequency* $A\Omega$ *of the rotation,* $B = min(|U|, A\Omega, CA\Omega)$*, are sufficiently large,* δ *is small, and the non-resonance conditions (48) are satisfied, so that the a priori bound (82) holds, then the integral equation (81) has a unique global solution* $u(x,t)$ *in the space* $C((0,\infty); C^\infty(\mathbb{T}^3))$*, satisfying the bound*

$$\int_0^t |u|_\infty(s)ds < C_1(1 - C_2(\frac{1}{B^2} + \delta^{(2-n)^-}))^{-1}|\mathbf{U}|\, t \qquad (84)$$

Proof: With the bound

$$\text{ess sup}_{t\in[0,\infty)}|u|_\infty(t) < C$$

from Theorem 6.1 it is easy to show that

$$\sup_{t\in[0,\infty)}|\nabla u|_2(t) < C$$

also. It is well-know that this implies the existence of unique strong solutions to the Navier-Stokes equations and that these solutions are smooth. **QED**

It seems contradictory that one can have two solutions of the same initial value problem (18), namely the laminar solution in Theorem 6.2 and the turbulent (stochastic) solution in Theorem 5.2. A consideration of a typical simulation for turbulent flow show that there is no contradiction and both of those solutions play a role although the latter is of greater physical importance.

Consider a simulation where we have modeled the torus as a box with periodic boundary conditions and we start with quiescent flow but gradually increase the uniform flow velocity U and the amplitude and angular velocity of the rotation $A\Omega$. To begin with the flow is laminar and it continues to be laminar although the Reynolds number $R = \frac{B}{\nu}$ becomes greater than 2000 which is the typical boundary for fully developed turbulence. But then suddenly a small ambient noise such as (11) grows exponentially as explained in Section 2 and the flow immediately becomes fully turbulent. We have witnessed the transition from the classical laminar solution in Theorem 6.2 (laminar flow) to the turbulent solution in Theorem 5.2 (turbulent flow).

The above transition can also seen in many experiments, see for example [10, 9], but usually at lower Reynolds numbers. As discussed in the introduction the laminar solution is not blowing up but it is unstable at high Reynolds numbers and a sudden noise-induced roughening of it takes place, as it undergoes a phase transition into fully developed turbulence.

7 The Existence of the Invariant Measure

In this section we will consider the stochastic Navier-Stokes equation

$$dw = (\nu\Delta w - w\cdot\nabla w + \nabla\Delta^{-1}\mathrm{tr}(\nabla w)^2)dt + \sum_{k\neq 0} h_k^{1/2} d\beta_t^k e_k \tag{85}$$

with initial data

$$w(x,0) = U_1 j_1 - A\sin(\Omega t + \theta) j_2 + A\cos(\Omega t + \theta) j_3 + u^0(x)$$

We will use that the solutions $u(x,t)$, where $w(x,t) = U j_1 - A\sin(\Omega t + \theta) j_2 + A\cos(\Omega t + \theta) j_3 + u(x,t)$, exist in $L(\Omega, \mathcal{F}, \mathbb{P}; H^{\frac{11}{6}^+})$, by Theorem 5.2. $H^{\frac{11}{6}^+}(\mathbb{T}^3) = W^{(\frac{11}{6}^+,2)}$ is the Sobolev space based on L^2. By Theorem 5.2 the equation (85) defines a flow on the complete metric space

$$W = \{u \in L(\Omega, \mathcal{F}, \mathbb{P}; H^{\frac{11}{6}^+}) | E(\|u\|^2_{\frac{11}{6}^+}) \leq (1 - C(\frac{1}{B^2} + \delta^{\frac{1}{6}^-}))^{-1}[\sum_{k\neq 0} \frac{(1+(2\pi|k|)^{\frac{11}{3}^+})}{\pi^2\nu|k|^2} h_k + \frac{C'}{B}]\}$$

This is the physical situation we are interested in, namely fully developed turbulence with nontrivial mean flow and rotation, see (20), and it applies to many if not most turbulent fluids, see [21, 22].

Since by Corollary 5.1, we can even take the initial data $u^0(x) \in \dot{L}^2(\mathbb{T}^3)$,[3] the integral equation

$$u(x,t) = K(t) * u^0(x) + u_o(x,t) - \int_0^t K(t-s) * (u \cdot \nabla u - \nabla \Delta^{-1} \mathrm{tr}(\nabla u)^2)\, ds, \quad (86)$$

with $u^0 \in \dot{L}^2(\mathbb{T}^3)$ and $u_o = \sum_{k \neq 0} h_k^{1/2} A_t^k e_k$, defines a map from a bounded set in $\dot{L}^2(\mathbb{T}^3)$ onto W. We define V to be the preimage of W in $\dot{L}^2(\mathbb{T}^3)$. V is also a complete metric space with the distance on V defined by the $\dot{L}^2(\mathbb{T}^3)$ norm.

More concretely, we can consider the initial value problem on V,

$$\begin{aligned} du = (\nu\Delta u &- U\partial_{x_1} u + A\sin(\Omega t + \theta)\partial_{x_2} u - A\cos(\Omega t + \theta)\partial_{x_3} u \\ &- u \cdot \nabla u + \nabla\Delta^{-1}\mathrm{tr}(\nabla u)^2)dt + \sum_{k \neq 0} h_k^{1/2} d\beta_t^k e_k \\ u(x,0) &= u^0(x) \in V \subset \dot{L}^2(\mathbb{T}^3) \end{aligned} \quad (87)$$

Then by Theorem 5.2 and Corollary 5.1 the initial value problem (87) defines a flow on V.

If ϕ is a bounded function on V then the invariant measure $d\mu$ for the SPDE (20) is given by the limit

$$\lim_{t \to \infty} E(\phi(u(\omega,t))) = \int_V \phi(u) d\mu(u) \quad (88)$$

In this section we proof that this limit exists and is unique. We prove below that the limit exist in the $H^{\frac{11}{6}^+}(\mathbb{T}^3)$ norm on W but since it dominates the $\dot{L}^2(\mathbb{T}^3)$ norm on V the conclusions will follow for V.

Theorem 7.1 *The integral equation (75) possesses a unique invariant measure.*

Corollary 7.1 *The invariant measure $d\mu$ is ergodic and strongly mixing.*

The corollary follows immediately from Doob's Theorem on invariant measures, see for example [26].

[3] dot denotes mean zero

We prove the theorem in three lemmas. First we define a transition probability

$$P_t(u^0,\Gamma) = \mathcal{L}(u(u^0,t))(\Gamma), \quad \Gamma \subset \mathcal{E},$$

where $\mathcal{L}$ is the law of $u(t)$, u^0 is the initial condition and $\mathcal{E}$ is the natural σ algebra of V. The action of P_t on the bounded function ϕ on V can be written as

$$P_t\phi = BM(\phi(u(u^0,t))) = \int_V \phi(u)\pi_t(u^0,du),$$

BM denoting the Brownian mean over the Brownian motions in equation (87) and π_t is the corresponding measure on V. Then

$$R_T(u^0,\cdot) = \frac{1}{T}\int_0^T P_t(u^0,\cdot)dt$$

is a probability measure on V. By the Krylov-Bogoliubov theorem, see [26], if the sequence of measures R_T is tight then the invariant measure $d\mu$ is the weak limit

$$d\mu(\cdot) = \lim_{T\to\infty}\frac{1}{T}\int_0^T P_t(u^0,\cdot)dt$$

Namely,

$$R_T^* d\nu(\Gamma) = \int_V R_T(u^0,\Gamma)d\nu(u^0)$$

and

$$< R_T^*\nu, \phi >= \int_V \phi(u^0)R_T(u^0,\Gamma)d\nu(u^0) \to \int_V \phi(u^0)d\mu(u^0)$$

as $T \to \infty$.

Lemma 7.1 *The sequence of measures*

$$\frac{1}{T}\int_0^T P_t(u^0,\cdot)dt \tag{89}$$

is tight.

Proof: By the inequality (52)

$$\frac{1}{T}\int_0^T E(\|u\|^2_{\frac{11}{6}+})(t)dt \le C$$

The complete metric space W is relatively compact in V so it suffices to show that $u(t)$ lies in a bounded set in W almost surely, or for all $\varepsilon > 0$ there exists an R such that,

$$\frac{1}{T}\int_0^T \mathbb{P}(\|u(t)\|^2_{\frac{11}{6}+} < R)dt > 1-\varepsilon$$

for $T \geq 1$. But this follows from Chebychev's inequality, similarly as in Lemma 4.7, namely,

$$\frac{1}{T}\int_0^T \mathbb{P}(\|u(t)\|^2_{\frac{11}{6}+} \geq R)dt \leq \frac{1}{R}C < \varepsilon$$

for R sufficiently large. By Corollary 5.1 we can take the initial data in V. This proves that the sequence of measures (89) is tight. **QED**

Next we prove the strong Feller property, see [26].

Lemma 7.2 *The Markovian semigroup P_t generated by the integral equation (75) on V is strongly Feller.*

Proof: Let $\mathbf{U} = U j_1 - A\sin(\Omega t+\theta) j_2 + A\cos(\Omega t+\theta) j_3$ denote the velocity of the uniform flow with rotation (17). Consider the variational equation,

$$\text{(90)} \qquad \begin{aligned} w_t = \nu\Delta w - (\mathbf{U}+u)\cdot\nabla w \quad &- \quad w\cdot\nabla u + 2\nabla\Delta^{-1}\text{tr}(\nabla u\nabla w) \\ w(0) \quad &= \quad \delta(x) \end{aligned}$$

for the functional derivative $w(x,t) = \frac{\partial u(x,t)}{\partial u(y,0)}$. We first show that the equation (90) generates a contraction semigroup. Consider the linear operators $Aw = \nu\Delta w$ and

$$Sw = -(\mathbf{U}+u)\cdot\nabla w - w\cdot\nabla u + 2\nabla\Delta^{-1}\text{tr}(\nabla u\cdot\nabla w)$$

on the space $L^2(\Omega, \mathcal{F}, \mathbb{P}; L^2(\mathbb{T}^3))$. A generates a contraction semigroup and we now show that S is A bounded or that there exists a constant C such that

$$\|Sw\|_2 \leq C\|w\|_2 + \frac{1}{2}\|Aw\|_2$$

where $\|\cdot\|_2$ denotes the norm on $L^2(\Omega, \mathcal{F}, \mathbb{P}; L^2(\mathbb{T}^3))$. First we show that there exists a constant C such that

$$|Sw|_2 \leq C|w|_2 + \frac{1}{2\sqrt{2}}|Aw|_2$$

where $|\cdot|_2$ denotes the usual L^2 norm. By Minkowski's inequality

$$
\begin{aligned}
|Sw|_2 &\leq |(\mathbf{U}+u)\cdot\nabla w|_2 + |w\cdot\nabla u|_2 + 2|\nabla\Delta^{-1}\mathrm{tr}(\nabla u\cdot\nabla w)|_2 \\
&\leq (|\mathbf{U}|+|u|_\infty)|\nabla w|_2 + |\nabla u|_4|w|_4 + \frac{1}{\pi}|\nabla u|_4|\nabla w|_4 \\
&\leq (|\mathbf{U}|+\|u\|_{\frac{11}{6}+})|\nabla w|_2 + C\|u\|_{\frac{11}{6}+}\|w\|_{\frac{11}{6}+}
\end{aligned}
$$

by Schwartzes and Sobolev's inequalities

$$
\leq C(|\mathbf{U}|+\|u\|_{\frac{11}{6}+})|w|_2 + \frac{1}{2\sqrt{2}}|\nu\Delta w|_2
$$

by interpolation. Now dividing by $|w|_2$ we get

$$
|S|_2 \leq C(|\mathbf{U}|+\|u\|_{\frac{11}{6}+}) + \frac{1}{2\sqrt{2}}|A|_2
$$

where $|\cdot|_2$ now denotes the operator norm for each ω. We square the last inequality and take the expectation, this gives

$$
E(|S|_2^2) \leq 2C^2E((|\mathbf{U}|+\|u\|_{\frac{11}{6}+})^2) + \frac{1}{4}|A|_2^2 \leq C' + \frac{1}{4}|A|_2^2
$$

since A is deterministic and by the a priori estimate (52). We also used the inequality $(a+b)^2 \leq 2a^2+2b^2$ and C' is a new constant. Then taking the square root of the last inequality and multiplying through by $\|w\|_2$ gives

$$
\|Sw\|_2 \leq C'\|w\|_2 + \frac{1}{2}\|Aw\|_2
$$

using again that A is deterministic and the inequality $\sqrt{a^2+b^2} \leq a+b$ for a and b positive. This shows that S is A bounded. Moreover in the space of divergence free function S is dissipative, namely

$$
\langle w, Sw\rangle = -\langle w,(\mathbf{U}+u)\cdot\nabla w\rangle - \langle w, w\cdot\nabla u\rangle + 2\langle u, \nabla\Delta^{-1}\mathrm{tr}(\nabla u\nabla w)\rangle = 0
$$

by use of the periodic boundary conditions. Since S is dissapative and A bounded $A+S$ also generates a contration semigroup $e^{(A+S)t}$, see Kato [13], and

$$
\|e^{(A+S)t}\| \leq 1 \qquad (91)
$$

The solution w of the variational equation (90) is just the kernal of the semigroup $e^{(A+S)t}$ and evidently by (91)

$$|w(x,t)| \leq 1 \tag{92}$$

The rest of the proof follows McKean [18],

$$\begin{aligned} P_t\phi(u) - P_t\phi(v) &= \int_{L^2(\mathbb{T}^3)} \phi(z)(\pi_t(u,dz) - \pi_t(v,dz)) \\ &= \int_{\mathbb{T}^3}\int_{\mathbb{T}^3} BM\{\phi \int_0^1 w(x = x_t, t)(u-v)dr\}dxdy \end{aligned}$$

where BM denotes the Brownian mean. Thus

$$\begin{aligned} |P_t\phi(u) - P_t\phi(v)| &\leq |\phi|_\infty |u-v|_2 \\ &\leq |\phi|_\infty \|(u-v)\|_{\frac{11}{6}^+} \end{aligned} \tag{93}$$

since $|w| \leq 1$. **QED**

Finally we prove irreducibility, see [26], of P_t. The proof of this lemma is an application of stochastic control theory.

Lemma 7.3 *The Markovian semigroup P_t generated by the integral equation (75) is irreducible.*

Proof: We first consider the linear deterministic equation

$$\begin{aligned} z_t + \mathbf{U}\cdot\nabla z &= \nu\Delta z + w(x,t) \\ z(x,0) = 0, &\ \ z(x,T) = b(x) \end{aligned} \tag{94}$$

and the deterministic equation

$$\begin{aligned} y_t + \mathbf{U}\cdot\nabla y = \nu\Delta y - y\cdot\nabla y + \nabla\Delta^{-1}\mathrm{tr}(\nabla y)^2 + Qh(x,t) \\ y(x,0) = 0, \ \ y(x,T) = b(x) \end{aligned} \tag{95}$$

where $Q: H^{-1} \to H^{\frac{11}{6}^+}$, both spaces have mean zero and kernel Q is empty. We will define the operator Q by a map from the coefficients (vectors) in an element $\sum_{k\neq 0} f_k e_k$ in $H^{-1}(\mathbb{T}^3)$ to the coefficients in the sum $\sum_{k\neq 0} h_k^{1/2} A_t^k e_k$, where the A_t^k are the oscillatory Ornstein-Uhlenbeck-type processes from (25). This map can be defined by an invertible matrix $h_k^{1/2} = Q_k f_k$, for example $Q_k = |k|^{-p} I_3$ where

I_3 is the three by three identity matrix and p is a positive rational number, since all the coefficients in the latter sum satisfy $h_k^{1/2} \neq 0$. Then it is easy to check that kernel $Q = 0$.

We can pick a function $w \in C([0,T];W)$ such that $z(x,T) = b(x)$ and a corresponding function $h \in L^2([0,T];H^{-1}(\mathbb{T}^3))$. Namely, $Qh = z\cdot\nabla z - \nabla\Delta^{-1}\mathrm{tr}(\nabla z)^2 + w$, since the kernel of Q is empty; then $y = z$ is a solution of the deterministic Navier-Stokes equation (95) above. This means that (95) is exactly controllable on W, see Curtain and Zwartz [11].

Now we compare y and the solution u of the integral eqution (69). By Lemma 5.1 we get that

$$\begin{aligned}\|u-y\|^2_{\frac{11}{6}^+}(t) \quad &\leq \quad [3\|u^0 - y^0\|^2_{\frac{11}{6}^+} + 3\|\sum_{k\neq 0} h_k^{1/2} A_t^k e_k - Qh\|^2_{\frac{11}{6}^+} \\ &+ \quad \delta^2 C_1 \text{ess sup}_{s\in[t-\delta,t]}(\|u\|^2_{\frac{11}{6}^+} + \|y\|^2_{\frac{11}{6}^+})^2] e^{C_2\int_0^{t-\delta}(1+\|u\|^2_{\frac{11}{6}^+}+\|y\|^2_{\frac{11}{6}^+})ds}\end{aligned}$$

By Lemma 4.7, for $\gamma > 0$

$$\mathbb{P}(\|u\|^2_{\frac{11}{6}^+} + \|y\|^2_{\frac{11}{6}^+}) \leq R) > 1 - \frac{\gamma}{2}$$

if

$$E(\|u\|^2_{\frac{11}{6}^+} + \|y\|^2_{\frac{11}{6}^+})/R \leq \frac{\gamma}{2}$$

Then

$$\begin{aligned}E(\|u-y\|^2_{\frac{11}{6}^+}(T)) \leq [3E(\|\sum_{k\neq 0} h_k^{1/2} A_T^k e_k - Qh\|^2_{\frac{11}{6}^+}) + \delta^2 C_1 R^2] e^{C_2(1+R)(T-\delta)} \\ \leq \frac{\varepsilon\gamma}{4}\end{aligned} \tag{96}$$

if δ is small enough, since $\sum_{k\neq 0} h_k^{1/2} A_t^k e_k$ is an oscillatory Ornstein-Uhlenbeck-type process with a non-degenerate covariance whose (Gaussian) measure is full in $L^2([0,T];H^{\frac{11}{6}^+})$. This implies that the probability

$$\begin{aligned}&\mathbb{P}(\|u(T)-b\|_{\frac{11}{6}^+} \leq \varepsilon \quad \text{and} \quad \|u\|^2_{\frac{11}{6}^+} + \|y\|^2_{\frac{11}{6}^+}) \leq R) \geq \\ &\mathbb{P}\left(\|u(T)-y(T)\|_{\frac{11}{6}^+} \leq \frac{\varepsilon}{2} \quad \text{and} \quad \|y(T)-b(T)\|_{\frac{11}{6}^+} \leq \frac{\varepsilon}{2} \text{ and } \|u\|^2_{\frac{11}{6}^+} + \|y\|^2_{\frac{11}{6}^+} \leq R\right) \\ &\geq \quad 1 - \frac{\gamma}{2} - \frac{\gamma}{2} = 1 - \gamma > 0\end{aligned}$$

by (96) and Chebychev's inequality, since (95) is exactly controllable. It also implies that

$$\mathbb{P}(|u(T)-b|_2^2) \leq \varepsilon) > 0$$

QED

Proof of Theorem 7.1 and Corollary 7.1

Proof: Theorem 7.1 and Corollary 7.1 are now easily proven in the following manner. If the Markovian semigroup P_t is strongly Feller and invariant, as it is by Lemmas 7.2 and 7.3, it is also t-regular. This means that the probability measures $P(u_o(s),\cdot)$ are all equivalent for $s \geq t$, and then by Doob's Theorem for invariant measures, see [26], the invariant measure is unique and strongly mixing. **QED**

7.1 Kolmogorov's Scaling

In 1941, Kolmogorov [14] formulated his famous scaling theory of the inertial range in turbulence, stating that the second-order structure function, scales as

$$S_2(x) = \langle |u(y+x)-u(y)|^2 \rangle \sim (\varepsilon|x|)^{2/3},$$

where $y, y+x$ are points in a turbulent flow field, u is the component of the velocity in the direction of x, ε is the mean rate of energy dissipation, and the angle brackets denote an (ensamble) average. A Fourier transform yields the Kolmogorov-Obukhov power spectrum in the inertial range

$$E(k) = C\varepsilon^{2/3}k^{-5/3},$$

where C is a constant, k is the wave number and $E(k)$ denotes the energy density in Fourier space. These results form the basis of turbulence theory. The following theorem proves the basic statement in Kolmogorov's statistical theory of turbulence.

Theorem 7.2 *The second structure function of turbulence satisfies the estimate*

$$\begin{aligned} S_2(x,t) &= E[|u(x+y,t)-u(y,t)|_2^2] \\ &= \int_V |u(x+y,t)-u(y,t)|_2^2 d\mu(u) \leq C|x\cdot(\mathbf{L}-x)|^{2/3} \end{aligned} \tag{97}$$

where C is a constant and $\mathbf{L}$ is a three vector giving the dimensions of the torus $\mathbb{T}^3$.

Proof: The proof is basically an amplification of Corollary 4.2. We write the difference as a Fourier series

$$u(x+y,t) - u(y,t) = \sum_{k\neq 0} \hat{u}(k) e^{2\pi i k\cdot y}(e^{2\pi i k\cdot x} - 1)$$

By the Cauchy-Schwartz inequality

$$\begin{aligned}|u(x+y,t) - u(y,t)| &\leq (\sum_{k\neq 0} |k|^{3+2\gamma}|\hat{u}(k)|^2)^{1/2}(\sum_{k\neq 0} |k|^{-3-2\gamma}|e^{2\pi i k\cdot x} - 1|^2)^{1/2} \\ &\leq \|u\|_{\frac{3}{2}+\gamma}(\sum_{k\neq 0} |k|^{-3-2\gamma}|e^{2\pi i k\cdot x} - 1|^2)^{1/2}\end{aligned}$$

We use the integral test to estimate the last series

$$\begin{aligned}(\sum_{k\neq 0} |k|^{-3-2\gamma}|e^{2\pi i k\cdot x} - 1|^2) &\leq C\int_{\mathbb{R}^3} |k|^{-3-2\gamma}|e^{2\pi i k\cdot x} - 1|^2 dk \\ = C(4\pi^2\int_{|k|\leq\frac{1}{|x|}} |x|^2|k|^2|k|^{-3-2\gamma}dk &+ 4\int_{|k|\geq\frac{1}{|x|}} |k|^{-3-2\gamma}dk) \\ &= C(4\pi^2\frac{|x|^{2\gamma}}{2-2\gamma} + 4\frac{|x|^{2\gamma}}{2\gamma})\end{aligned}$$

for $x_j \leq L_j/2,\ j = 1,2,3$. Now squaring and taking the expectation we get that

$$E[|u(x+y,t) - u(y,t)|^2] \leq CE[\|u\|^2_{\frac{3}{2}+\gamma}]\ |x|^{2\gamma}$$

for $x_j \leq L_j/2,\ j = 1,2,3$. Moreover by making the same estimate for the variable $z = \mathbf{L} - x$, where the three-vector $\mathbf{L}$ has the entries $L_j,\ j = 1,2,3$, we get the estimate

$$E[|u(z+y,t) - u(y,t)|^2] \leq CE[\|u\|^2_{\frac{3}{2}+\gamma}]\ |\mathbf{L} - x|^{2\gamma}$$

for $x_j \geq L_j/2,\ j = 1,2,3$. Combining the two estimates we obtain the estimate

$$E[|u(x+y,t) - u(y,t)|^2] \leq CE[\|u\|^2_{\frac{3}{2}+\gamma}]\ |x\cdot(\mathbf{L} - x)|^{2\gamma}$$

and then choosing $\gamma = \frac{1}{3}^+$ and applying the estimated (52) to $E[\|u\|^2_{\frac{11}{6}^+}]$ we get the estimate (97). **QED**

Remark 7.1 The estimate (97) is not sharp due to intermittency, as pointed out by Landau and discussed by Kolmogorov [15].

Theorem 7.3 *There exist solutions of the stochastic Navier-Stokes equation (20) with an expectation of the* $H^{\frac{11}{6}^+}$ *norm that is uniformely bounded for every* $t \in \mathbb{R}^+$*, but whose expectation of the* H^{2^-} *norm is infinite for every* $t \in \mathbb{R}^+$*.*

Proof: Suppose that the expectation of the $H^{\frac{11}{6}+}$ norm of u is finite by Theorem 4.1. Then a similar argument as lead to inequality (62) gives the inequality

$$\begin{aligned} \|u\|^2_{\frac{11}{6}+\sigma} &\geq 9\sum_{k\neq 0}(1+(2\pi|k|)^{\frac{11}{3}+2\sigma})h_k|A^k_t|^2 \\ &- C(\varepsilon+\delta^{\frac{1}{6}-\sigma})\text{ess sup}_{s\in[0,t]}(\|u\|^4_{\frac{11}{6}+}+\|u\|^6_{\frac{11}{6}+}) \end{aligned} \tag{98}$$

with $\varepsilon = \frac{1}{B^2}$ and δ small. Now if

$$E[\sum_{k\neq 0}(1+(2\pi|k|)^{\frac{11}{3}+2\sigma})h_k|A^k_t|^2] = \sum_{k\neq 0}\frac{(1+(2\pi|k|)^{\frac{11}{3}+2\sigma})}{4\pi^2|k|^2}h_k = \infty$$

for $0 < \sigma < \frac{1}{6}$ then it follows that

$$E[\|u\|^2_{\frac{11}{6}+\sigma}] = \infty$$

also. **QED**

A Appendix

Lemma A.1 (4.6) *The functions* H, K, L *in the proof of Theorem 4.1 satisfy the estimate*

$$E(H+K+L) \leq \frac{C}{|B(k)|^2}E(\text{ess sup}_{t\in[0,\frac{1}{2B}]}\|u\|^2_{\frac{11}{6}+}) + \frac{C'}{B} \tag{99}$$

with $B = min(U, A\Omega, CA\Omega)$.

Proof: First notice that by Lemma 3.1, $E(K)$ is bounded by the same quantity as $4\nu^2\lambda_1 E(H)$ and by Lemma 3.2, $E(L)$ is bounded by the same quantity as $2\nu\lambda_1 E(H)$. This implies that the bound on $E(H+K+L)$ is the same as that on $(1+2\nu\lambda_1(1+2\nu))E(H)$ and it suffices to estimate $E(H)$.

The functions H is computed by multiplying together the a priori estimates in Lemmas 3.1, 3.2, 4.4 and 4.5, using (63) and that $u(0)=0$,

$$
\begin{aligned}
H &= (\int_0^{t-\delta} |u(s)-u(s+\frac{1}{2B})|_2^2 ds \int_0^{t-\delta} e^{-4\pi^2\nu|k|^2 s}|\nabla u|_2^2(s)ds) \\
&\leq (\frac{1}{2\nu\lambda_1}|u-u_{\frac{1}{2B}}|_2^2(0) \\
&+ 2\sum_{k\neq 0}\int_0^{t-\delta}\int_0^s e^{2\lambda_1 2\nu(s-r)}\langle u-u_{\frac{1}{2B}}, h_k^{1/2}e_k\rangle d(\beta_r^k-\beta_{r+\frac{1}{2B}}^k)ds) \int_0^{t-\delta} e^{-4\pi^2\nu|k|^2 s}|\nabla u|_2^2(s)ds) \\
&\leq (\frac{1}{\nu\lambda_1}\sum_{j\neq 0}|A^j_{\frac{1}{2B(k)}}|^2 + \frac{C}{|B(k)|}\text{ess sup}_{t\in[0,\frac{1}{2B}]}\|u\|^2_{\frac{11}{6}+}\int_0^{\frac{1}{2B}}|\nabla u|_2^2(s)ds \\
&+ 2\sum_{j\neq 0}\int_0^{t-\delta}\int_0^s e^{-2\nu\lambda_1(s-r)}\langle u-u_{\frac{1}{2B}}, h_j^{1/2}e_j\rangle d(\beta_r^j-\beta^j_{r+\frac{1}{2B(k)}})ds) \\
&\times (\frac{1}{2\nu^2}\sum_{k\neq 0}(\frac{1}{4\pi^2|k|^2}+\frac{1}{2\lambda_1})h_k + \frac{1}{\nu}\sum_{k\neq 0}\int_0^{t-\delta} e^{-4\pi^2|k|^2\nu(t-s)}\int_0^s e^{-2\nu\lambda_1(s-r)}\langle u, h_k^{1/2}e_k\rangle d\beta_r^k\, ds \\
&+ \frac{1}{\nu}\sum_{k\neq 0}\int_0^{t-\delta} e^{-4\pi^2|k|^2\nu(t-s)}\langle u, h_k^{1/2}e_k\rangle d\beta_s^k)
\end{aligned}
$$

Thus using the a priori estimate in Lemma 3.1 and rearranging the terms we get that

$$
\begin{aligned}
H &= \\
&\leq (\frac{1}{\nu\lambda_1}\sum_{j\neq 0}|A^j_{\frac{1}{2B(k)}}|^2 + \frac{C}{|B|}\text{ess sup}_{t\in[0,\frac{1}{2B(k)}]}\|u\|^2_{\frac{11}{6}+}(\frac{1}{4|B|\nu}\sum_{k\neq 0}h_k + \frac{1}{\nu}\sum_{k\neq 0}\int_0^{\frac{1}{2B}}\langle u, h_k^{1/2}e_k\rangle d\beta_s^k) \\
&+ 2\sum_{j\neq 0}\int_0^{t-\delta}\int_0^s e^{-2\nu\lambda_1(s-r)}\langle u-u_{\frac{1}{2B}}, h_j^{1/2}e_j\rangle d(\beta_r^j-\beta^j_{r+\frac{1}{2B(k)}})ds) \\
&\times (\frac{1}{2\nu^2}\sum_{k\neq 0}(\frac{1}{4\pi^2|k|^2}+\frac{1}{2\lambda_1})h_k + \frac{1}{\nu}\sum_{k\neq 0}\int_0^{t-\delta} e^{-4\pi^2|k|^2\nu(t-s)}\int_0^s e^{-2\nu\lambda_1(s-r)}\langle u, h_k^{1/2}e_k\rangle d\beta_r^k\, ds \\
&+ \frac{1}{\nu}\sum_{k\neq 0}\int_0^{t-\delta} e^{-4\pi^2|k|^2\nu(t-s)}\langle u, h_k^{1/2}e_k\rangle d\beta_s^k)
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \frac{1}{2\nu^2}\sum_{k\neq 0}\left(\frac{1}{4\pi^2|k|^2}+\frac{1}{2\lambda_1}\right)h_k\left(\frac{1}{\nu\lambda_1}\sum_{j\neq 0}|A^j_{\frac{1}{2B(k)}}|^2+\frac{C}{|B|}\text{ess sup}_{t\in[0,\frac{1}{2B(k)}]}\|u\|^2_{\frac{11}{6}+}+\frac{1}{4|B|\nu}\sum_{k\neq 0}h_k\right)\right) \\
+ \quad & \frac{C}{|B|}\text{ess sup}_{t\in[0,\frac{1}{2B(k)}]}\|u\|^2_{\frac{11}{6}+}\frac{1}{\nu}\sum_{k\neq 0}\int_0^{\frac{1}{2B}}\langle u,h_k^{1/2}e_k\rangle d\beta^k_s\left(\frac{1}{\nu}\sum_{k\neq 0}\int_0^{t-\delta}e^{-4\pi^2|k|^2\nu(t-s)}\langle u,h_k^{1/2}e_k\rangle d\beta^k_s\right. \\
+ \quad & \left.\frac{1}{\nu}\sum_{k\neq 0}\int_0^{t-\delta}e^{-4\pi^2|k|^2\nu(t-s)}\int_0^s e^{-2\nu\lambda_1(s-r)}\langle u,h_k^{1/2}e_k\rangle d\beta^k_r\, ds\right)+D
\end{aligned}
$$

where D are terms that vanish when the expectation is applied. Now applying the expectation we get

$$
\begin{aligned}
E(H) \quad & \leq \frac{1}{8\nu^3\lambda_1}\sum_{k\neq 0}\left(\frac{1}{4\pi^2|k|^2}+\frac{1}{2\lambda_1}\right)h_k\sum_{k\neq 0}h_k\frac{1}{|B(k)|}+\frac{C}{|B|^2}E(\text{ess sup}_{t\in[0,\frac{1}{2B(k)}]}\|u\|^2_{\frac{11}{6}+}) \\
+ \quad & \frac{C}{|B|}\frac{1}{\nu}\left(\sum_{k\neq 0}\int_0^{\frac{1}{2B}}e^{-4\pi^2|k|^2\nu(t-s)}E(\text{ess sup}_{t\in[0,\frac{1}{2B(k)}]}\|u\|^2_{\frac{11}{6}+}\langle u,h_k^{1/2}e_k\rangle^2)\, ds\right. \\
+ \quad & \left.\sum_{k\neq 0}\int_0^{t-\delta}e^{-4\pi^2|k|^2\nu(t-s)}\int_0^{\frac{1}{2B}}e^{-2\nu\lambda_1(s-r)}E(\text{ess sup}_{t\in[0,\frac{1}{2B(k)}]}\|u\|^2_{\frac{11}{6}+}\langle u,h_k^{1/2}e_k\rangle^2)\, ds\right)
\end{aligned}
$$

by the Ito isometry. Bringing the sum into the expectation in the last two integrals and estimating $|u|_2^2$ by Lemma 3.1, we get that

$$
\begin{aligned}
E(\text{ess sup}_{t\in[0,\frac{1}{2B}]}\|u\|^2_{\frac{11}{6}+}\sum_{k\neq 0}\langle u,h_k^{1/2}e_k\rangle^2) &\leq 2E(\text{ess sup}_{t\in[0,\frac{1}{2B}]}\|u\|^2_{\frac{11}{6}+}|u|_2^2) \\
&\leq \frac{1}{\nu\lambda_1}E(\text{ess sup}_{t\in[0,\frac{1}{2B}]}\|u\|^2_{\frac{11}{6}+})\sum_{k\neq 0}h_k
\end{aligned}
$$

We substitute this estimate into the integrals and get that

$$
E(H)\leq\frac{C}{|B|^2}E(\text{ess sup}_{t\in[0,\frac{1}{2B}]}\|u\|^2_{\frac{11}{6}+})+\frac{C'}{|B|}
$$

with new constants C and C'. **QED**

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