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Equilibrium in Marine Mutual Insurance Markets with Convex Operating Costs

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Abstract

The paper analyzes the possibility of reaching an equilibrium in a market of marine mutual insurance syndicates, called Protection and Indemnity Clubs, or P&I Clubs for short, displaying economies of scale. Our analysis rationalizes some empirically documented findings, and points out an interesting future scenario.

We find an equilibrium in a market of mutual marine insurers, in which some smaller clubs, having operating costs above average, may grow larger relative to the other clubs in order to become more cost effective, and where medium to larger cost efficient clubs may stay unchanged or some even downsize relative to the others. Some of the very large clubs suffering from diseconomies of scale may have a motive to further increase relative to the other clubs.

According to observations, most clubs have, during the last decade, expanded significantly in size measured by gross tonnage of entered ships, some clubs have merged, but very few seem to have decreased their underwriting activity, in particular none of the really large ones.

The analysis points to the following future scenario: The small and the medium to large clubs converge in size, while there is a possibility for some very large clubs to be present as well.

KEYWORDS: Marine Insurance, syndicated market, P&I clubs, equilibrium, economy of scale, diseconomy of scale

1   Introduction

The paper takes as given two scale effects observed in the marine mutual insurance industry, see e.g., Li and Shan (2004), and investigates if his is consistent with a partial equilibrium model. While the insurance products offered by the P&I Clubs are rather similar, the ability to lower costs seems like an important factor for the competitiveness of these clubs. This can be achieved by utilizing scale economies.

Economies of scale due to uncertainty apparently exists in the marine mutual industry, as has been documented by several authors, e.g., by Katrishen and Scordis (1998). The explanation for this, shared by many different insurance markets, is that of diversification. The more ships in a club, the less is the risk per ship, as follows essentially from the Law of Large Numbers. Even if the liabilities are not independently and identically distributed, diversification has the effect of lowering the overall risk in the portfolio in relation to its size, also observed in other industries, like e.g., banking (e.g., Baltensperger (1972)). The effect this has on the operating cost per ship is typically that of lowering this type of cost as a function of the number of ships in a club. These costs involve opportunity costs from holding reserves to deal with future claims payments. Since the uncertainty of cash flows decreases relatively with the number of policies, the costs caused by holding cash reserves and capital accounts also decrease relatively with size.

Next are the effects on the operating costs that are not directly related to uncertainty of the insurance portfolio. These can either decrease or increase relative to an increase in size. For mutual marine clubs the operating costs have been documented to increase at a lower rate than their output. However multinational insurers can only achieve this effect up to a certain size, and those firms with size above a particular level typically display diseconomies of scale in their operating costs. This may be due to the fact that an increase in size is accompanied by an increase in the complexity of firms' operations and the cost of coordinating those operations. It may come to the point where this increase is also offsetting the scale economies due to diversification, in which case the overall operating costs may display diseconomies of scale, as found by Li and Shan (2004). These authors looked at data from 13 major

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P&I Clubs for the years 2002 and 2003 and found a U-shaped operating cost function, with a turning point at about 80 million gross tons.

The objective of this paper is to investigate if such findings are consistent with an economic equilibrium. And if so, can we learn from the nature of the competitive equilibrium in what direction this market is likely to move in the future?

The paper is organized as follows: In Section 2 we formulate the theory of syndicates for a single P&I Club consisting of a certain number of ship owners. In Section 3 we consider a market of such syndicates and find an optimal risk sharing arrangement in this market when each P&I Club has a operating cost depending on its size. Section 4 discusses existence of various types of equilibriums in such markets. In Section 5 we develop market insurance premiums for different loss distributions. In Section 6 we provide three different interpretations of an equilibrium in the market of P&I Clubs, in increasing order of realism. Here we present estimates of the risk tolerances of the various clubs in the International Group. Section 7 concludes.

2   A P&I-Club Considered as a Syndicate

If a group of businessmen is not satisfied with the offers received from insurance companies, the members of the group could set up a mutual insurance scheme of their own. All that is needed is really an informal agreement on how losses caused by specific random events and hitting some member, shall be shared by all. A P&I Club, which offers "Protection and Indemnity" to ship owners, is such an arrangement. The old standard marine policy of Lloyds covered only three-fourths of the liability which a ship could incur in a collision, and left a number of other liabilities completely uncovered. To cover these risks ship owners formed the P&I Clubs, which in reality are mutual insurance companies.

It should be possible to include the risks covered by the P&I policy in an ordinary marine insurance contract, but it seems that the Clubs have certain advantages. The members are ship owners, and their number is, even on a world wide basis, fairly limited. This means that some simplifications and some informality is possible. The premium paid is proportional to the gross tonnage of the vessel, and the number of votes a member can cast is proportional to the number of gross tons he has registered in the Club.

Based on these observations, let us first formulate a model for such a Club. Since it is a mutual arrangement, the theory initiated by Karl Borch (1960a), (1960b) and (1962) seems appropriate. It goes as follows:

Given is a group of I ship owners, each one facing a certain risk repre-

sented by a random variable U_i, $i \in \mathcal{I} = \{1, 2, \ldots, I\}$. We model ship owner i's random endowments X_i by

$$X_i = w_i - U_i, \tag{1}$$

where w_i = ship owner i's wealth, assumed to be a constant here, and U_i is the potential loss facing ship owner i. The representation (1) is supposed to hold after the ship owner has insured his fleet in the commercial marine insurance market. The residual risk U_i can be viewed as the risk not covered above some cap, or reinsurance layer, often found in XL -reinsurance contracts.

The ship owners are assumed risk averse with marginal utility functions $u_i'(x) = e^{-x/a_i}, i \in \mathcal{I}$, and therefore they seek further insurance. Not being able to obtain this in the commercial marine insurance market, these individuals are then forced to share these residual risks between themselves, rather than facing them in splendid isolation. The random endowment of ship owner i is denoted by Y_i after the exchange has taken place. Let us denote the sum of the initial endowments X_i by X_M, $X_M = \sum_{j=1}^{I} X_j$. Then the Pareto optimal sharing rules, also known to be the partial equilibrium allocations of the ship owners, are known to have the following form

$$Y_i = \frac{a_i}{\alpha} X_M + b_i, \qquad \text{where} \qquad b_i = a_i \ln \lambda_i - a_i \frac{k}{\alpha}, \qquad i \in \mathcal{I}. \tag{2}$$

This follows from the first order conditions of optimal risk exchange, given here by equating the marginal utilities multiplied by positive constants λ_i:

$$\lambda_i e^{-Y_i/a_i} = \xi, \qquad a.s., \qquad i \in \mathcal{I}, \tag{3}$$

where ξ is the state price deflator, or marginal utility of the representative agent. After taking logarithms in this relation, and summing over i, market clearing implies

$$\xi = e^{(k-X_M)/\alpha}, \qquad \text{where} \qquad k = \sum_{j=1}^{I} a_j \ln \lambda_j, \qquad \alpha = \sum_{j=1}^{I} a_j. \tag{4}$$

Thus the optimal sharing rules are *affine* in X_M. The constants of proportionality a_i/α are simply equal to to each ship owner's risk tolerance, measured relative to the other members. In order to compensate for the fact that the least risk-averse ship owner will hold the larger proportion of the total fleet, zero-sum side payments occur between the ship owners, here represented by the terms b_i. Without these side payments a ship owner, with a small initial wealth but with a large risk tolerance, would end up with a

large final wealth, but this could not possibly be consistent with his budget constraint.

In order to determine the ray $\lambda = (\lambda_1, \ldots, \lambda_I)$, we employ precisely the said budget constraints:

$$E(Y_i e^{(k-X_M)/\alpha}) = E(X_i e^{(k-X_M)/\alpha}), \qquad i \in \mathcal{I}, \tag{5}$$

from which side payments b_i are found as

$$b_i = \frac{E\{X_i e^{-X_M/\alpha} - \frac{a_i}{\alpha} X_M e^{-X_M/\alpha}\}}{E\{e^{-X_M/\alpha}\}}, \qquad i \in \mathcal{I}. \tag{6}$$

Now the optimal sharing rules Y_i are completely determined in terms of the given primitives of the model.

The ray λ can also be determined modulo a normalization. Letting $k = \sum_{j=1}^{I} a_j \ln \lambda_j$ denote this normalization, then

$$\lambda_i = e^{b_i/a_i} e^{k/\alpha}, \qquad i \in \mathcal{I}.$$

If we impose that $E\{\xi\} = 1/(1+r)$ where r is the risk free interest rate, we obtain $e^{-k/\alpha} = (1+r)E\{e^{-X_M/\alpha}\}$, in which case the constants λ are given by

$$\lambda_i = \frac{e^{b_i/a_i}}{(1+r)E\{e^{-X_M/\alpha}\}}, \qquad i \in \mathcal{I}.$$

In this model market prices are given by

$$\pi(Z) = \frac{1}{1+r} \frac{E\{Z \cdot e^{-X_M/\alpha}\}}{E\{e^{-X_M/\alpha}\}}, \qquad \text{for any} \qquad Z \in L^2, \tag{7}$$

where Z is any risk having a finite variance, i.e., being in the set L^2 for short. Alternatively this can be written

$$\pi(Z) = \frac{1}{1+r} \left\{ E(Z) + \frac{\text{cov}(Z, e^{-X_M/\alpha})}{E\{e^{-X_M/\alpha}\}} \right\}, \qquad \text{for any} \qquad Z \in L^2. \tag{8}$$

The last term in the expression (8) is the risk premium, which would disappear under risk neutrality.

The results related to the explicit form of the side payments given in (6) can be found in Aase (1993), where also the pricing rule (7) is derived. Further results regarding this type of models can be found in Aase (2002, 2004a and 2004b).

3 Equilibrium in a Market of P&I Clubs

The model of the previous section can be interpreted along the following lines. The group of ship owners have formed a syndicate, and the objective function of this syndicate is given by $u_\lambda(x)$, where $u'_\lambda(x) = e^{-x/\alpha}$. It should be noted that this function has the same form as the ship owners' individual marginal utility functions, being of the negative exponential type, and hence displaying constant absolute risk aversion. Hence, the function $u_\lambda(x)$ can be interpreted as the objective function of the P&I Club, which is really a syndicate.

Notice that we do not impose a "utility" function on the clubs. Rather the objective function $u_\lambda(x)$ is endogenously determined through the formation of the syndicate. When we later talk about the "risk aversion of the P&I Club", what we mean is the parameter $(1/\alpha)$, and likewise for the reciprocal "risk tolerance of the P&I Club" α. As indicated above, $\alpha = \sum_{i=1}^I a_i$, so the syndicate's risk tolerance is the sum of the risk tolerances of the individual ship owners that constitute this P&I Club. When the ship owners in a club are all risk averse, so is the P&I Club, and when the ship owners are all risk tolerant, so is the P&I Club.

Consider a market of N different P&I Clubs formed this way, indexed by $n \in \mathcal{N} = \{1, 2, \cdots, N\}$, having "objective functions" $v_n(x)$ of the forms $v'_n(x) = e^{-x/\alpha_n}, n \in \mathcal{N}$. That is, the objective for each Club n to is solve

$$\sup_{Y \in L^2} E\big(v_n(Y)\big) \qquad s.t. \qquad \pi(Y) \leq \pi(X_n). \qquad (9)$$

By an equilibrium we mean the simultaneous determination of a linear price functional $\pi(\cdot)$, and optimal portfolios $(Y_1, Y_2, \cdots, Y_n)$ such that Y_n solves the problem (9) and markets clear: $\sum_{k=1}^N Y_k = \sum_{k=1}^N X_k = X_M$. We intend to find the optimal risk sharing arrangement in this market, and some of its economic implications.

The motivation behind this construction is that the residual risks the ship owners retain after commercial marine insurance contracts have been underwritten, can not be further insured in the commercial marine insurance market. Recall that this is simply the original reason for the existence of these clubs. If the clubs are going to trade risks, it seems like a logical consequence of these market structures that this trade will have to take place among themselves. Viewed this way, the clubs can be thought of as members in a syndicate with objective functions $v_n(x), n \in \mathcal{N}$, and we may apply the theory of optimal risk sharing in a syndicate on this particular market of P&I Clubs. According to this definition the International Group of P&I Clubs, containing the major insurers in this category at present, is a syndicate.

The portfolio X_n of Club n is

$$X_n = w_n + m_n(p_n - c_n(m_n)) - U_n, \qquad n \in \mathcal{N}, \qquad (10)$$

where w_n = the reserves, m_n = the number of ships in the n-th club's portfolio, p_n = premiums per ship, $c_n(m_n)$ = the operating costs per ship for club n, and U_n = random loss incurred by the n-th club, where $E(U_n) = \mu_n, n \in \mathcal{N}$.

To make the role of size predominant, we may assume that the ships are homogeneous. The initial premiums p_n are by definition varying with n, a variation which may or may not be compensated in equilibrium (we return to this issue later), and the mean losses $\mu_n = \mu$ for all $n \in \mathcal{N}$. The cost function $c_n(m_n) = c(m_n)$ of P&I Club n is here assumed to be a convex function of size m_n only, decreasing sharply in the beginning, then flattening out towards some minimum (around 80 million gross tons in the data), and finally increasing slightly as the club grows bigger. The function looks like a left skewed smile.

In general we do not need any independence assumptions of the various random losses U_n, which are assumed to have an arbitrary, as long as the model is well defined, joint distribution, only satisfying certain moment restrictions to become clear later. However, the distribution of U_n will depend upon the number of ships m_n of club n, and typically the standard deviation of U_n will increase slower than the number of ships m_n grows, stemming from the diversification effect of increasing the number of ships in the portfolio of a club. For the case where $U_n = \sum_{j=1}^{m_n} Z_j^{(n)}$, where the $Z_j^{(n)}$ represents the potential loss incurred from ship j in the portfolio of Club n, if these losses are mutually independent, then the standard deviation of U_n is proportional to $\sqrt{m_n}$.

Since the perils of the sea often induce several ships to suffer simultaneous losses caused by the same incident, like a collision, it is an advantage that the model allows for dependencies between the losses U_n.

Employing the theory of the previous section, we notice that this implies that the optimal portfolios Y_n after trade among the clubs are given by

$$Y_n = \frac{\alpha_n}{a}\big(w + M(p - c_M) - U\big) + b_n, \qquad n \in \mathcal{N}. \qquad (11)$$

Here $M = \sum_{k=1}^N m_k$, $p = \frac{1}{M}\sum_{k=1}^N m_k p_k$, $c_M = \frac{1}{M}\sum_{k=1}^N m_k c(m_k)$, $w = \sum_{k=1}^N w_k$, $a = \sum_{k=1}^N \alpha_k$, $U = \sum_{k=1}^N U_k$ and b_n are the side payments. The market portfolio $X_M = (w + M(p - c_M) - U)$, and $EU = \mu M$, where $\mu = \frac{1}{M}\sum_{k=1}^N \mu_k$.

Club n's original risk U_n has been replaced by the fraction α_n/a of the diversified risk U, and the reserves, premiums and operating costs have been

replaced by smooth versions at this same ratio[1]. The fraction α_n/a is to be interpreted as Club n's risk tolerance relative to the risk tolerance of the market. In addition we have the side payments b_n, which can be written, by virtue of (6) and (7)

$$
\begin{aligned}
b_n = \big(w_n - \frac{\alpha_n}{a}w\big) + \big(m_n p_n - \frac{\alpha_n}{a}Mp\big) \\
+ \big(\frac{\alpha_n}{a}Mc_M - m_n c(m_n)\big) + \big(\frac{\alpha_n}{a}\pi(U) - \pi(U_n)\big).
\end{aligned}
\tag{12}
$$

The two first terms adjust for reserve and premium smoothing, the third for operation costs smoothing and the last for costs of diversification. These side payments are transfers that take place between the clubs internally, and between the syndicate and the ship owners. In one of the scenarios discussed, the mutual one, these adjustments compensate for the varying initial premiums p_n, so that after pooling all the ship owners face the same premium p per ship.

In Section 6 we consider three different scenarios for the premiums p_n. In the first the initial premium covers, among other things, the costs $c(m_n)$ of each Club n, in the second all the p_n are equal, and in the third they depart only because of credit risk differences.

The fact that the above allocations $Y_n, n \in \mathcal{N}$, together with the pricing functional given in (7) constitute a competitive equilibrium (if it exists), and that these equilibrium allocations are Pareto optimal, follow from the references cited in Section 2.

In the next section we explore the implications of the optimal risk sharing arrangement given in equations (11). We return to the side payments (12) in Section 6.

4 Feasibility of Equilibria

First we investigate if the equilibrium allocations derived in the previous section are feasible. If no trade takes place between the P&I Clubs, the premium p per ship must cover (i) the operating costs, (ii) net expected value of the losses, and (iii) a compensation for risk bearing. This principle really dates back to Adam Smith (1776), who wrote in his *Wealth of Nations* (Book I Chapter 10) that the insurance "premiums must be sufficient to compensate the common losses, to pay the expense of management, and to afford such a

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[1]This smoothing could be seen in light of the current (Spring 2005) discussion between regulators and American Insurance Group regarding "smoothing of accountancy results". Presumably the latter kind of smoothing, partly using derivatives, is different from (11).

profit as might have been drawn from an equal capital employment in any common trade".

Using the market pricing rule given in equation (8), this principle gives the following expression for the premium p_n per ship for Club n

$$p_n^{(1)} = c(m_n) + \pi\big(Z_j^{(n)}\big) = c(m_n) + \frac{1}{1+r}\left(\mu + \frac{\text{cov}(Z_j^{(n)}, e^{U/a})}{E(e^{U/a})}\right),$$

which is the principle we consider in the first scenario in Section 6. Here $Z_j^{(n)}$ represents the potential loss from any of the ships $j, j = 1, 2, \cdots, m_n$, in Club n's portfolio, the individual losses all having expected value μ, and we have used that the first three terms in the market portfolio are constants, which all cancel in the above pricing formula. The last term is the risk premium of each ship in the club's portfolio. This formula can also be written more simply as

$$p_n^{(1)} = c(m_n) + \frac{1}{m_n}\pi(U_n). \tag{13}$$

From these expressions we see that if the clubs do not pool, there can not be any equilibrium at the prices given in (13), since these premiums vary with the clubs through the operating cost term $c(m_n)$. The various clubs would then offer different premiums for otherwise identical risks, since we have assumed that the ships are homogeneous, and this is not compatible with an equilibrium.

There could, perhaps, be an equilibrium of this type if all the premiums were equal, i.e., $p_1 = p_2 = \cdots = p_N$, in which case $m_1 = m_2 = \cdots = m_N$. The present market structure is not, however, consistent with this uniformity in size across the clubs, as can be seen from Table 1 is Section 6. The remaining possibility is then that each club charges $p = \frac{1}{N}\sum_k p_k$, in which case there could, perhaps, be a no pooling equilibrium with varying profits among the clubs. No trade between the clubs hinges upon affine objective functions $v_n(x) = a_n x + c_n$ of all the clubs, which could only be a consequence of risk neutral ship owners in the first place (see sections 2 and 3). If these individuals were risk neutral, they would not seek any insurance solutions, and the P&I Clubs would simply not exist.

When the objective functions are strictly concave, it is clear from the results of Section 3 that there are gains from trade even if all the clubs had the same costs. In particular there will be gains from diversification with strictly concave objective functions. We elaborate further on this point in Section 5.

Turning to the pooling equilibrium, consider the optimal portfolios Y_n given in equation (11). Notice that Club n now holds the fraction $\left(\frac{\alpha_n}{a}\right)$ of

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the market portfolio X_M. Originally it faced total operating costs $m_n c(m_n)$, and now it appears to be facing the weighted average operating costs $c_M = \frac{1}{M} \sum_{k=1}^{N} m_k c(m_k)$. We may rewrite the expression for Y_n as follows

$$Y_n = \hat{w}_n + \frac{\alpha_n}{a} \big(M(p - c_M) - U \big), \qquad n \in \mathcal{N}, \tag{14}$$

where $\hat{w}_n = (\frac{\alpha_n}{a} w + b_n)$ can be interpreted as the reserves of Club n after pooling. Using the pricing principle outlined above, it must be the case that

$$\frac{\alpha_n}{a} Mp = \pi \Big(\frac{\alpha_n}{a} (M c_M + U) \Big),$$

or, since $\pi(\cdot)$ is linear,

$$p = c_M + \frac{1}{M} \pi(U). \tag{15}$$

Notice that, given the above premium principle, the difference between the premiums before and after pooling is

$$p_n^{(1)} - p = (c(m_n) - c_M) + (\frac{1}{m_n} \pi(U_n) - \frac{1}{M} \pi(U)).$$

Since the ships are homogeneous, the $Z_j^{(n)}$'s have the same distributions for all $j = 1, 2, \cdots, m_n; n = 1, 2, \cdots, N$. By the linearity of the pricing functional π, the last term in the above difference cancels out. Notice that this is true regardless of the dependence between these losses. Thus

$$p_n^{(1)} - p = c(m_n) - c_M. \tag{16}$$

Thus these price differences are caused solely by the operating cost differences in this model.

From the formula (8), the premium p per ship after pooling can also be written

$$p = c_M + \frac{1}{1+r} \Big(\mu + \frac{1}{M} \frac{\text{cov}(U, e^{U/a})}{E(e^{U/a})} \Big). \tag{17}$$

From this expression we notice that the market premium per vessel does not depend upon the particular Club n after poling, and is thus a viable premium. This means that an equilibrium exists in the above model under our assumptions, i.e., that of operational costs varying across the different P&I Clubs. We formulate our findings this far as follows:

Theorem 1 *An equilibrium exists in the market of P&I Clubs outlined above, where pooling takes place, the resulting equilibrium allocations Y_n, $n \in \mathcal{N}$*

given in (14) are Pareto optimal, and the premium per vessel p is given in equation (17).

A "no pooling" equilibrium could exist at premium p if all P&I clubs were risk neutral. Risk neutrality is, on the other hand, not consistent with ship owners demanding insurance.

The core insurance product offered by International Group Clubs is very similar, complying with the above model. In the hypothetical situation with no trade, presumably the ship owners would seek out insurers that offer the lowest rates, implying that they would end up in the clubs of size near the optimal, from an operating costs point of view. As a consequence these clubs would grow away from the optimal cost level, and more clubs would have to be formed. In the end one should observe a market where all the clubs were identical in size, corresponding to the optimal operating cost level. This is not consistent with the market structure in this industry today, as is illustrated in Table 1 of Li and Shan (2004).

They present a list of 15 P&I clubs ranging from the two largest, the London based "United Kingdom" at 120 million g.t and the Oslo based "Gard" at 97.7 million g.t. to the London based "Shipowners" at 8.8 million g.t. and the Haren based "NNPC" at 0.064 million g.t. The average among these clubs are found by the authors to be 45.6 million g.t.

Most of the clubs in their investigation appear to have experienced an increase during the last decade, some up to 10% annual rates, with a weighted average of about 7% per annum. Only the Oslo based "Skuld" is displayed with a negative annual growth rate of -5.23% during the period 2000-2003. The increase has largely been achieved through mergers and an expansion in membership. The "Liverpool & London Club" has disappeared, and the "North of England Club" absorbed the "Newcastle Club" in 1999.

5 Computation of Risk Premia for various Probability Distributions

The reader is perhaps wondering at this stage how the risk premiums appearing in the formulas (13) and (17) look like. To answer this one needs the probability distributions of the random losses $U_n, n \in \mathcal{N}$ to be specified. In this section we present two different distributional assumptions, and demonstrate the required computations.

5.1 Multinormally Distributed Claims

First we consider the case where the risks U_n's are multinormally distributed. Here we may take as a starting point the insurance version of the CAPM which states that (see Aase (2002))

$$\pi(X_n) - \frac{1}{1+r}EX_n = \frac{\text{cov}(X_n, X_M)}{\text{var}X_M}\big(\pi(X_M) - \frac{1}{1+r}EX_M\big), \qquad (18)$$

where the covariance term in units of the variance of the market portfolio can be interpreted as portfolio n's beta. One could of course question the validity of the normal assumption applied to insurance claims, which can not take on negative values. To this there is to say that the normal distribution is applied with success to a vide variety of situation where, say, a negative value is meaningless, such as models of heights, or weights of human sub-populations. The variance and the mean will then adjust the probability of observing values in the forbidden region to be negligible.

In economics we often have the added difficulty that the preferences may not be defined for negative values of wealth, which is here a problem with the right hand tails of the loss distribution. In our model this is not a formal problem, since the negative exponential function is defined for all real numbers. This distribution allows us to model bankruptcy risk, which we will take into account in Scenario 3. The normal distribution suffers from having too heavy left tails and too light right tails as compared to various types of loss data, since this distribution is symmetrical around its mean. It has many other convenient properties though, in particular are we able to model dependencies with ease. Assuming this joint distribution for the moment, the CAPM relation can alternative be written in terms of the random losses U_n

$$\pi(U_n) - \frac{1}{1+r}EU_n = \frac{\text{cov}(U_n, U)}{\text{var}(U)}\big(\pi(U) - \frac{1}{1+r}EU\big). \qquad (19)$$

Both in this relation and in the formula for the market premium of each vessel in (17), everything is determined as soon as we are able to compute the market value of U, i.e.,

$$\pi(U) = \frac{1}{1+r}\big(EU + \frac{\text{cov}(U, e^{U/a})}{E(e^{U/a})}\big).$$

To this end, we use Stein's lemma to conclude that

$$\text{cov}(U, e^{U/a}) = \frac{\text{var}(U)}{a}E(e^{U/a}).$$

From this we obtain that the market value of any of the losses U_n is given by

$$\pi(U_n) = \frac{1}{1+r}\big(E(U_n) + \frac{1}{a}\text{cov}(U_n, U)\big), \qquad n \in \mathcal{N}, \tag{20}$$

which leads to

$$p_n^{(1)} = c(m_n) + \frac{1}{1+r}\big(\mu + \frac{1}{a}\text{cov}(Z_j^{(n)}, U)\big), \qquad n \in \mathcal{N},$$

and the premium p of each vessel in equilibrium is

$$p = c_M + \frac{1}{1+r}\Big\{\mu + \frac{1}{aM}\Big(\sum_k \text{var}(U_k) + 2\sum_{k>l}\text{cov}(U_k, U_l)\Big)\Big\}. \tag{21}$$

We notice that as the market becomes more risk tolerant, i.e., a increases, the risk premium decreases, and when the market risk aversion $(1/a)$ increases, the risk premium increases.

Notice also that since the covariances $\text{cov}(U_n, U_l)$ grow slower than $m_n m_l$ as these sizes of the fleets increase, a consequence of diversification, the risk premium in (21) decreases as the number of vessels increases, a reasonable property in a competitive market. This property does not hold in the polar case of independence: Suppose each $U_n = \sum_{j=1}^{m_n} Z_j^{(n)}$, where the individual losses from each ship are all i.i.d. normally distributed with mean μ and variance σ^2, and assume that all the U_n's are mutually independent. It follows that

$$p = c_M + \frac{1}{1+r}\big\{\mu + \frac{1}{a}\sigma^2\big\}. \tag{22}$$

In this case we readily observe the effects from the probability distribution of the individual losses on the equilibrium premium p of each vessel, as well as the effect from the attitude towards risk in the market represented by the parameter a. The effects from the covariances are no longer present, so the premium is invariant to the size of the fleet. This is likely to be an artifact also of the assumption about constant absolute risk aversions. [2]

5.2 Gamma Distributed Claims

We noticed in the previous section that one undesirable property of the normal distribution is that it allows for negative claims, which is not possible in real markets the way we model such insurance claims. For this reason we now present the relevant analysis in a situation when the claims are gamma

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[2]We are keenly conscious about the limitations of this assumptions, of which the wealth effect will be of some concern to us.

distributed, and consequently only take on positive values. Also this distribution allows us to model varying credit risks. The undesirable property now, however, is that we retain the independence assumption leading to the premium formula (22). The tails of this distribution has sometimes been found to be too light in some lines of insurance, but obviously it is a better candidate than the normal for many insurance problems.

To this end, assume that the individual losses $Z_j^{(n)}$ from each ship are all i.i.d. exponentially distributed with parameter λ. This distribution has its support on the positive real axis, and thus overcomes the potential problem of the normal distribution in this regard. Since the loss facing P&I Club n is given by $U_n = \sum_{j=1}^{m_n} Z_j^{(n)}$, it is a consequence of the properties of the exponential distribution that U_n is gamma distributed with parameters (m_n, λ).

In this situation we do not have a simple version of the insurance CAPM like the one in (20), but we intend to find the risk premium of each individual contract in formula (17). For this we need the probability distribution of $U = \sum_{n=1}^N U_n$. Under our assumptions the accumulated loss U is gamma distributed with parameters (M, λ), where $M = \sum_{k=1}^N m_k$. It is precisely here we need the independence assumption of the U_n's. Only independent gammas having the same λ-parameter convolute to a gamma distribution, whereas arbitrary normals sum to a normal distribution (provided they are jointly normal).

In order to derive the relevant formulas for the market premiums, we need to compute the quantities $\mathrm{cov}(U, e^{U/a})$ and $E(e^{U/a})$. Since $\mathrm{cov}(U, e^{U/a}) = E(Ue^{U/a}) - EU E(e^{U/a})$, we attempt to find the terms on the right hand side of this equality. Notice that we must require that $\lambda > a^{-1}$ for the following formulas to be well defined.

First recall that $E(U) = M/\lambda$. Second we find $E(Ue^{U/a})$ as follows

$$E(Ue^{U/a}) = \int_0^\infty ue^{u/a}\frac{\lambda^M}{\Gamma(M)}u^{M-1}e^{-\lambda u}du = \frac{M}{\lambda}\Big(\frac{\lambda}{\lambda - a^{-1}}\Big)^{M+1}.$$

Third we compute $E(e^{U/a})$. It is given by

$$E(e^{U/a}) = \int_0^\infty e^{u/a}\frac{\lambda^M}{\Gamma(M)}u^{M-1}e^{-\lambda u}du = \Big(\frac{\lambda}{\lambda - a^{-1}}\Big)^M.$$

Putting this together and inserting into the expression for the market premium in (17), we obtain the formula

$$p = c_M + \frac{1}{1+r}\Big\{\mu + \frac{1}{\lambda}\Big(\frac{\lambda}{\lambda - a^{-1}} - 1\Big)\Big\},$$

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where the last term is the risk premium. Since $\mu = 1/\lambda$, this can be written simply as

$$p = c_M + \frac{1}{1+r}\Big(\frac{1}{\lambda - a^{-1}}\Big). \tag{23}$$

This formula displays some of the same basic properties as the formula (22) of the previous section, for example it does not depend on the number of ships in the market fleet. Some desirable properties are the following: (i) When the risk tolerance a of the market increases, the risk premium decreases, (ii) When the risk aversion a^{-1} in the market increases, the risk premium increases, (iii) Under risk neutrality $a^{-1} = 0$, and the risk premium is equal to zero, (iv) The risk premium decreases as the parameter λ increases.

Property (iv) needs some further explanation. Since $E(Z_1^{(n)}) = 1/\lambda$ (for any n) and $\mathrm{var} Z_1^{(n)} = 1/\lambda^2$, an increase in the parameter λ implies that the expected loss per ship decreases, and so does its variance. Thus the risk of insuring this ship ought to decrease, and so should the risk premium. Accordingly we find property (iv) rather natural as well.

We can also calculate $\pi(U_n)$. By the independence between the U_n's, we find that

$$\frac{\mathrm{cov}\big(U_n, e^{U/a}\big)}{E(e^{U/a})} = \frac{m_n}{\lambda}\Big(\frac{\lambda}{\lambda - a^{-1}} - 1\Big),$$

which gives that

$$p_n^{(1)} = c(m_n) + \frac{1}{1+r}\Big(\frac{1}{\lambda - a^{-1}}\Big).$$

As a summary of this section's results, we have the following

Theorem 2 *When the claims U_n against the various P&I Clubs are jointly multinormally distributed, an insurance version of the CAPM applies to these claims, given in the relation (20).*

The market premium p of the individual ships is given in equation (21) for an arbitrary correlation structure, and in equation (22) for the special case that the losses U_n's are mutually independent.

When the claims $Z_j^{(n)}$ caused by the individual ships belonging to Club n are independent and exponentially distributed for all j and n, the resulting claims U_n against the clubs are gamma distributed (m_n, λ) and mutually independent.

As a result the aggregate risk U also has a gamma distribution (M, λ). The market premium p of an individual vessel in this market is given in equation (23).

It would be interesting to test out the results of this section on real data. The risk premiums we have obtained are all positive, well in accordance with

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theory. Insurance practice sometimes tells us otherwise. For example, some lines of insurance may be compared to the banking industry: The premiums paid by the insurance customers may be interpreted as bank deposits requiring positive returns. For this to function, the insurance companies must, like banks, invest the premium reserves in, say, the financial markets and obtain a rate of return higher then the one required by the insurance buyers.

The assets of a bank consists of the debts of a large number of people, and the banks require a higher return on these assets than they pay to the depositors. Competition between the insurance companies may then have the effect that the risk premiums of the insurance claims can be negative, but still the insurance companies may make a profit when the returns on the invested premium reserves are taken into account.

This picture of the insurance industry is of course very much at odds with classical risk theory, but is quite consistent with economic theory. These effects are not incorporated in the above simple model, as it is unclear to the author if this plays an important role in the market for P&I Clubs.

5.3 Maximizing the Objectives of the Clubs

Since we are fortunate enough to have solved the optimization problem (9), we are in position to calculate the optimal objective function of each club. It is easy to see that the inequality

$$E(v_n(X_n)) \leq E(v_n(Y_n))$$

is equivalent to the following inequality

$$\frac{1}{\alpha_n}\pi(U_n) - \log\Big(E\big(e^{\frac{1}{\alpha_n}U_n}\big)\Big) \leq \frac{1}{a}\pi(U) - \log\Big(E\big(e^{\frac{1}{a}U}\big)\Big). \qquad (24)$$

The budget constraint $\pi(X_n) = \pi(Y_n)$ is automatically satisfied, since Y_n solves problem (9). This can be written

$$\frac{\alpha_n}{a} \geq \frac{m_n}{M} + \frac{\alpha_n}{\pi(U)}\log\bigg(\frac{E\big(e^{U/a}\big)}{E\big(e^{U_n/\alpha_n}\big)}\bigg). \qquad (25)$$

Depending on the sign of the last term in this inequality, the final fraction of the risky wealth of Club n may increase or, perhaps, decrease relative to the initial fraction. From the above, what appears to determine this is diversification and risk aversion, but not the operating costs, nor the reserves. More precisely, suppose

$$\frac{E\big(e^{U/a}\big)}{E\big(e^{U_n/\alpha_n}\big)} > 1. \qquad (26)$$

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This means that the risk tolerance (α_n/a) of Club n relative to the syndicate is larger than the ratio of the club's initial insurance risk to the total risk, appropriately measured. Then the inequality (25) says that the final equilibrium portfolio fraction of the club is larger than its initial fraction of ships insured.

For example, in the multinormal case the inequality (25) can be written

$$
\frac{\alpha_n}{a} \geq \frac{m_n}{M} + \frac{(1/2)\alpha_n}{(\pi(U) - EU)}\Big\{\Big(\frac{\sigma}{a}\Big)^2 - \Big(\frac{\sigma_n}{\alpha_n}\Big)^2\Big\}. \tag{27}
$$

This inequality shows that $\frac{\alpha_n}{a} > \frac{m_n}{M}$ if $\frac{\alpha_n}{a} > \frac{\sigma_n}{\sigma}$, i.e., if the risk tolerance of the Club relative to the syndicate is larger than the ratio of the standard deviation of the club's initial risk U_n to the standard deviation of the total risk U, then the final portfolio fraction of the club is larger than its initial fraction of ships insured.

For the gamma distribution the mixed measure of risk and risk aversion appearing in (26) is given by

$$
\frac{E\big(e^{U/a}\big)}{E\big(e^{U_n/\alpha_n}\big)} = \frac{\big(\frac{\lambda}{\lambda - a^{-1}}\big)^M}{\big(\frac{\lambda}{\lambda - \alpha_n^{-1}}\big)^{m_n}},
$$

leading to the the analogous inequality

$$
\frac{\alpha_n}{a} \geq \frac{m_n}{M} + \frac{\alpha_n}{\pi(U)}\Big\{M\mathrm{log}\Big(\frac{\lambda}{\lambda - a^{-1}}\Big) - m_n\mathrm{log}\Big(\frac{\lambda}{\lambda - \alpha_n^{-1}}\Big)\Big)\Big\}.
$$

To a second order Taylor series approximation, this inequality can be written

$$
\frac{\alpha_n}{a} \geq \frac{m_n}{M} + \frac{(1/2)E(U)}{(\pi(U) - E(U))}\frac{1}{\lambda\alpha_n}\Big\{\frac{m_n}{M} - \Big(\frac{\alpha_n}{a}\Big)^2\Big\}.
$$

For any distribution a rather crude approximation is given by

$$
\frac{\alpha_n}{a} \geq \frac{m_n}{M} + \frac{(1/2)\alpha_n}{(\pi(U) - EU)}\Big\{\frac{E(U^2)}{a^2} - \frac{E(U_n^2)}{\alpha_n^2}\Big\}.
$$

Already for the normal we see that it is only accurate if the means of U and U_n are zero. It still provides roughly the the same type of qualitative result as for the normal, but replacing σ_n by $\sqrt{E(U_n^2)}$ and σ by $\sqrt{E(U^2)}$, and can easily be further refined.

The fact that the reserves and the costs fell out of this comparison, is due the properties of the felicity index, being the negative exponential. The management still faces the task of maximizing the objectives of the club, which naturally depend also on these factors. We return to this issue in Section 6.3.

6 Implications of the Economic Analysis

6.1 General observations

In Section 4 we demonstrated that after the optimal risk exchanges have taken place between the P&I Clubs, the operating cost of each Club is the weighted average operating cost c_M of all the clubs, and the premium of each identical ship is the same in this market. At least this is the picture as long as we ignore the side payments b_n. The premium contains an operation cost component, a net premium and a risk premium, and since the latter two must be identical for the same type of risk, the cost components of the equilibrium market premium p must also be identical across this market.

At first sight this may seem to imply that the clubs have no motivation to improve their cost efficiency, since the mere pooling of risks will leave all the clubs with the same level of the operating costs. In general this is a hastened conclusion since, among other things, the liabilities have also changed, and there are the side payments. In Scenario 1, however, where the initial premiums are given by formula (13), this conclusion is about right. For this reason we consider two additional interpretations of the equilibrium analysis, in increasing order of realism, and each assuming an increased degree of competitiveness among the P&I Clubs.

We start by taking a closer look at the side payments. Recall from Section 3 that these can be written

$$
\begin{aligned}
b_n = {} & \left(w_n - \frac{\alpha_n}{a}w\right) + \left(m_n p_n - \frac{\alpha_n}{a}Mp\right) \\
& + \left(\frac{\alpha_n}{a}Mc_M - m_n c(m_n)\right) + \left(\frac{\alpha_n}{a}\pi(U) - \pi(U_n)\right).
\end{aligned}
\tag{28}
$$

The reserve adjustment can be rewritten as

$$
\left(w_n - \frac{\alpha_n}{a}w\right) = \left(w_n - \frac{m_n}{M}\right)w + \left(\frac{m_n}{M} - \frac{\alpha_n}{a}\right)w.
$$

Likewise, the premium adjustment is

$$
\left(m_n p_n - \frac{\alpha_n}{a}Mp\right) = m_n(p_n - p) + p\left(m_n - \frac{\alpha_n}{a}M\right),
$$

and the cost adjustment is

$$
\left(\frac{\alpha_n}{a}Mc_M - m_n c(m_n)\right) = m_n(c_M - c(m_n)) + \left(\frac{\alpha_n}{a}M - m_n\right)c_M.
$$

Finally, as observed in Section 4, the adjustment for diversification can be written

$$
\left(\frac{\alpha_n}{a}\pi(U) - \pi(U_n)\right) = \left(\frac{\alpha_n}{a} - \frac{m_n}{M}\right)\pi(U).
$$

Before we proceed, let us take a closer look at the cost data presented by Li and Shan (2004). These authors defined the operating costs as the sum of the general expenses and acquisition costs, and the average cost per ton is the total operating costs divided by the size of the club. The estimated operating costs range from US $ 2.25 to $ 0.25 per ton for the 20 observations corresponding to P&I clubs of sizes less than 80 million gross tons, and range from $ 0.180 to $ 0.258 per ton for the 5 observations corresponding to P&I clubs of sizes larger that 80 million gross tons, with a weighted average around $ 0.34 per ton (not reported). Thus the estimated cost function is U-shaped, but rather skewed to the left, while the right part of the cost curve does not climb above the weighted average. Let us assume that the operating cost function has this form.

6.2 Calibrating the Risk Aversions

So far we have taken the initial portfolios X_n of each Club n as exogenously given, and derived the optimal portfolios Y_n endogenously. Consider the data of Table 1. The three first columns of the table contain information that can be found in various publicly available sources, and agree with parts of Table 1 of Li and Shan (2004), except that we focus attention on the clubs in the International Group. [3]

Since the data in this table can be thought of as the result of some form of equilibrium, the sizes in column four, when transformed to relative sizes, can be interpreted as the final fractions (α_n/a) of each club's share of the total fleet in the market, at least under certain conditions. Those are that the reserves, initial numbers of insured ships, premiums, costs and risks are all "in the same proportions". This partly follows from the expressions for the final portfolios Y_n after pooling, since the side payments are all zero under these conditions. With this interpretation in mind, we have tentatively estimated the proportions (α_n/a) in Column 5 in Table 1.

Small clubs typically strive to become bigger, whereas big clubs rarely try to size down their operations, despite possible diseconomies of scale. The only big P&I Club that has decreased during recent years is "Skuld", but its size in 2003 was 51.5 million g.t., well below the observed "optimal" size of 80 million g.t. Otherwise all the big clubs have also increased during the last years, and the largest club in the sample, "United Kingdom", has displayed a growth rate of 10% per annum during the period 2000-03.

Of the 13 clubs all members of the International Group, three of the smaller and medium sized clubs have growth rates even larger than this,

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[3]Consult Li and Shan (2004) for further information.

Club	Period	Growth	Size	Risk tolerance	Adjust.
United Kingdom	2000-03	10.06%	120.0	0.176	+
Gard	1996-03	4.76%	97.7	0.143	0
Britannia	1994-03	5.68%	80.0	0.117	0
Steamship	2001-03	1.38%	64.5	0.094	0
Standard	2001-03	13.03%	58.0	0.085	-
Japan	1993-03	2.24%	54.0	0.079	-
Skuld	2000-03	-5.23%	51.5	0.075	-
West of England	1998-03	3.73%	46.0	0.067	-
North of England	2001-03	19.72%	43.0	0.063	0
London	2000-03	2.46%	28.1	0.041	0
American	2000-03	19.33%	17.5	0.026	+
Swedish	2000-03	2.36%	14.8	0.022	+
Shipowners	1998-03	9.86%	8.8	0.013	++
Aggregate			683.9	1.00	

Table 1: *Estimates of the final portfolio fractions $\frac{\alpha_n}{a}$ of the International Group, with suggested adjustments. These are also the ratios of the risk tolerances α_n of the various clubs to the total risk tolerance a of the International Group.*

where "North of England" with above 19% has grown fastest during the period 2001-03.

In our one period model we can not capture this yearly growth, but we would like to say something about the relative distribution of ships between the clubs from the beginning to the end of the period.

Based on the results of Section 5.3 on may, perhaps, think of calculating the fraction α_n/a based on an expression like

$$\frac{\widehat{\alpha_n}}{a} = \frac{m_n}{M} + \frac{\alpha_n}{\pi(U)}\log\left(\frac{E\left(e^{U/a}\right)}{E\left(e^{U_n/\alpha_n}\right)}\right) \tag{29}$$

motivated by the inequality (25). The right hand side depends on the unknown parameter (α_n/a) and is a lower bound, not an "unbiased estimate" in any statistical sense, of course (it is not even a random variable). Moreover, this value takes as given that there are no gains from trade. If the initial risk to the total risk in the market is well balanced to the risk tolerance of the club relative the risk tolerance in the syndicate, the last term on the right of

this equation disappears, in which case

$$\frac{\widehat{\alpha_n}}{a} = \frac{m_n}{M} \tag{30}$$

results. Adopting this as the base case, let us attempt to study the effects, of the operating costs and the reserves.

The parameters α_n play two different roles in our theory. Being given exogenously and determined by the risk tolerances of the ship owners belonging to Club n, they can not be thought of as decision variables. The ratios (α_n/a), on the other hand, can be interpreted as Club n's fraction of the total syndicate wealth after pooling, excluding the side payments. It is likely that the management of Club n has an opinion about what this fraction ought to be.

In the next section we will try to answer this question.

6.3 The Effects of the Side Payments and the Objective Criterion on the Final Portfolios

In this section we analyze the three different scenarios mentioned in Section 3. In the first two we ignore bankruptcy, i.e., we assume that $Y_n(s) \geq 0$ for all states s in the set of possible states $\mathcal{S}$ of the world, but allow for it in the last scenario.

We assume as the *base case* that the clubs are initially well balanced. What we mean by this can perhaps best be illustrated by an example. In the normal case, suppose that $(\alpha_n/a) = (\sigma_n/\sigma)$, i.e., insurance risks and risk tolerances are in harmony initially. Then (27) implies that $(\alpha_n/a) \geq (m_n/M)$.

We then ask the question: When management considers the reserves and the operating costs of Club n, should they alter this fraction, and if so, in what direction?

In a choice between two alternatives, our objective function picks the preferred outcome independent of the "wealth" level of the club. Still our objective criterion depends on both reserves, premium income and costs, and can be used to analyze the effects of changing the final portfolio weights. In addition we can argue directly from the expressions for the optimal portfolios Y_n, which we have closed formulas for. This may work well, except for exploring the possible effects of diversification, but here the result (25), or (27), is useful.

6.3.1 The Non-Profit Mutual

In the first scenario we imagine that the ship owners initially pay p_n in premium per ship to Club n, and are later compensated $(p - p_n)$ per ship by the syndicate after pooling has taken place. In the model there is no time lag between these two events. Any additional premium compensations are shared among the clubs. The key here is that Club n is acknowledged by the syndicate to have obtained the premium p_n per ship in its portfolio. In this scenario the premium p_n is given by

$$p_n^{(1)} = c(m_n) + \frac{1}{m_n}\pi(U_n) \qquad n \in \mathcal{N}.$$

The side payments can be written

$$b_n = m_n(p_n^{(1)} - p) + m_n(c_M - c(m_n)) + \left(w_n - \frac{\alpha_n}{a}w\right), \qquad (31)$$

where the first term is the part of the initial premium adjustment which is exchanged between the Club n and its customers, while the second is the corresponding part of the cost adjustment. The last term is exchanged between Club n and the other clubs.

In mutual insurance *experience rating* is common, where the insured pays an extra premium if the company is doing worse than expected, and is compensated by a "dividend" in the opposite case. The above practice of first paying $p_n^{(1)}$ and then being reimbursed $(p - p_n^{(1)})$ can be considered as a mild form of experience rating.

Since $(p_n^{(1)} - p) = (c(m_n) - c_M)$ as observed is Section 4,

$$b_n = \left(w_n - \frac{\alpha_n}{a}w\right). \qquad (32)$$

After pooling the portfolio of Club n is the following

$$Y_n = \frac{\alpha_n}{a}\left(w + M(p - c_M) - U\right) + b_n = w_n + \frac{\alpha_n}{a}(\pi(U) - U), \qquad (33)$$

where we have used (32). The objective criterion at Y_n is

$$E\left(v_n(Y_n)\right) = 1 - \exp\left\{-\frac{1}{a}\left(\frac{a}{\alpha_n}w_n + \pi(U) - a\log E\left(e^{U/a}\right)\right)\right\}. \qquad (34)$$

Since $w_n > 0$, decreasing (α_n/a) increases the objective function of Club n. Not all the clubs can decrease their final portfolios at the same time, since market clearing requires $\sum_k(\alpha_k/a) = 1$, so only those with the highest bargaining power will manage this.

We shall discuss the following three categories:
(i) a relatively small club with $c(m_n) > c_M$,
(ii) a medium to large, cost effective club where $c(m_n) < c_M$, and
(iii) a very large club where diseconomies of scale implies that $c(m_n) > c_M$.
Note that none of the clubs in Table 1 are in the latter category, only two to tree clubs are in category (i), the rest in category (ii).

Consider a club in category (i). In this case $p_n^{(1)} > p$ and the two first terms in the side payment in (31) are equal but of opposite sign. The first term is the additional premium that this club is allowed to charge initially, but since the ship owners must all pay p in equilibrium, they are compensated by Club n for this additional premium, the compensation being given exactly by the second term. Thus the sum of the two first terms are zero. If this is taken into account, the premium per ship after pooling is p, and the cost per ship after pooling is c_M, so there seems to be no motivation for this club to take more or less risk than it did initially.

The reserve after pooling is still w_n, and the diversification effect can be seen from (25), or even more transparent from (27): If the initial proportion of ships (m_n/M) in the club is well balanced to the reserves and to the initial risks, it seems as the final proportions (α_n/a) ought to remain unchanged in this case.

Consider next a club in category (ii). For a medium to large, cost efficient club with operating cost $c(m_n) < c_M$, the premium $p_n^{(1)}$ satisfies the inequality $p_n^{(1)} < p$, and the two first terms in (31) are still equal, of opposite signs and cancel out. The first term is the premium rebate this club is supposed to offer initially, but since the ship owners must all pay p in equilibrium, they are entitled to reimburse their Club n for this initial discount, the reimbursement being given by the second term.

Finally consider a club in category (iii). Because of diseconomies of scale $c(m_n) > c_M$, and the cash flows from the first two terms in (31) are the same as in category (i).

Summing up, all the different clubs, small, medium or large, have the same motivation to decrease or retain their underwriting after pooling relative to their initial underwriting, provided these are balanced according to the clubs reserves and initial risks held. A likely outcome is that all the side payments b_n are close to zero, and the final portfolios Y_n are given by

$$Y_n \approx \frac{m_n}{M}(w + M(p - c_M) - U), \qquad \text{for each club } n \in \mathcal{N}.$$

This means that the various clubs hold approximately the same fraction of the total fleet after pooling as they held originally, they all receive the same premium p per ship and they all have the same average operating costs.

A more natural solution would seem like one in which the small clubs were motivated to grow in order to improve their operating cost efficiency, and the medium clubs were motivated to size down in order to further capitalize on their cost efficiency. Thus the medium clubs would be able to lend to the small clubs. However, the weakness of Scenario 1 is that it does not give these clubs the right kind of motives to pursue either of these strategies.

In reality the cost efficient medium clubs are subsidizing the cost inefficient small clubs, as well as the cost inefficient large clubs, after pooling, a picture that fits well with the standard view that marine mutual clubs are non-profit and provide services to their members at cost. A truly mutual insurance arrangement typically displays this feature, namely that the "strong" helps the "weak", although it may be hard to visualize the very large club as worthy of any subsidy.

6.3.2 More Competition

In order to overcome the weaknesses obove, we now assume that the syndicate of marine mutual insurance companies is more competitive in that initially the various clubs all receive the same premium p per ship from the ship owners, with no further transfers between these customers and the clubs. In the above formulas for the side payments the premiums $p_n = p_n^{(2)}$ for $n = 1, 2, \cdots, N$, where

$$p_n^{(2)} = p = c_M + \frac{1}{M}\pi(U), \qquad n \in \mathcal{N}. \tag{35}$$

In this case the side payments after pooling are exchanged only between the clubs in the syndicate, and these are given by

$$b_n = m_n(c_M - c(m_n)) + \left(w_n - \frac{\alpha_n}{a}w\right), \tag{36}$$

since now the premium adjustment term is zero for all the clubs. After pooling the final portfolios can be written

$$
\begin{aligned}
Y_n = &- \left(\frac{\alpha_n}{a}Mc_M - m_n(c_M - c(m_n))\right) + \left(\frac{\alpha_n}{a}Mp\right) \\
&+ \left(\frac{\alpha_n}{a}w + (w_n - \frac{\alpha_n}{a}w)\right) - \frac{\alpha_n}{a}U.
\end{aligned}
\tag{37}
$$

The first term on the right hand side can be considered as the negative of the cost plus cost adjustments, the second term is the premium, the third as the reserve plus reserve adjustments, and the last term is the liabilities. Consider first the cost term. The number of ships in Club n after pooling

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is $(\alpha_n/a)M$, so the operating costs of Club n per ship after pooling can be written

$$\Big(\frac{m_n}{M}\frac{a}{\alpha_n}\Big)c(m_n) + \Big(1 - \frac{m_n}{M}\frac{a}{\alpha_n}\Big)c_M, \tag{38}$$

a convex combination of the operating costs $c(m_n)$ of Club n before pooling and the weighted average operating costs of all the clubs in the syndicate.

Consider a small club in category (i). According to our assumptions this club has higher operating costs per ship than the weighted average, i.e., $c(m_n) > c_M$, implying that the cost adjustment term in (36) is negative. The club is now being penalized for its cost inefficiency, which appears more realistic than in Scenario 1.

Exploring the cost component given in equation (38), from an operating cost perspective this formula for the cost per ship after pooling suggests that a small club should try to become larger after pooling in order to get the cost per ship down, since $c(m_n) > c_M$. The premium per ship after pooling is just p, so unlike in the previous scenario, there is no need for the club to decrease in size after pooling to get the premium income per ship up. Recalling the dual interpretation of (α_n/a), increasing the *risk tolerance* (α_n/a) from (σ_n/σ) is consistent with increasing the *portfolio fractions* (α_n/a) from (m_n/M) in equilibrium, as seen by (27) for the normal case, a similar reasoning holding in general.

The club will have incentive to become more cost efficient, which it can achieve by growing. The total premium income will increase in direct proportion to the risk, but since the reserve is unchanged, the regulators are likely to limit the club's increase in risk exposure.

Consider the objective of the club. The optimal portfolio after pooling is

$$Y_n = w_n + m_n(c_M - c(m_n)) + \frac{\alpha_n}{a}(\pi(U) - U), \tag{39}$$

where we have used (36). The objective criterion at Y_n is thus

$$E\big(v_n(Y_n)\big) = 1 - \exp\Big\{-\frac{1}{a}\Big(\frac{a}{\alpha_n}D_n + \pi(U) - a\log E\big(e^{U/a}\big)\Big)\Big\}, \tag{40}$$

where

$$D_n = \big(w_n + m_n(c_M - c(m_n))\big). \tag{41}$$

If $D_n > 0$, decreasing (α_n/a) increases the objective function of Club n, and if $D_n < 0$, increasing (α_n/a) increases the objective of the club. From this it follows that the club will only want to increase its final portfolio if its reserves are sufficiently small, i.e., if $w_n < m_n(c(m_n) - c_M)$, otherwise it wants to decrease risk.

Moving to a club in category (ii), the first term in (36) is now positive, so this club is finally rewarded for its cost efficiency. From an operating cost perspective, expression (38) for the cost per ship after pooling suggests that a medium club should try to downsize its underwriting relative to the other clubs. By doing so, we see that the cost given in (38) will be smaller than $c(m_n)$, so even the most cost efficient club can decrease its operating cost this way. This conclusion is supported by the objective criterion of the club as well, since this time $D_n > 0$, which calls for a decrease in (α_n/a).

Since the club is, after all, making its money in the insurance business by obtaining a risk premium per ship, there will obviously be a trade-off between profit and cost efficiency. For example, it could well be that it is not profitable for such a club to downsize at all.

Unlike in Scenario 1, this time the premium per ship after pooling is just p, so in this situation there is no motive for the club to take more relative risk to get the premium income per ship up.

In this situation the club may have a motive to become more cost efficient or stay where it is, which it can achieve after pooling by downsizing relative to the others or staying put. The total premium income will remain the same, or decrease in direct proportion to the risk, and since the reserve after pooling is unchanged, the regulators are likely to have no objections to either of these strategies.

Finally consider case (iii). A very large club is penalized from cost inefficiency through the side payment given in (36) if its cost is larger that the weighted average c_M of all the clubs. Intuitively one might think that such a club should size down its relative underwriting operations in order to become more cost efficient. This is also consistent with the objective criterion if the reserves are large, i.e., if $w_n > m_n(c(m_n) - c_M)$.

As seen from the expression (38) for the cost per ship after pooling, however, a very large club could alternatively try to *increase* its relative underwriting in order to get the cost per ship down, since $c(m_n) > c_M$. From (38) we notice that if the equality $(\alpha_n/a) = (m_n/M)$ holds, the cost per ship after pooling is just $c(m_n)$, the initial cost. The only way to reduce this cost is to increase the fraction (α_n/a). Again the regulators are likely to set limits for how large this increase could be, since the reserves are still w_n after pooling. From the objective criterion we see, on the other hand, that the club will only pursue this strategy if the reserves satisfy the inequality $w_n < m_n(c(m_n) - c_M)$.

Summing up, Scenario 2 displays a situation where the cost efficient clubs are rewarded after pooling, and the cost inefficient clubs are being penalized through the side payments b_n. From analyzing the final portfolios after pooling, and the objective criterions of the clubs, we see that there is an incentive

for the small, leveraged clubs to grow, the medium clubs will make a profit and act as net lenders, and the very large clubs, experiencing negative side payments, will have a motive to *further increase* their underwriting.

If the reserves of the clubs in categories (i) and (iii) are sufficiently large, however, all the clubs will have a motive to downsize their underwriting.

6.3.3 Credit Risk

Until now we have ignored the possibility that the losses may bring a P&I club in distress. If for Club n, $Y_n(s) < 0$ in some state s in the state space $\mathcal{S}$, we have not been specific about what could happen. One possibility is that this loss is shared between the other clubs in their respective proportions (α_n/a) of the market. Another possibility is that the ship owners behind Club n are asked to pay in the residual as an addition to the premium, after the losses have materialized, as a direct form of experience rating. If $\sum_k Y_k(s) < 0$, the ship owners must in any case cover this residual market loss on own accounts.

In this section we consider a situation where the clubs claim limited liability. A ship owner will then judge the club according to this possibility, and only be willing to pay a premium that reflects the total risk. The situation is now that a club with a larger risk exposure than what corresponds to its relative size in the market, may have to offer a discount on its premium.

Since the club is owned by the ship owners, this means that they have to cover parts of the risk theselves, and are in return only willing to pay a premium for the risk that is actually covered. Let us denote the assets of Club n by A_n, i.e.,

$$A_n = \frac{\alpha_n}{a}\big(w + M(p - c_M)\big) + b_n, \tag{42}$$

and let

$$B_n = A_n a/\alpha_n = w + M(p - c_M) + \frac{a}{\alpha_n}b_n. \tag{43}$$

Then the premium must now be determined by the relation

$$p_n = c_M + \frac{1}{M}\Big(\pi(U|U \leq B_n)P(U \leq B_n) + \pi(U|U > B_n)P(U > B_n)\Big). \tag{44}$$

Here $\pi(U|U > B_n) = \frac{A_n}{1+r}$, and

$$\pi(U|U \leq B_n) = \frac{1}{1+r}\Big(E(U|U \leq B_n) + \frac{\text{cov}(U, e^{U/a}|U \leq B_n)}{E(e^{U/a}|U \leq B_n)}\Big). \tag{45}$$

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The actual computation of these quantities is straightforward for the models considered in Section 5. By continuity there will exist some discount factor $d_n < 1$ such that

$$p_n := p_n^{(3)} = c_M + \frac{d_n}{M(1+r)}\Big(EU + \frac{\text{cov}(U, e^{U/a})}{E(e^{U/a})}\Big). \tag{46}$$

Here $d_n < 1$ for all $n \in \mathcal{N}$, and $d < 1$ as well, where $d = \frac{1}{N}\sum_k d_k$. Notice from (42) and (43) that when the side payments are all zero, all the discount factors $d_n = d$ for all $n \in \mathcal{N}$.

This situation is analogous to a reinsurance market where the reinsurers, here the the P&I clubs, can not get coverage above a certain XL-layer, here A_n for Club n, where A_n serves as a cap. The average premium $p^{(d)}$ in the syndicate reflects this. In real life the clubs should be able to obtain an umbrella, or catastrophe coverage, of this residual risk.

Also in this scenario the side payments after trade are exchanged only between the clubs in the syndicate, since this time p_n is what the ship owners really pay per ship to Club n. The side payments are given by

$$b_n = m_n(c_M - c(m_n)) + \frac{m_n}{M}\pi(U)(d_n - d) + \\ \Big(\frac{\alpha_n}{a} - \frac{m_n}{M}\Big)\pi(U)(1 - d) + (w_n - \frac{\alpha_n}{a}w), \tag{47}$$

where the premium adjustment term is

$$m_n(p_n^{(3)} - p^{(d)}) = \frac{1}{M}m_n\pi(U)(d_n - d). \tag{48}$$

In the syndicate each ship is accounted for the average premium $p^{(d)}$, and the different clubs will "receive" the difference $(p_n^{(3)} - p^{(d)})$ per ship after pooling. In addition we have the term

$$\Big(\frac{\alpha_n}{a} - \frac{w_n}{M}\Big)\pi(U)(1 - d) \tag{49}$$

stemming from adjustment costs for diversification of the insurance risk when credit risk is present. If the latter risk does not exist, $d = 1$ and this term disappears. This correction to the side payment will, ceteris paribus, make it more profitable for a club to increase its risk exposure, because it now enjoys limited liability.

The final portfolio Y_n can be written

$$Y_n = -\Big(\frac{\alpha_n}{a}Mc_M - m_n(c_M - c(m_n))\Big) + \Big(\frac{\alpha_n}{a}Mp^{(d)} + \frac{m_n}{M}\pi(U)(d_n - d)\Big) \\ + \Big(\frac{\alpha_n}{a}w + (w_n - \frac{\alpha_n}{a}w)\Big) + \Big((\frac{\alpha_n}{a} - \frac{m_n}{M})\pi(U)(1 - d) - \frac{\alpha_n}{a}U\Big). \tag{50}$$

The two new terms are the second and the fourth on the right hand side, the second being the premium term after pooling. Dividing by the number of ships, this term can be written

$$\left(p^{(d)} + \left(\frac{m_n}{M}\frac{a}{\alpha_n}\right)\frac{1}{M}\pi(U)(d_n - d)\right). \tag{51}$$

The objectives of the clubs can be found as follows. First we need the optimal portfolio after pooling, which is

$$Y_n = w_n + m_n(c_M - c(m_n)) + \frac{m_n}{M}\pi(U)(d_n - 1) + \tag{52}$$
$$\frac{\alpha_n}{a}\pi(U)(1 - d) + \frac{\alpha_n}{a}(\pi(U) - U),$$

where we have used (47). The objective criterion of Club n at Y_n is thus

$$E\big(v_n(Y_n)\big) = 1 - \exp\Big\{-\frac{1}{a}\Big(\frac{a}{\alpha_n}\tilde{D}_n + \pi(U)(2 - d) - a\log E\big(e^{U/a}\big)\Big)\Big\} \tag{53}$$

where

$$\tilde{D}_n = \big(\tilde{w}_n + m_n(c_M - c(m_n))\big), \tag{54}$$

and

$$\tilde{w}_n := w_n - \frac{m_n}{M}\pi(U)(1 - d_n). \tag{55}$$

If $\tilde{D}_n > 0$, decreasing (α_n/a) increases the objective function of Club n, and if $\tilde{D}_n < 0$, increasing (α_n/a) increases the objective of the club. Since $\tilde{w}_n < w_n$, the constants $\tilde{D}_n < D_n$ and the situations with increasing risk taking in categories (i) and (iii) is more likely here than in Scenario 2.

Starting with a small club in category (i), according to our assumptions this club has higher operating costs per ship than the weighted average, i.e., $c(m_n) > c_M$, implying that the cost adjustment term in (47) is negative. The club is still being penalized for its cost inefficiency. If $(\alpha_n/a) = (m_n/M)$, the last two terms in (47) are zero. This leads to a negative side payment, in which case $d_n < d$. Thus the two first terms in (47) are negative, and consequently the club is penalized both for being cost inefficient, and for having a credit risk higher than the average. This is now the base case, giving a negative b_n.

In this situation the club will have motivation to grow more cost efficient. As in Scenario 2, from an operating cost perspective, expression (38) for the cost per ship after pooling suggests that a small club should try to become larger in order to get the cost per ship down.

Since $d_n < d$, we notice from (51) that also the premium term increases when (α_n/a) increases, but this effect may eventually be reduced, since a

higher exposure can led to a decreased credit rating. Finally, the term (49) calls for more risk taking of this small club, except from the last term in (50) which is the very risk itself.

From the objective criterion it follows that increasing risk only increases the objective function if $\tilde{D}_n < 0$, which means that the reserves w_n must be relatively small, but not as small as in Scenario 2, since $\tilde{w}_n < w_n$. This effect is enforced if the total risk U has a large market value, the initial ratio of ships is large, and the credit rating is low, in fact, $\tilde{w}_n$ may then be negative. The likely outcome for a small club is that of more risk taking than in Scenario 2.

Moving to a medium club in category (ii), the first term in (47) is now positive since $c(m_n) < c_M$, so this club is still rewarded for its cost efficiency. If $(\alpha_n/a) = (m_n/M)$ were to hold, the last two terms in (47) are zero. This leads to a positive b_n, which gives a more favorable credit rating than the small club could obtain, i.e., $d_n > d$. As a consequence the side payment is further increased by the second term in (47), and ends up being clearly positive.

As in Scenario 2, from an operating cost perspective, expression (38) for the cost per ship after pooling suggests that a medium club should downsize its underwriting. We notice from the expression (51) for the premium per ship after pooling that this premium increases as (α_n/a) decreases. The likely effect of this is to further strengthen the club's credit rating, making this premium even larger. The downsizing may is reduced, however, by the risk taking effect from the term (49), not to forget the profits from underwriting.

While $D_n > 0$ in Scenario 2, here $\tilde{D}_n$ may be negative, and relative downsizing is not the only alternative according to the objective criterion.

The overall outcome for a medium to large club depends on the relative sizes of all these effects, and may vary between downsizing, more risk taking and staying at about the same level of risk as initially.

Finally consider category (iii). A very large club is penalized for its diseconomies of scale since its cost is larger that the weighted average c_M of all the clubs. If $(\alpha_n/a) = (m_n/M)$, then the cost per ship after pooling is $c(m_n)$, the initial cost, and the club is motivated to improve its cost result by increasing its risk exposure. This can again be seen from the expression (38) for the cost per ship after pooling. A very large club might thus be inclined to *increase* its underwriting in the pooling negotiations in order to get the cost per ship down.

If $(\alpha_n/a) = (m_n/M)$ were to hold, the last two terms in (47) are zero. As a consequence the club may have a negative side payment, which will be further lowered from the credit risk effect of the second term. Thus its side payment in this *base case* may be negative.

From the expression for the premium per ship after pooling, we notice the same effect is true for a large club as for a club in category (i). However, if the club has reserves large enough for its credit rating to satisfy $d_n > d$, not an unlikely situation for such a club, the premium will be reduced by increasing (α_n/a). This conclusion is also supported by the objective criterion.

The diversification effect from (49) calls for taking more risk, ceteris paribus, when there is credit risk. The outcome for a very large club thus seems to depend upon its credit rating: If $d_n < d$ it is is about the same situation as in Scenario 2, or, perhaps, leaning towards a bit more risk taking behavior based on the objective criterion. If $d_n > d$ on the other hand, the premium reduction may be roughly counterbalanced by the positive effect from the term (49). From the objective criterion, in the latter case we notice that $\tilde{w}_n$ is larger, which calls for downsizing.

The overall outcome for a very large club depends on the relative sizes of all these different effects, and may vary between downsizing, more risk taking and staying at about the same level of risk as initially. The surprising conclusion is that in many situations such a club may find it best to increase in size. This is in particular true if the cost effect dominates in a club with not too large reserves. Such a club will take more risk if it is allowed to by the regulators and the other participants in the syndicate.

Finally let us recall that the positions taken must "adds up", i.e., $\sum_k b_k = 0$, and $\sum_k \alpha_k/a = \sum_k m_k/M = 1$, since we have assumed that markets clear, and $\sum_k w_k/w = 1$ by definition.

Based on the analysis in this section we have added a sixth column in Table 1 indicating possible adjustments to the estimates of the ratios (α_n/a) of the final portfolios, excluding side payments, interpreting the values in Column 5 as the initial (m_n/M)-values.

Summing up, with credit risk we still have a situation where the cost efficient clubs are rewarded after pooling, and the cost inefficient clubs are being penalized. The small clubs tend to be further penalized by credit risk, and the medium clubs are now typically more rewarded.

The medium clubs will make a profit and could act as net lenders, and the very large clubs, may either act as lenders or as borrowers, depending upon their credit ratings, and the degree of diseconomies of scale. The very large clubs may have a motive to grow relative to the other clubs if the cost effect dominates, and they do not have too large reserves. Clubs in this latter category are not observed in the sample of Table 1.

For a small club with a levered, risky position, if the perils of the sea are in its disfavor, in order to grow more cost efficient, and hence more competitive, a strategy of merger and acquisition could be explored, rather than the one outlined above. Such strategies have been observed in the market for P&I

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Clubs during the last couple of decades.

Since we only consider a one period model, all clubs cannot grow simultaneously, and we can only explore the relative changes of the various clubs. If the exogenous growth in the market for P&I coverage is strong enough, all the clubs may grow, but to a varying degree.

Our analysis indicates one possible scenario: The small and the medium clubs converge in size, and some very large clubs grow even larger.

This conclusion may be somewhat surprising. Considering the development in this market during the last few decades, however, this scenario seems to conform to the current trend among the P&I clubs.

A dynamic model would be valuable to capture the growth element in the market for P&I coverage, preferably one that makes this growth endogenous. This is the topic for a future investigation.

7   Conclusions

The paper analyzes the possibility of reaching an equilibrium in a market of Protection and Indemnity Clubs, displaying economies of scale. Our analysis rationalizes some empirically documented findings, and points out new possible directions for this market.

Of the three scenarios we have analyzed, the one with credit risk, and side payments only transfered between the clubs, seems to be the most promising. Being more realistic, it is still parsimonious enough for us to draw some interesting conclusions. This arrangement is close to a competitive equilibrium between corporate insurance companies. This is following a trend of demutualization, where mutual companies either are transfered into corporations, or they act more and more like stock owned companies. The main reasons for this seem to be that the insurance companies want to attract sufficient equity capital to meet asset-to-liability limits set by regulators, and this can be difficult for a mutual company to carry out because of its special ownership structure. For a P&I Club this may be less of a problem, however, since many of its owners are rather wealthy. The situation we consider with credit risk will typically induce some of the clubs to take more risk.

In particular we find an equilibrium in a market of such mutual marine insurers, in which smaller clubs, having operating costs above average, may grow larger relative to the other clubs in order to become more cost effective, and where among the medium to larger cost efficient clubs there may be some that want to downsize their underwriting relative to the other clubs. Some of the very large clubs suffering from diseconomies of scale may have a motivation to further increase from an operating cost perspective, if allowed

to do so, while others may downsize their underwriting relative to the other clubs.

According to observations, most clubs have, during the last decade, expanded significantly in size measured by gross tonnage of entered ships, some clubs have merged, but very few seem to have decreased their underwriting activity, in particular none of the really large ones. Combining these observations with our model, the analysis points to the following possible future scenario: The small clubs and the medium to large clubs will tend to converge in size, while there may still be some very large clubs present as well.

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