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# Turbulence of a Unidirectional Flow

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**Abstract**

Recent advances in the theory of turbulent solutions of the Navier-Stokes equations are discussed and the existence of their associated invariant measures. The statistical theory given by the invariant measures is described and associated with historically-known scaling laws. These are Hack's law in one dimension, the Bachelor-Kraichnan law in two dimensions and the Kolmogorov's scaling law in three dimensions. Applications to problems in turbulence are discussed and applications to Reyolds Averaged Navier Stokes (RANS) and Large Eddy Simulation (LES) models in computational turbulence.

## 1 Introduction

Everyone is familiar with turbulence in one form or another. Airplane passengers encounter it in wintertime as the plane begins to shake and is jerked in various directions. Thermal currents and gravity waves in the atmosphere create turbulence encountered by low-flying aircraft. Turbulent drag also prevents the design of more fuel-efficient cars and aircrafts. Turbulence plays a role in the heat transfer in nuclear reactors, causes drag in oil pipelines and influence the circulation in the oceans as well as the weather.

In our daily lives we encounter countless other examples of turbulence. Surfers use it to propel them and their boards to greater velocities as the wave breaks and

becomes turbulent behind them and they glide at great speeds down the unbroken face of the wave. This same wave turbulence shapes our beaches and carries enormous amount of sand from the beach in a single storm, sometime to dump it all into the nearest harbor. Turbulence is harnessed in combustion engines in cars and jet engines for effective combustion and reduced emission of pollutants. The flow around automobiles and downtown buildings is controlled by turbulence and so is the flow in a diseased artery. Atmospheric turbulence is important in remote sensing, wireless communication and laser beam propagation through the atmosphere, see [35] and [36]. The applications of turbulence await us in technology, biology and the environment. It is one of the major problems holding back advances of our technology.

Turbulence has puzzled and intrigued people for centuries. Five centuries ago a fluid engineer by the name of Leonardo da Vinci tackled it. He did not have modern mathematics or physics at his disposal but he had a very powerful investigative tool in his possession. He explored natural phenomena by drawing them. Some of his most famous drawings are of turbulence.

Leonardo called the phenomenon that he was observing ”la turbolenza” in 1507 and he gave the following description of it:

”Observe the motion of the surface of the water, which resembles that of hair, which has two motions, of which one is caused by the weight of the hair, the other by the directions of the curls; thus the water has eddying motions, one part of which is due to the principal current, the other to the random and reverse motion.”

This insightful description pointed out the separation of the flow into the average flow and the fluctiations that plays an important role in modern turbulence theory. But his drawings also led Leonardo to make other astute observations that accompany his drawings, in mirror script, such as:

- Where the turbulence of water is generated
- Where the turbulence of water maintains for long
- Where the turbulence of water comes to rest

These three observations are well-known features of turbulence and they are all illustrated in Leonardo’s drawings.

One reason why turbulence has not been solved yet is that the mathematics or the calculus of turbulence has not been developed until now. This situation is analogous to the physical sciences before Newton and Leibnitz. Before the physical sciences could bloom into modern technology the mathematics being the language that they are expressed in had to be developed. This was accomplished by Newton and Leibnitz and developed much further by Euler. Three centuries later we are at a similar threshold regarding turbulence. The mathematics of turbulence is being

born and the technology of turbulence is bound to follow.

The mathematics of turbulence is rooted in stochastic partial differential equations. It is the mathematical theory that expresses the statistical theory of turbulence as envisioned by the Russian mathematician Kolmogorov, one of the fathers of modern probability theory, in 1940. The basic observation is that turbulent flow is unstable and the white noise that is always present in any physical system is magnified in turbulent flow. In distinction, in laminar flow the white noise in the environment is suppressed. The new mathematical theory of turbulence expresses how the noise is magnified and colored by the turbulent fluid. This then leads to a computation or an approximation of the associated invariant measure for the stochastic partial differential equation. The whole statistical theory of Kolmogorov can be expressed mathematically with this invariant measure in hand.

The problems that mathematicians have with proving the existence of solutions of the Navier-Stokes equations in three dimensions has lead to the mistaken impression that turbulence is only a three dimensional phenomenon. Nothing is further from the truth. Turbulence thrives in one and two dimensions as well as in three dimensions. We will illustrate this by describing one dimensional turbulence in rivers.

Although we will coach it in terms of river flow in his paper, this type of modeling and theory have many other applications. One such application is to the modeling of fluvial sedimentation that gives rise to sedimentary rock in petroleum reserves. The properties of the flow through the porous rock turn out to depend strongly on the structure of the meandering river channels, see [20]. Another application is to turbulent atmospheric flow. Contrary to popular belief, in the presence of turbulence, the temperature variations in the atmosphere my be highly anisotropic or stratified. Thus the scaling of the fluid model corresponding to a river or a channel may have a close analog in the turbulent atmosphere, see [32].

Two dimensionless numbers the Reynolds number and the Froude number are used to characterize turbulent flow in rivers and streams. If we model the river as an open channel with $x$ parameterizing the downstream direction, $y$ the horizontal dept and $U$ is the mean velocity in the downstream direction, then the Reynolds number

$$R = \frac{f_{\text{turbulent}}}{f_{\text{viscous}}} = \frac{Uy}{\nu}$$

is the ratio of the turbulent and viscous forces whereas the Froude number

$$F = \frac{f_{\text{turbulent}}}{f_{\text{gravitational}}} = \frac{U}{(gy)^{1/2}}$$

is the ratio of the turbulent and gravitational forces. $\nu$ is the viscosity and $g$ is the gravitational acceleration. Other forces such as surface tension, the centrifugal force and the Coriolis force are insignificant in streams and rivers.

The Reynolds number indicates whether the flow is laminar or turbulent with the transition to turbulence starting at $R = 500$ and the flow usually being fully turbulent at $R = 2000$. The Froude number measures whether gravity waves, with speed $c = (gy)^{1/2}$ in shallow water, caused by some disturbance in the flow, can overcome the flow velocity and travel upstream. Such flow are called tranquil flows, $c > U$, in distinction to rapid or shooting flows, $c < U$, where this cannot happen; they correspond to the Froude numbers

1. $F < 1$, subcritical, $c > U$
2. $F = 1$, critical, $c = U$
3. $F > 1$, supercritical, $c < U$

Now for streams and rivers the Reynolds number is typically large $O = 10^5 - 10^6$, whereas the Froude numbers is small typically $O = 10^{-1} - 10^{-2}$, see [24]. Thus the flows are highly turbulent and ought to be tranquil but this is not the whole story as we will now explain.

In practice streams and rivers have varied boundaries which are topologically equivalent to a half-pipe. These boundaries are rough and resist the flow and this had lead to formulas involving channel resistance. The most popular of these are Chézy's law, where the average velocity $V$ is

$$V = u_c C r^{1/2} s_o^{1/2}, \;\; u_c = 0.552 m/s$$

and Manning's law, with

$$V = u_m \frac{1}{n} r^{2/3} s_o^{1/2}, \;\; u_m = 1.0 m/s$$

where $s_o$ is the slope of the channel and $r$ is the hydraulic radius. $C$ is called Chézy's constant and measures inverse channel resistance. $n$ is Manning's roughness coefficient, see [24]. We get new effective Reynolds and Froude numbers with these new averaged velocities $V$,

$$R^* = \frac{g}{3u_c^2 C^2} R, \quad F^* = \left( \frac{g}{u_c^2 C^2 s_o} \right)^{1/2} F$$

from Chézy's law.

It turns out that in real rivers the effective Froude number is approximately one and the effective Reynolds number is also one, when $R = 500$ for typical channel roughness $C = 73.3$. Thus the transition to turbulence typically occurs in rivers when the effective turbulent forces are equal to the viscous forces.

The reason for the transition to turbulence is that at this value of $R^*$ the amplification of the noise that grows into fully developed turbulence is no longer damped by the viscosity of the flow. The damping by the effective viscosity is overcome by the turbulent forces.

Now let us ignore the boundaries of the river. The point is that in a straight segment of a reasonably deep and wide river the boundaries do not influence the details of the river current in the center, except as a source of flow disturbances. We will simply assume that these disturbances exist, in the flow at the center of the river and not be concerned with how they got there. For theoretical purposes we will conduct a thought experiment where we start with an unstable uniform flow and then put the disturbances in as small white noise. Then the mathematical problem is to determine the statistical theory of the resulting turbulent flow. The important point is that this is now a theory of the water velocity $u(x)$ as a function of the one-dimensional distance $x$ down the river. Thus if $u$ is turbulent it describes one-dimensional turbulence in the downstream direction of the river.

The flow of water in streams and rivers is a fascinating problem with many application that has intrigued scientists and laymen for many centuries, see Levi [26]. Surprisingly it is still not completely understood even in one or two-dimensional approximation of the full three-dimensional flow. Erosion by water seems to determine the features of the surface of the earth, up to very large scales where the influence of earthquakes and tectonics is felt, see [37, 38, 34, 7, 5, 39]. Thus water flow and the subsequent erosion gives rise to the various scaling laws know for river networks and river basins, see [12, 8, 9, 10, 11].

One of the best known scaling laws of river basins is Hack's law [17] that states that the area of the basin scales with the length of the main river to an exponent that is called Hack's exponent. Careful studies of Hack's exponent, see [11] show that it actually has three ranges, depending on the age and size of the basin, apart from very small and very large scales where it is close to one. The first range corresponds to a spatial roughness coefficient of one half for small channelizing (very young) landsurfaces. This has been explained, see [5] and [13], as Brownian motion of water and sediment over the channelizing surface. The second range with a roughness coefficient of 2/3 corresponds to the evolution of a young surface forming a convex (geomorphically concave) surface, with young rivers,

that evolve by shock formation in the water flow. These shocks are called bores (in front) and hydraulic jumps (in rear), see Welsh, Birnir and Bertozzi [39]. Between them sediment is deposited. Finally there is a third range with a roughness coefficient 3/4. This range that is the largest by far and is associated with what is called the mature landscape, or simply the landscape because it persists for a long time, is what this paper is about. This is the range that is associated with turbulent flow in rivers and we will develop the statistical theory of turbulent flow in rivers that leads to Hack's exponent.

Starting with the three basic assumption on river networks: that the their structure is self-similar, that the individual streams are self-affine and the drainage density is uniform, see [8], river networks possess several scalings laws that are well documented, see [31]. These are self-affinity of single channels, which we will call the meandering law, Hack's law, Horton's laws [30] and their refinement Tokunaga's law, the law for the scaling of the probability of exceedance for basin areas and stream lengths and Langbein's law. The first two laws are expressed in terms of the meandering exponent $m$, or fractal dimension of a river, and the Hack's exponent $h$. Horton's laws are expressed in terms of Horton's ratio's of link numbers and link lengths in a Strahler ordered river network, Tokunaga's law is expressed in term of the Tokunaga's ratio's, the probability of exceedance is expressed by decay exponents and Langbein's law is given by the Langbein's exponents, [8].

In a series of paper's Dodds and Rothman [12, 8, 9, 10, 11] showed that all the above ratios and exponents are determined by $m$ and $h$, the meandering and Hack's exponents, see [17], [12]. The origin of the meandering exponent $m$ has recently been explained, see [6], but in this paper we discuss how it and Hack's exponent are determined by the scaling exponent of turbulent one-dimensional flow. Specifically, $m$ and $h$ are determined by the scaling exponent of the second structure function, see [14], in the statistical theory of the one-dimensional turbulent flow.

The break-through that initiated the theoretical advances discussed above was the proof of existence of turbulent solutions of the full Navier-Stokes equation driven by uniform flow, in dimensions one, two and three. These solutions turned out to have a finite velocity and velocity gradient but they are not smooth instead the velocity is Hölder continuous with a Hölder exponent depending on the dimension, see [4]. These solutions scale with the Kolmogorov scaling in three dimensions and the Batchelor-Kraichnan scaling in two dimension. In one dimensions they scale with the exponent $3/4$, that is related to Hack's law [17] of river basins, see [7] and [5]. The existence of these turbulent solutions is then

used to proof the existence of an invariant measure in dimensions one, two and three, see [4]. The invariant measure characterizes the statistically stationary state of turbulence and it can be used to compute the statistically stationary quantities. These include all the deterministic properties of turbulence and everything that can be computed and measured. In particular, the invariant measure determines the probability density of the turbulent solutions and this can be used to develop accurate sub-grid modeling in computations of turbulence, bypassing the problem that three-dimensional turbulence cannot be fully resolved with currently existing computer technology.

# 2 The Initial Value Problem

Consider the Navier-Stokes equation

$$\begin{aligned} w_t + w\cdot\nabla w &= \nu\Delta w - \nabla p \\ w(x,0) &= w_o, \end{aligned} \tag{1}$$

where $\nu = \nu_0/VL$, $V$ being a typical velocity, $L$ the length of a segment of the river and $\nu_0$ the kinematic viscosity of water, with the incompressibility conditions

$$\nabla\cdot w = 0 \tag{2}$$

Eliminating the pressure $p$ using (2) gives the equation

$$w_t + w\cdot\nabla w = \nu\Delta w + \nabla\{\Delta^{-1}[trace(\nabla w)^2]\} \tag{3}$$

We want to consider turbulent flow in the center of a wide and deep river and to do that we consider the flow to be in a box and impose periodic boundary conditions on the box. Since we are mostly interested in what happens in the direction along the river we take our $x$ axis to be in that direction.

We will assume that the river flows fast and pick an initial condition of the form

$$w(0) = U_o e_1 \tag{4}$$

where $U_o$ is a large constant and $e_1$ is a unit vector in the $x$ direction. Clearly this initial condition is not sufficient because the fast flow may be unstable and the white noise ubiquitous in nature will grow into small velocity and pressure oscillations, see for example [3]. But we perform a thought experiment where white noise is introduced into the fast flow at $t = 0$. This experiment may be hard

to perform in nature but it is easily done numerically. It means that we should look for a solution of the form

$$w(x,t) = U_o e_1 + u(x,t) \tag{5}$$

where $u(x,t)$ is smaller that $U_o$ but not necessarily small. However, in a small initial interval $[0,t_o]$ $u$ is small and satisfies the equation (3) linearized about the fast flow $U_o$

$$\begin{aligned} u_t + U_o \partial_x u &= \Delta u + f \\ u(x,0) &= 0 \end{aligned} \tag{6}$$

driven by the noise

$$f = \sum_{k \neq 0} h_k^{1/2} d\beta_t^k e_k$$

The $e_k = e^{2\pi i k \cdot x}$ are (three-dimensional) Fourier components and each comes with its own independent Brownian motion $\beta_t^k$. None of the coefficients of the vectors $h_k^{1/2} = (h_1^{1/2}, h_2^{1/2}, h_3^{1/2})$ vanish because the turbulent noise was seeded by truly white noise (white both is space and in time). $f$ is not white in space because the coefficients $h_k^{1/2}$ must have some decay in $k$ so that the noise term in (6) makes sense. However to determine the decay of the $h_k^{1/2}$s will now be part of the problem. The form of the turbulent noise $f$ expresses the fact that in turbulent flow there is a continuous sources of small white noise that grows and saturates into turbulent noise that drives the fluid flow. The decay of the coefficients $h_k^{1/2}$ expresses the spatial coloring of this larger noise in turbulent flow. We have set the kinematic viscosity $\nu$ equal to one for computational convenience, but it can easily be restored in the formulas.

This modeling of the noise is the key idea that make everything else work. The physical reasoning is that the white noise ubiquitous in nature grows into the noise $f$ that is charateristic for turbulence and the differentiability properties of the turbulent velocity $u$ are the same as those of the turbulent noise.

The justification for considering the initial value problem (6) is that for a short time interval $[0,t_o]$ we can ignore the nonlinear terms

$$-u \cdot \nabla u + \nabla \{\Delta^{-1}[trace(\nabla u)^2]\}$$

in the equation (3). But this is only true for a short time $t_o$, after this time we have to start with the solution of (6)

$$u_o(x,t) = \sum_{k \neq 0} h_k^{1/2} \int_0^t e^{(-4\pi^2|k|^2 + 2\pi i U_o k_1)(t-s)} d\beta_s^k e_k(x) \tag{7}$$

as the first iterate in the integral equation

$$u(x,t) = u_o(x,t) + \int_{t_o}^{t} K(t-s) * [-u \cdot \nabla u + \nabla \Delta^{-1}(trace(\nabla u)^2)]ds \tag{8}$$

where $K$ is the (oscillatory heat) kernal in (7). In other words to get the turbulent solution we must take the solution of the linear equation (6) and use it as the first term in (8). It will also be the first guess in a Picard iteration.The solution of (6) can be written in the form

$$u_o(x,t) = \sum_{k \neq 0} h_k^{1/2} A_t^k e_k(x)$$

where the

$$A_t^k = \int_0^t e^{(-4\pi^2|k|^2 + 2\pi i U_o k_1)(t-s)} d\beta_s^k \tag{9}$$

are independent Ornstein-Uhlenbeck processes with mean zero, see for example [29].

Now it is easy to see that the solution of the integral equation (8) $u(x,t)$ satisfies the driven Navier-Stokes equation

$$u_t + U_o \partial_x u = \Delta u - u \cdot \nabla u + \nabla \Delta^{-1}(trace(\nabla u)^2) + \sum_{k \neq 0} h_k^{1/2} d\beta_t^k e_k, \quad t > t_0$$

(10)

$$u_t + U_o \partial_x u = \Delta u + \sum_{k \neq 0} h_k^{1/2} d\beta_t^k e_k, \quad u(x,0) = 0, \quad t \leq t_0$$

and the above argument is the justification for studying the initial value problem (10). We will do so from here on. The solution $u$ of (10) still satisfies the periodic boundary conditions and the incompressibility condition

$$\nabla \cdot u = 0 \tag{11}$$

The mean of the solution $u_o$ of the linear equation (6) is zero by the formula (7) and this implies that the solution $u$ of (10) also has mean zero

$$\overline{u}(t) = \int_{\mathbb{T}^3} u(x,t)dx = 0 \tag{12}$$

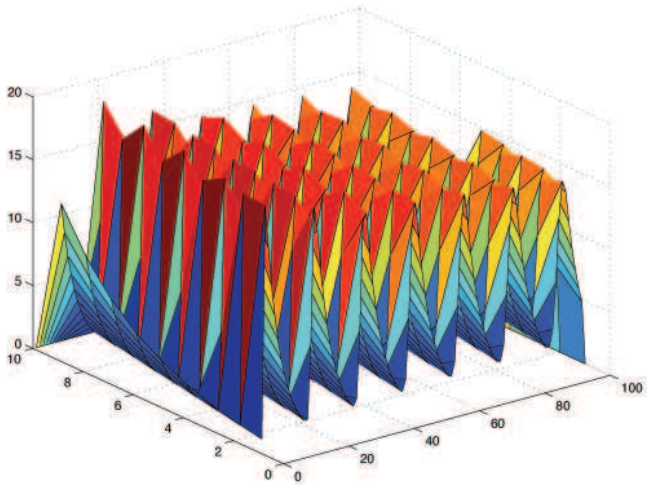


Figure 1: The traveling wave solution of the heat equation for the flow velocity $U_o = 85$. The perturbations are frosen in the flow. The $x$ axis is space, the $y$ axis time and the $z$ axis velocity $u$.

## 2.1 Stability

The uniform flow $w = U_o e_1$ seem to be a stable solution of (6) judging from the solution (7). Namely, all the Fourier coefficients are decaying. However, this is deceiving, first the Brownian motion $\beta_k$ is going to make the amplitude of the kth Fourier coefficient large in due time with probability one. More importantly if $U_o$ is large then (6) has traveling wave solutions that are pertubations "frozen in the flow", and for $U_o$ even larger these traveling waves are unstable and start growing. For $U_o$ large enough this happens after a very short initial time interval and makes the flow immediately become fully turbulent. The role of the white noise is then not to cause enough growth eventually for the nonlinearities to become important, but rather to immediately pick up (large) perturbations that grow exponentially. These are the large fluctuations that are observed in most turbulent flows. In Figure 1, we show the traveling wave solution of the transported heat equation (6), with $U_o = 85$. In Figure 2, where the flow has increased to $U_o = 94$, the traveling wave has become unstable and grows exponentially. Notice the difference in vertical scale between the figures.

Thus the white noise grows into a traveling wave that grows exponentially. This exponential growth is saturated by the nonlinearities and subsequently the flow becomes turbulent. This is the mechanism of explosive growth of turbu-

lence of a uniform stream and describes what happens in our thought experiment described in Section 2.

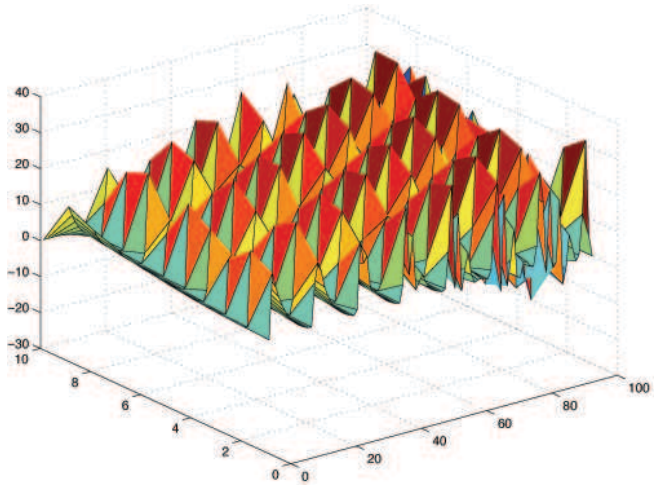


Figure 2: The traveling wave solution of the heat equation for the flow velocity $U_o = 94$. The perturbations are growing exponentially. The $x$ axis is space, the $y$ axis time and the $z$ axis velocity $u$.

# 3 One-dimensional Turbulence

In a deep and wide river it is reasonable to think that the directions transverse to the main flow, $y$ the direction across the river, and $z$ the horizontal direction, play a secondary role in the generation of turbulence. As a first approximation to the flow in the center of a deep and wide, fast-flowing river we will now drop these directions. Of course $y$ and $z$ play a role in the motion of the large eddies in the river but their motion is relatively slow compared to the smaller scale turbulence. Thus our initial value problem (10) becomes

$$u_t + U_o u_x = u_{xx} - u u_x + \partial_x^{-1}((u_x)^2) - \int_0^1 \partial_x^{-1}((u_x)^2)dx + \sum_{k\neq 0} h_k^{1/2} d\beta_t^k e_k, \tag{13}$$

We still have periodic boundary condition on the unit interval but the incompressibility condition can be dropped at the price of subtracting the term

$$b = \int_0^1 \partial_x^{-1}((u_x)^2)dx$$

from the right hand side of the Navier-Stokes equation. This term keeps the mean of $u$, $\overline{u} = \int_0^1 u dx = 0$, equal to zero, see Equation (12). This equation (13) now describes the turbulent flow in the center of relative straight section of a fast river. The full three-dimensional flow will be treated in a subsequent publication.

The following theorem and corollaries are proven in [4]. It states the existence of turbulent solutions in one dimension. First we write the initial value problem (13) as an integral equation

$$u(x,t) = u_o(x,t) + \int_{t_o}^t K(t-s) * [-\frac{1}{2}(u^2)_x + \partial_x^{-1}(u_x)^2 - b]ds \tag{14}$$

Here $K$ is the oscillatory heat kernal (7) in one dimension and

$$u_o(x,t) = \sum_{k\neq 0} h_k^{1/2} A_t^k e_k(x)$$

the $A_t^k$s being the Ornstein-Uhlenbeck processes from Equation (9).

If $\frac{q}{p}$ is a rational number let $\frac{q}{p}^+$ denote any real number $s > \frac{q}{p}$, and $E$ the expectation with a probability measure $P$ on a set of events $\Omega$.

**Theorem 3.1** *If the solution of the linear equation (6) satisfies the condition*

$$E(\|u_o\|^2_{(\frac{5}{4}^+,2)}) = \sum_{k\neq 0}(1+(2\pi k)^{(5/2)^+})h_k E(|A_t^k|^2) \tag{15}$$
$$\leq \frac{1}{2}\sum_{k\neq 0}\frac{(1+(2\pi ik)^{(5/2)^+})}{(2\pi k)^2}h_k < \infty$$

*and $U_0$ is sufficiently large, then the integral equation (14) has a unique solution in the space $C([0,\infty); \mathcal{L}^2_{(\frac{5}{4}^+,2)})$ of stochastic processes with*

$$\|u\|^2_{\mathcal{L}^2_{(\frac{5}{4}^+,2)}} = E(\int_0^T \|u\|^2_{(\frac{5}{4}^+,2)}dt) < \infty$$

*for any finite $T$.*

**Corollary 3.1** *The solution of the linearized equation (6) uniquely determines the solution of the integral equation (14).*

**Corollary 3.2** *Onsager's Conjecture*
*The solutions of the integral equation (14) are Hölder continuous with exponent* $3/4$.

**Remark 3.1** The hypothesis (15) is the answer to the question we posed in Section 2 how fast the coefficients $h_k^{1/2}$ had to decay in Fourier space. They have to decay sufficiently fast for the supremum in $t$ of the expectation of the $H^{\frac{5}{4}^+} = W^{(\frac{5}{4}^+,2)}$ Sobolev norm of the initial function $u_o$, to be finite. In other words the sup in $t$ of the $\mathcal{L}^2_{(\frac{5}{4}^+,2)}$ norm has to be finite.

# 4 The Existence and Uniqueness of the Invariant Measure

We can define the invariant measure $d\mu$ for a stochastic partial differential equation (SPDE) by the limit

$$\lim_{t\to\infty} E(\phi(u(t))) = \int_{L^2(\mathbb{T}^1)} \phi(u) d\mu(u) \tag{16}$$

where $E$ denotes the expectation, $u(t)$ is the solution of the SPDE, parametrized by time, and $\phi$ is any bounded function on $L^2(\mathbb{T}^n)$. $L^2(\mathbb{T}^n)$ is the space of square integrable functions on a torus $\mathbb{T}^n$ which means that we are imposing periodic boundary conditions on an interval, rectangle or a box, respectively $n = 1,2,3$ dimensions. However, the theory also carries over to other boundary conditions. One first uses the law $\mathcal{L}$ of the solution $u(t)$

$$P_t(w,\Gamma) = \mathcal{L}(u(w,t))(\Gamma), \quad \Gamma \subset \mathcal{E},$$

where $w = u_o$ is the initial condition for the SPDE and $\mathcal{E}$ is the $\sigma$ algebra generated by the Borel subsets $\Gamma$ of $L^2(\mathbb{T}^n)$, to define transition probabilities $P_t(w,\Gamma)$ on $L^2(\mathbb{T}^n)$. A stochastically continuous Markovian semi-group is called a Feller semi-group, see [29], and for such Feller semi-groups

$$\frac{1}{T}\int_0^T P_t(w,\cdot)dt$$

defines a probability measure. This is how one forms probability measures on $L^2(\mathbb{T}^n)$ by taking these time averages of the transition probabilities and then one uses the Krylov-Bogoliubov Theorem, see [29], to show that the sequence of the resulting probability measures, indexed by time $T$, is tight. This is the first step, then the invariant measure exists and is the (weak) limit

$$d\mu(\cdot) = \lim_{T\to\infty} \frac{1}{T} \int_0^T P_t(w,\cdot)dt$$

Once the existence of the invariant measure has been established, one wishes to prove that it is unique. To prove this one first has to prove that $P_t$ is in fact a strong Feller semi-group or that for all $T > 0$ there exists a constant $C > 0$, such that for all $\varphi \in B(L^2)$, the space of bounded functions on $L^2$, and $t \in [0,T]$

$$|P_t\varphi(x) - P_t\varphi(y)| \leq C\|\varphi\|_\infty \|x-y\|, \;\; x,y \in L^2.$$

Here $\|\cdot\|$ denotes the norm in $L^2$. Then one must prove the irreducibility of the $P_t$, namely that for any $\Gamma \subset L^2$ and $w \in \Gamma$

$$P_t(w,\Gamma) = P_t\chi_\Gamma(w) > 0,$$

where $\chi_\Gamma$ is the characteristic function of $\Gamma$. The strong Feller property and irreducibility are usually defined for a fixed $t$ but by the semi-group property, if these hold at one $t$ they also hold at any other $t$. Now if the transition semi-group $P_t$ associated with the equation (17) below is a strong Feller semi-group and irreducible, then by Doob's Theorem on Invariant Measures, see [29],

1. The invariant measure $\mu$ associated with $P_t$ is unique.

2. $\mu$ is strongly mixing and

$$\lim_{t\to\infty} P_t(w,\Gamma) = \mu(\Gamma),$$

   for all $w \in L^2$ and $\Gamma \in \mathcal{E}$ where $\mathcal{E}(L^2)$ denotes the sigma field generated by the Borel subsets of $L^2$.

3. $\mu$ is equivalent to all measures $P_t(w,\cdot)$, for all $w \in L^2$ and all $t > 0$.

# 5 The Statistical Theory

The invariant measure can be used to compute statistical quantities characterizing the turbulent state. The mathematical model consists of the Navier-Stokes equation where we have used the incompressibility condition to eliminate the pressure,

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = \nu \Delta u + \nabla \Delta^{-1} [trace(\nabla u)^2] + f, \tag{17}$$

$\nu$ is the kinematic viscosity and $f$ represents turbulent noise as in Equation (6). The velocity also satisfies the incompressibility condition

$$\nabla \cdot u = 0. \tag{18}$$

In one dimension, modeling a fast turbulent flow in a relatively narrow river, one can ignore the dimension transverse to the flow and the equation becomes,

$$u_t + u u_x = \nu u_{xx} + \partial_x^{-1} (u_x)^2 - b + f, \tag{19}$$

as discussed above. The existence of turbulent solutions of this equation and their associated invariant measures was established in [4], following the method of McKean [27]. The existence of invariant measures for the one-dimensional Navier-Stokes equation (dissipative Burger's equation) with stochastic forcing was established by Sinai [33], see also [23], and McKean [27]. The existence in the two-dimensional case was established by Mattingly, see for example [18] and [19]. If one considers the second structure function

$$S_2(y) = E[|u(x+y) - u(x)|^2]$$

of the solution, one can show that it scales with the power 3/2 in the lag variable $y$ for the equation (19), in one dimenson, see [7, 5, 4], and 2/3 for the equation (17), in three dimension, the latter is Kolmogorov's theory. The Kolmogorov scaling of the second structure function is usually written as

$$S_2(y) = C\varepsilon^{2/3} y^{2/3}.$$

where $\varepsilon$ is the dissipation rate. In two dimension the scaling is more complicated due to the existence of the inverse cascade, see Kraichnan [22], and two scaling regimes may exist (Kraichnan and Batchelor [22, 1], and Kolmogorov [21]). It is still an open problem to examine the higher moments for different scalings or multifractality, see [14] and [25], and the scalings at very small scales below the

Kolmogorov scale. The latter is the scale below which dissipation and dissipative scaling is supposed to dominate. Finally, one needs to examine the scaling in time, that we have suppressed in the above formula, to see if one can characterize the transients to the stationary (fully developed turbulence) state.

If $\phi$ is a bounded function on $L^2(\mathbb{T}^1)$, then the invariant measure $d\mu$ for the SPDEs (13) is given by the limit

$$\lim_{t\to\infty} E(\phi(u(t))) = \int_{L^2(\mathbb{T}^1)} \phi(u) d\mu(u), \tag{20}$$

see Equation (16). In [4] it is proven that this limit exists and is unique. We get the follwing theorem, as explained in Section 4,

**Theorem 5.1** *The integral equation (14) possesses a unique invariant measure.*

**Corollary 5.1** *The invariant measure $d\mu$ is ergodic and strongly mixing.*

The corollary follows immediately from Doob's Theorem for Invariant Measures above, see for example [29].

The equations describing the erosion of a fluvial landsurface consist of a system of PDEs, one ($u$) equation describing the fluid flow, the other equation describing the sediment flow, see [7]. Using these equations, Hack's law is proven in the following manner. In [5] the equations describing the sediment flow are linearized about convex (concave in the terminology of geomorphology) surface profiles describing mature surfaces. Then the colored noise generated by the turbulent flow (during big rainstorms) drives the linearized equations and the solutions obtain the same color (scaling), see Theorem 5.3 in [5]. The resulting variogram (second structure function) of the surfaces scales with the roughness exponent $\chi = \frac{3}{4}$, see Theorem 5.4 in [5]. This determines the roughness coefficient $\chi$ of mature landsurfaces.

The final step is the following derivation of Hack's law is copied from [7].

## 5.1 The Origin of Hack's Law

The preceding results allow us to derive some of the fundamental scaling results that are known to characterize fluvial landsurfaces. In particular, the avalanche dimension computed in [7] and derived in [5], given the roughness coefficient $\chi$, allows us to derive Hack's Law relating the length of a river $l$ to the area $A$ of

the basin that it drains. This is the area of the river network that is given by the avalanche dimensions

$$A \sim l^D$$

and the avalanche dimensions is $D = 1 + \chi$. This relation says that if the length of the main river is $l$ then the width of the basin in the direction, perpendicular to the main river, is $l^\chi$. Stable scalings for the surface emerge together with the emergence of the separable solutions describing the mature surfaces, see [7]. We note that in this case $\chi = \frac{3}{4}$, hence we obtain

$$\begin{aligned} l &\sim A^{\frac{1}{1+\chi}} \\ &\approx A^{0.57} \end{aligned} \tag{21}$$

a number that is in excellent agreement with observed values of the exponent of Hack's law of 0.58, see [16].

It still remains to explain how the roughness of the bottom and boundary of a river channel gets spread to the whole surface of the river basin over time. In [6] it is shown that the mechanism for this consists of the meanderings of the river. As the rivers meanders over time it sculpts a roughness of the surface with the roughness exponent $3/4$.

# 6 Invariant Measures and Turbulent Mixing

Now how does the existence of the invariant measure help in determining the turbulent mixing properties on a small scale? First, it is know that the invariant measure is not only ergodic but in fact strongly mixing, see [4]. Secondly, the invariant measure allows one to compute the statistical properties, in particular the mixing rates. This, of course sounds, a little too good to be true so what is the problem?

The main problem one has to tackle first is that no explicit formula exist for the invariant measure, such as the explicit formula one has for the Gaussian invariant measure of Brownian motion. Indeed no such formula can exist, no more than one can have an explicit formula for a general turbulent solution of the Navier-Stokes equation. However, since the invariant measure is both ergodic and weakly mixing, by Doob's theorem, see for example [29], one can use the ergodic theorem and approximate the invariant measure by taking the long-time time average. In practice this means that we take the limit of the expectation of a computed solution or rather it substitute: an ensamble average of many computed solutions and

the time average of this ensamble average, when time becomes large. Roughly speaking this means that we can approximate the invariant measure to the same accuracy as the computed solution. However, this means that we also have an approximation of the probability density and this can be used to make a sub-grid model for (LES) computations.

It is desirable to go beyond the above approximation and develop approximations of the invariant measure that are independent of the computational accuracy. This requires one to find an approximations of the invariant measure by a sequence of measures that can be computed explicitly and an estimate of the error one makes by each approximation. There are some proposals for doing this that need to be explored. One also needs to investigate the properties of the invariant measure, what its continuity properties are with respect to other measures, etc. The discovery of these properties that now are completely unknown will help in determining good and efficient approximations to the invariant measure and the probability density.

If methods are found to efficiently approximate the invariant measure then there are no limits to the spatial and temporal scales that can be resolved except the theoretical one given by the Kolmogorov and dissipative scales. In other words with good methods to approximate the invariant measures the turbulent mixing problem can be solved and the mixing rates of the various components due to the turbulence computed. Furthermore, at least theoretically this can be done to any desired accuracy.

# 7 Approximations of the Invariant Measure

It is imperative for application to be able to approximate the invariant measure up a high order. This permits the computation of statistical quantities to within the desired accuracy in experiments or simulations. The first step in the approximation procedure is to use the same method that was used to construct the solutions to construct approximations of the invariant measure. If we linearize the Navier-Stokes equation (17) around a fast unidirectional flow $U_o e_1$ where $e_1$ is a unit vector in the $x$ direction and include noise then we get a heat equation with a convective term that has the solution

$$u_o(x,t) = \sum_{k\neq 0} h_k^{1/2} \int_0^t e^{(-4\pi^2|k|^2 + 2\pi i U_o k_1)(t-s)} d\beta_s^k e_k(x) \tag{22}$$

as explained in Section 2. The $\beta_t^k s$ are independent Brownian motions and the $e_k s$ are Fourier components. Then if we look for a solution of (17) of the form $U = U_o e_1 + u$ then $u$ satisfies the integral equation

$$u(x,t) = u_o(x,t) + \int_{t_o}^{t} K(t-s) * [-u \cdot \nabla u + \nabla \Delta^{-1}(trace(\nabla u)^2)] ds \tag{23}$$

where $K$ is the (oscillatory heat) kernal in (7). The solution of the integral equation is constructed by subsituting $u_o$ as the first guess into the integral and then iterating the result. This produces a sequence of (Picard) iterates that one proves converges to the solution of the integral equation. No explict formula can exist for the limit in general but one can iterate the integral equation as often as desired to produce an approximate solution. The formulas get more and more complicated but it is possible that one quickly get a good approximation to the real solutions. This obviously depends on the rate of convergence. In any case the $m$th iterate $u_m$ of the integral equation with $u_0 = u_o$ is an approximate solution that can be compared to a numerical solution of the equation (17).

It is conceivable that these approximations can be implemented by a symbolic or partially symbolic and partially numerical computation.

By the ergodic theorem the time average of the solution

$$\frac{1}{T}\int_0^T u(t)dt$$

converges to the invariant measure. In fact,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \phi(u(t))dt = \int_{L^2(\mathbb{T}^1)} \phi(u) d\mu(u) \tag{24}$$

where $\phi \in B(L^2)$ is any bounded function on $L^2$. Thus we can find approximations $\mu_m$ to the invariant measure $\mu$ by considering the sequence

$$\frac{1}{T_m}\int_0^{T_m} u_m(t)dt \sim \int_{L^2(\mathbb{T}^1)} u d\mu_m(u)$$

$u_0$ in these formulas is simply the solution of the linear equation (6) for uniform flows and the invariant measure $\mu_0$ is obtained in the limit is a weighted Gaussian, see [5]. The higher Picard iterates will give more complicated limits. Again, these approximations can probably be implemented by a symbolic or partially symbolic and partially numerical computation.

The problem is that this way of approximating the invariant measure may not be very inefficient. Thus it is important to seek more efficient ways of implementing these approximations first theoretically and then numerically.

# 8 RANS and LES Models

The objective of RANS (Reynolds Averaged Navier Stokes) computations is to compute the spatial distribution of the mean velocity of the turbulent flow. To do this the velocity and pressure are decomposed into the mean $\overline{u}$ and the deviation from the mean $u = U - \overline{u}$ (or fluctuation)

$$U(x,t) = \overline{u}(x,t) + u(x,t)$$

The average denoted here by a bar is an ensamble average. Then, by definition, the mean of $u$ is equal to zero. Similarly, the pressure is decomposed as

$$P(x,t) = \overline{p}(x,t) + p(x,t)$$

The divergence condition (18) gives that

$$\nabla \cdot \overline{u} = 0 = \nabla \cdot u$$

and averaging the Navier-Stokes equation (17) gives the equation for the mean velocity

$$\frac{\partial \overline{u}}{\partial t} + \overline{u} \cdot \nabla \overline{u} + \nabla \cdot \overline{u \otimes u} = \nu \Delta \overline{u} - \nabla \overline{p} \tag{25}$$

Thus the mean satisfies an equation similar to (1) except for an additional term due the Reynolds stress

$$\mathbf{R} = \overline{u \otimes u}$$

The additional term in (25) acts as an effective stress on the flow due to momentum transport cause by turbulent fluctuation. Until recently it has been impossible to determine this term from first principle and various approximations have been used. The simplest formulation is to set the Reynolds stress tensor to

$$\mathbf{R} = -\nu_T(x) \nabla \overline{u}$$

where $\nu_T(x)$ is called the turbulent eddy viscosity. This makes the additional term in the equation act as an additional (viscous) diffusion term. A better approximation is to develop an evolution equation for $\mathbf{R}$. This equation turns out to depend on the $\overline{u \otimes u \otimes u}$ and so on. Thus an infinite sequence of evolution equations for higher and higher moments is obtained and it must be closed at some level. This is done by approximating some higher moment by a formula depending only on lower moments. The closure problem is the problem of how to implement this

moment truncation. A good recent exposition of the RANS models is contained in [2].

The approximate invariant measure discussed above gives us a new insight into RANS models. In particular the mean is nothing but the expectation

$$\overline{u}(x,t) \sim \int_{L^2} u d\mu_m(u)$$

This obviously does not determine $\overline{u}$ since $u$ is unknown but we can now work with the various closure approximations and improve them knowing what the spatial average actually means. This can be done and the result simulated. The hope is to develop RANS models that are less dependent on the available data and the parameter regions covered by that data.

In LES, see [28], the velocity is decomposed into Fourier modes and then the expansion truncated at some intermediate scale that are usually given by the grid resolution. Then one computes the large scales explicitly and models the effects of the small scales, smaller than the cutoff, on the large scales with a *subgrid model*. The cutoff is usually done with a smooth Gaussian filter. LES thus assumes that the small scale turbulence structures are not significally dependent on the geometry of the flow and therefore can be respresented by a general model. This method is able to handle transition to turbulence and the resulting tubulent regimes in the flow better than RANS that usually needs to be told explicity where the transition occurs. Now if we let $\overline{u}$ and $\overline{u}$ denote resolved velocity and pressure then the Navier-Stokes equation for the resolve quantities can be written as

$$\frac{\partial \overline{u}}{\partial t} + \overline{u} \cdot \nabla \overline{u} + \nabla \cdot (\overline{u} \otimes \overline{u}) = \nu \Delta \overline{u} - \nabla \overline{p} - \nabla \cdot \tau \tag{26}$$

where $\tau$ represents the subgrid stress tensor (SGS)

$$\tau = \overline{u \otimes u} - \overline{u} \otimes \overline{u}$$

and the resolved scales are divergence free

$$\nabla \cdot \overline{u} = 0$$

$\tau$ describes the effects of the subgrid scales on the resolved velocity.

The most common subgrid models use a relationship between SGS and $\mathbf{e}$ the resolved strain tensor

$$\mathbf{e} = \frac{1}{2}(\nabla u + (\nabla u)^{\mathbf{T}})$$

where $(\nabla u)^{\mathbf{T}}$ denotes the transpose. The relationship between $\tau$ and $\mathbf{e}$ is

$$\tau - \frac{1}{3} trace\, \tau\, \delta_{ij} = -2\nu_T \mathbf{e}$$

Here $\delta_{ij}$ denotes the Kronecker's delta and the eddy viscosity is

$$\nu_T = C\varepsilon^2 \sqrt{2\mathbf{e}(\mathbf{e})^{\mathbf{T}}}$$

The $\varepsilon$ is a characteristic length scale for the subgrid. As it stands this subgrid model is purely dissipative and excessively so. If the constant in front of $|\mathbf{e}| = \sqrt{\mathbf{2e}(\mathbf{e})^{\mathbf{T}}}$ is allowed to vary with time, see [15], a much better result is obtained. Then the constant is computed dynamically during the simulation and with this modfication the so-called Smagorinsky subgrid model does not produce excessive dissipation. However, it only work with situations where the flow is homogeneous in at least one direction and thus does not permit general geometries.

In general when modeling an experiment we want the subgrid model to reproduce

- The Kolmogorov $k^{-5/3}$ energy spectrum of homogenous isotropic turbulence
- The statistics of turbulent channel flow

The advantage that we have with the approximate invariant measure is that we can base the cutoff on the approximately correct probability density function instead of a Gaussian that has nothing to do with the details of the small scale flow. This holds the promise that we can reproduce the correct scaling in the subgrid model. Ultimately this tests that the LES is producing the correct scaling down to the size of the computational grid.

# 9 Validation of the Numerical Methods

Turbulent fluids are highly unstable phenomena that are sensitive to noise and perturbation. Velocity trajectories depend sensitively on their initial conditions and it is not clear that they can be given a deterministic interpretation. This means that computations of such fluids are highly sensitive to truncation and even round-off errors. One must regards turbulent phenomena to be structurally unstable and stochastic. Statistical quantities associated to the turbulent fluids are deterministic and can be computed by taking appropriate statistical ensambles. However, one must be careful that the numerical methods one uses can be trusted to converge to the correct statistical quantity. It turns out that it is not enough to check that

the conventional quantities such as energy or momentum and make sure that they converge. One must also consider the scalings of the statistical quantities and check that they show the correct scalings over a sufficiently large parameter range. In doing this one must choose the numerical methods carefully.

In a series of papers the author and his collaborators, [38, 37, 7, 5], showed that whereas explicit methods generally fail to produce the correct scalings over a large parameter interval, implicit methods do. This reason for this is that in an implicit method the time step is independent of the spatial discretization and does not go to zero as the spatial discretization decreases. Explicit methods obtain stability by inserting artificial viscosity into the problem and this artificial viscosity destroys the small scale scalings. Before the scaling of the small scales is obtained the time step goes to zero in the explicit method and the computation grinds to a halt. This makes implicit methods the methods of choice. Although the implicit methods also induce some viscosity, it is much smaller and does not interfere with the small scale scaling to the same extent as for explicit methods. The problem is that implicit methods are much slower than explicit and although this is not a serious obstacle in one dimension it is in two dimensions and makes the turbulence problem intractable in three dimensions. Thus it becomes imperative to compute correct closure approximations for RANS and subgrid models for LES in order to be able to solve these by implicit methods and produce numerically the correct scalings. One way of implementing this is to use the (approximate) invariant measure to develop tests on numerical methods to see if they produce correct scalings down to the size of the numerical grid.

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