Title
A geometry of three-candidate elections

Author
Bergstrom, Ted

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Ted Bergstrom

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1 Preferences and reduced preference profiles

Assumption 1. *There are three candidates A, B, and C. All voters are assumed to have transitive strict preference orderings over these candidates.*

A preference profile for a community of voters lists the number of voters who have each of the possible preference orderings. Given Assumption 1, a voter could have any of six possible preference orderings over the three candidates. Table 1 displays a preference profile for an election that satisfies Assumption 1.

Table 1: A preference profile with 3 candidates

Ranking of candidates	Number of voters					
	n_1	n_2	n_3	n_4	n_5	n_6
1	A	C	B	A	C	B
2	B	B	C	C	A	A
3	C	A	A	B	B	C

Table 1 shows that for each candidate, there are two different preference orderings that rank that candidate second. For each pair of orderings with the same second choice, the candidate ranked first by one ordering is ranked last by the other. With three candidates, two different rankings with the same second choice are *opposite* in the following sense:

Definition 1 (Opposite preference orderings)**.** *Two strict preference orderings are said to be* opposite *if for any pair of candidates, the preferred choice of one ranking is the less preferred of the other.*

In majority voting, for any two candidates, the votes of two voters with opposite preference cancel each other. It follows that the outcome of majority voting is determined by a 3-vector showing the difference between the numbers with opposite preferences for whom candidates, B, C, and A are the second choice in each pair of opposites:

$$(m_1, m_2, m_3) = (n_1 - n_2, n_3 - n_4, n_5 - n_6)\,.$$

Table 2: A reduced form profile

	Number of voters		
Ranking	m_1	m_2	m_3
1	A	B	C
2	B	C	A
3	C	A	B

We describe a reduced form profile as in Table 2. In this profile, if $m(i) > 0$, then the number of voters with the preference ordering shown in the column below exceeds the number with the opposite ordering by $m(i)$. If $m(i)$ is negative, the number of voters with the preference below is exceeded by the number with the opposite preference by $|m(i)|$.

Since voting outcomes are determined by the proportions of voters with each preference ordering, it is convenient to define the fractions of voters having each preference ordering, Thus for $i = 1 \ldots 3$, we let

$$\alpha(i) = \frac{m(i)}{|m(A)| + |m(B)| + |m(C)|}$$

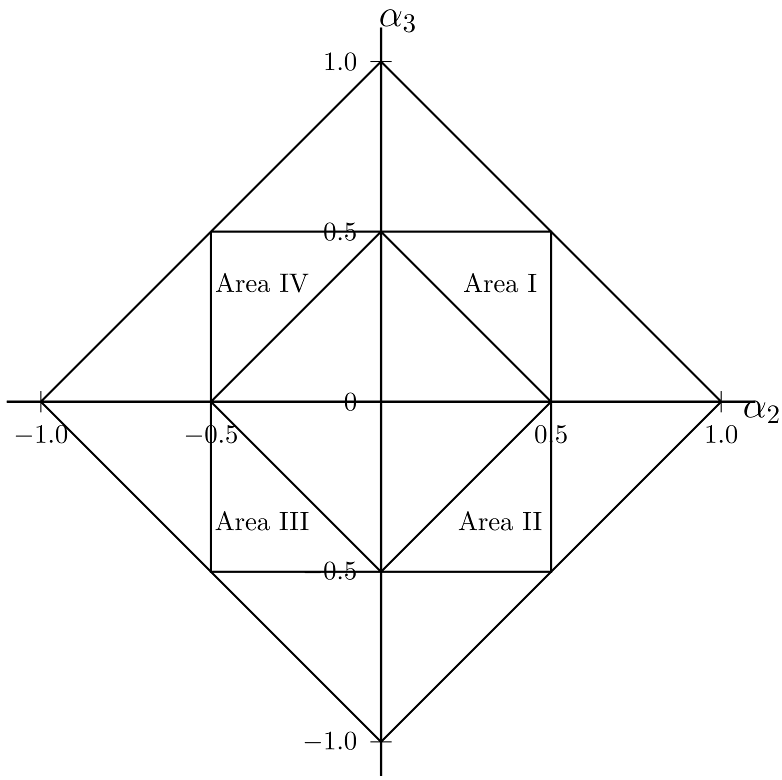
For example, the profile vector $(1/6, -1/3, -1/2)$ represents a profile in which 1/6 of the voters have the ordering ABC shown in the first column of Table 2. Those orderings for which $\alpha(i)$ is negative are the opposites of those shown in Table 2. Thus. 1/3 of voters have the profile ACB and 1/2 of voters have the ordering BAC.

2 A preference profile diagram

We arbitrarily choose one candidate to call B. Let us then give the name A to the top ranked candidate of the ordering that ranks B second. With this convention, a reduced preference profile has $\alpha_1 > 0$, while α_2 will be positive or negative depending on whether the ranking BCA is held by more voters than ACB or vice versa, and α_3 will be positive or negative depending on whether more voters rank the candidates CAB or BAC.

Every possible reduced form preference profile can be represented by a vector $(\alpha_1, \alpha_2, \alpha_3)$ such that $|\alpha_i| \leq 1$ for each i and $|\alpha_1| + |\alpha_2| + |\alpha_3| = 1$. A preference profiles can be represented by a point in the graph shown in Figure 1, where α_2 is on the horizontal axis and α_3 on the vertical axis. Then for any point (α_2, α_3) in this diagram, $\alpha_1 = 1 - |\alpha_2| - |\alpha_3|$ is the distance from that point to the nearest point on the base path of the outer baseball diamond.

Figure 1: A graph of profile types



For all points inside the inner baseball diamond, $\alpha_2 + \alpha_3 < 1/2$ and therefore $\alpha_1 > 1/2$. When this is the case, the majority voting relation is the same as the transitive preference relation ABC.

For all points lying between the outer baseball diamond and the large square, more than half of the voters share a single preference ordering. For these distributions of preferences, the majority voting relation is the same as the preference relation of the majority group and hence is transitive. For points that are above the line parallel to the horizontal axis at $\alpha_3 = .5$, more than half of the voters have the preference ordering CAB, and so that is also the majority voting ordering. For points lying to the right of the vertical line with $\alpha_2 = .5$, more than half of the voters have the preference ordering BCA and that is the majority voting ordering. For points in this region lying below the line $\alpha_3 = -.5$, more than half of the voters have the preference ordering BAC and that is the majority voting ordering. For points in this region to the left of the vertical line $\alpha_2 = -.5$, more than half of the voters have the preference ordering ACB and that is the majority voting relation.

For all points that lie between the inner baseball diamond and the square that encloses it, we have $\alpha_i < 1/2$ for $i = 1, \ldots 3$. This region consists of the four triangles labelled area I-IV.

For preference profiles represented by points in Area I, we have $0 < \alpha_i < 1$ for $i = A, B, C$. In this case the reduced preference profile is the Latin square seen in Table 2 and there is a Condorcet cycle.

Preference profiles represented by points in Areas II-IV are not Latin squares, but rather are consistent with single peaked preferences, in which the candidates are arrayed from left to right, with the "moderate" candidate being B in Tables 3 and 4, and A in Table 5.

If the point (α_2, α_3) is in area II, the reduced preference profile is as in Table 3. In this case, Candidate B defeats Candidates A and C, and candidate C defeats candidate A. Thus the majority voting relation is BCA.

Table 3: Area II profile

	Number of voters		
Ranking	α_1	α_2	α_3
1	B	C	B
2	A	B	C
3	C	A	A

If the point (α_B, α_C) is in area III, the reduced preference profile is as in Table 5. In this case Candidate B defeats A and C, and Candidate A defeats Candidate C. The majority voting relation is BAC.

Table 4: Area III profile

	Number of voters		
Ranking	α_1	α_2	α_3
1	B	A	B
2	A	B	C
3	C	C	A

If the point (α_B, α_C) is in area IV, the reduced preference profile is as in Table **??**. In this case, Candidate A defeate B and C, while Candidate B defeats Candidate C, so the majority voting relation is ABC.

I think the times call for extra attention to ways that the theory of externaities and public goods can help us to think about epidemics, medical supplies.

Figure 2: Majority voting orders by profile type

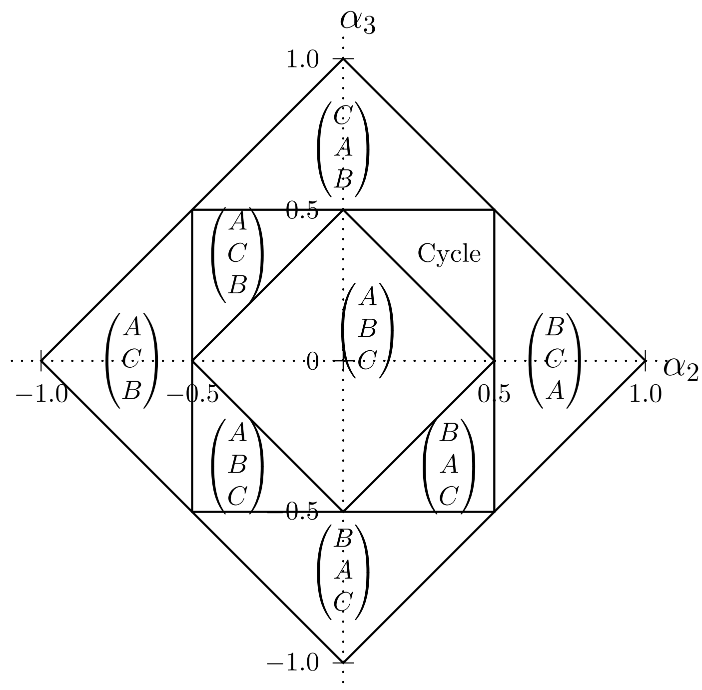


Figure 2 shows the majority voting orderings that apply for preference profiles represented in each region of the graph.

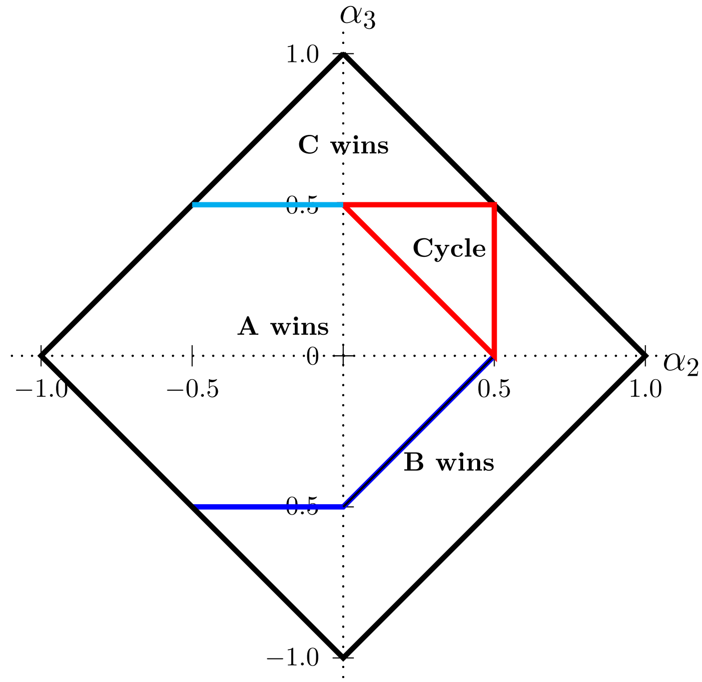
Figure 3 shows the regions of the graph representing preference profiles in which A is the winning candidate, B is the winning candidate, and C is the winning candidate, and for which the majority voting ordering is cyclic and there is no Condorcet winner. You may find it surprising that the regions for A, B, and C are not symmetric. The reason for this is that we chose to name the candidates in such a way that A is the first choice and C is the last choice for at least one of the preference orderings in the reduced form preference profile.

In Figure 3, the dark blue line segments show the border between the region where A wins and where B wins. For all profiles located on the dark blue line, there are two Condorcet winners, A and B. The light blue line shows the border between the regions where A wins and C wins. For all profiles located on this line, there are two Condorcet winners, A and C. The red triangle bounds the set of profiles for which the majority voting relation is cyclic. For all profiles represented by points on this boundary, the majority voting relation is also cyclic.

3 An application–Strategy proofness

We can use this diagrammatic representation to illuminate the question of when is it the case that if the candidate selected is a Condorcet winner, it will be in

Figure 3: Winning Candidate Regions



the interest of all voters to vote "sincerely", that is vote for their preferred candidate in every pairwise contest.

Partha Dasgupta and Eric Maskin [1] define a voting mechanism as strategy proof if when all voters vote sincerely, no coalition of voters can improve on the outcome by having its members vote in any contest for a candidate that is not their preferred candidate.

Proposition 1. *If the preference profile satisfies Assumption 1, and if there is a unique Condordet winner, then majority voting is strategy proof.*

Proof. Let x be the winning candidate if all voters vote sincerely. If majority voting is not strategy proof, then for some subset C of all voters, then some candidate y whom all voters in C prefer to x will win if the voters in C deviate from sincere voting. Since all voters in C prefer y to x, it must be that when all voters in C voted sincerely in the vote between x and y, they all voted for y. Since x is the winner when all voters vote sincerely and y is the winner with the proposed deviation, it must be that in the vote between x and y, the proposed deviation has some voters who previously voted for x changing their votes. But when they voted sincerely, all voters in C voted for y, so the outcome could not be changed by having some voters in C deviate from sincere voting.

□

A special case of Proposition 1 is where the only deviations from sincere voting to be considered are deviations by a single voter. In this case, a strategy proof equilibrium is one in which every player is using a weakly dominant stratgy.

Corollary 1. *If the preference profile satisfies Assumption 1, and if there is a unique Condordet winner, then voting one's true preference is a weakly dominant strategy for all voters.*

If the total number of voters in the reduced preference profile is even, then there will be preference profiles located on the boundary line between the area where where A wins and that where B wins. For these points, there are two Condorcet winners, A and B. Therefore Proposition 1 nor Corollary 1 do not apply to these distributions.

Let us consider preference profiles located along the horizontal segment of the border between the A wins region and the B wins region. All points along this line segment have preference profiles of the form found in Table 3.

Table 5: Some profiles with ties between A and B

	Number of voters		
Ranking	1/2	x	$1/2 - x$
1	B	A	A
2	A	C	B
3	C A	C	

References

[1] Partha Dasgupta and Eric Maskin. Elections and strategic voting: Condorcet and Borda. Online at `https://economics.harvard.edu/files/economics/files/maskin_fall_2019.pdf`, 2019.

With this profile, consider voters with preference ordering ABC.