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Abstract

We study multiline insurance companies with limited liability and limited capital, creating the possibility of insurer default. Insurance premiums are determined by no-arbitrage option pricing methods. The results are developed under the realistic assumption that the losses created by insurer default are allocated among policyholders following an ex post, pro rata, sharing rule. In general, the ratio of default costs to expected claims, and thus the ratios of premiums to expected claims vary across insurance lines. Moreover, capital and related costs are allocated across lines in proportion to each line's share of a digital default option on the insurer. Our results differ from those derived elsewhere in the literature.

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Abstract

We study multiline insurance companies with limited liability and limited capital, creating the possibility of insurer default. Insurance premiums are determined by no-arbitrage option pricing methods. The results are developed under the realistic assumption that the losses created by insurer default are allocated among policyholders following an ex post, pro rata, sharing rule. In general, the ratio of default costs to expected claims, and thus the ratios of premiums to expected claims vary across insurance lines. Moreover, capital and related costs are allocated across lines in proportion to each line's share of a digital default option on the insurer. Our results differ from those derived elsewhere in the literature.

1 Background

The optimal allocation of risk in an insurance market was studied in the seminal work of Borch (1962), who showed that without frictions a Pareto efficient outcome can be reached. Furthermore, in a friction-free setting, insurers can hold sufficient capital to guarantee they will pay all claims, since there is no lost opportunity cost to holding additional capital. Limited liability and limited capital, however, implies that insurers may fail to make the required payments to policyholders.

When markets are incomplete, in the sense that policyholders face a counterparty risk that cannot be independently hedged, the existence of limited liability and limited capital can have a significant impact. This leads to the following questions: For an insurance company offering insurance in multiple insurance lines under the assumptions of limited liability and limited capital, what will be the price structure across lines and how should the firm allocate its capital and associated costs between these lines?

The main contribution of this paper is to introduce a parsimonious model to answer these questions. Our results extend the analyses in earlier papers, e.g., in Phillips, Cummins, and Allen (1998) and Myers and Read (2001), but our results are quite different from theirs. The main source of the difference is that we assume that the insurer's available assets are distributed to the policyholders following what we call the *ex post pro rata rule*. Under this rule, the available assets are allocated to policyholder claims based on each claimant's share of the total claims. This rule has sensible properties and generally reflects the actual practice, as discussed below.

The risk of default by insurance firms has recently been highlighted by the precarious position of the so-called "monoline" bond and mortgage security insurers as a result of the subprime mortgage crisis. These insurers are monoline in that insurance laws prohibit them from offering various consumer lines such as homeowner and auto insurance.[1] The insurers are "multiline," however, in the sense that they provide coverage against the default risk of a wide range of debt issues, including individual mortgages, securitizations such as mortgage-backed securities and collateralized debt obligations, and debt issued by entities such as state and local governments. Given the potentially high default rates on these debt issues and the limited capital of the insurers, it is quite possible that these insurers will be unable to pay all of the claims they face.

A similar issue of insurer default risk arises with investment banks which function as

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[1] In a companion paper in process, we develop the conditions under which the equilibrium structure of a competitive insurance market may be either monoline or multiline.

the counterparties on derivative securities for a wide range of underlying instruments, including foreign exchange, interest rate swaps and options, and credit default swaps. Our results for pricing and capital allocations can be interpreted as applying to such security insurers and derivative counterparties, as well as more traditional insurance firms.

Most of our analysis can be carried out and understood under the simplifying assumption that there is no (excess) cost associated with holding capital within the firm. This is in the same spirit as Phillips, Cummins, and Allen (1998), but contrary to Myers and Read (2001). In practice, costs of internal capital appear to be important. For example, Froot, Scharsfstein, and Stein (1993) emphasize the importance of capital market imperfections for understanding a variety of corporate risk management decisions: the tax disadvantages to holding capital within a firm is an especially common and important factor. We therefore generalize our results to the situation when there are costs associated with holding internal capital.

The paper is organized as follows: In section 2 we give a review of related literature. In section 3, we introduce the basic framework for our analysis. In section 4, we analyze the pricing of insurance across lines for a multiline insurance company, and in section 5 we analyze capital and cost allocation across lines. Finally, section 6 provides concluding remarks.

2   Literature review

Financial models of insurance pricing and capital allocation were first developed by applying the principles of the capital asset pricing model (CAPM, see, for example, Fairley (1979) and Hill (1979)), or a discounted cash flow model (see, for example, Myers and Cohn (1987)). Both models, however, have significant drawbacks. The CAPM applications have the basic problem that they fail to incorporate the default risk faced by policyholders as a result of their insurer's limited liability. The CAPM is also not well suited to pricing risks with heavy tails, as would be plausible for various lines of catastrophic disasters and terrorist attacks. The discounted cash flow models must apply a risk-adjusted discount rate, but the derivations of this rate incorporate neither the frictional costs of holding capital nor the default risk for policyholders.

A major advance occurred by applying option valuation methods to the questions of insurance pricing and capital allocation, starting with the monoline insurance models

of Doherty and Garven (1986) and Cummins (1988). These papers specify the default risk faced by policyholders as a put option held by the equity owners of the insurer, which provides the equity owners the option to default on claims payments due to policyholders. The value of the option depends on the range of possible outcomes for policy claims and the amount of capital held by the insurer (which is the strike price of the option). The premium paid by the policyholders is then determined as the expected losses on the policy line minus the value of the default option. This means that the ratio of the premium to expected claims is less than 1, since the claims are not always paid in full.

The extension of option pricing methods to a multiline insurer was first provided by Phillips, Cummins, and Allen (1998) (PCA). Their analysis embeds the simplifying assumption that claims for all the lines are realized at the same date, which has the very useful implication that insurer default is simultaneously determined for all insurance lines. If an insurer's assets equal or exceed the actual policyholder claims, then all claims are paid in full. Whereas, if the actual claims exceed the available assets, the insurer defaults, and pro rated payments are made to policyholders following a loss allocation rule. The specific rule used by PCA is that each policyholder is allocated a share of the shortfall based on the amount of her initially expected claims relative to the total of all initially expected claims. Since the shortfall shares are allocated based on the expected claims as of the initial date, we will refer to this as the *ex ante* rule.

The PCA ex ante allocation rule implies that the default cost relative to expected claims is constant across lines, and therefore the premium to expected claims ratio will be constant across lines. While this result is very powerful, the ex ante rule is a very special case, with the undesirable feature that the allocation of the shortfall in case of default is allocated to policyholders on the basis of only the aggregate shortfall and the ex ante expected claim on each insurance policy. This means, for example, that when the insurer faces a shortfall in available funds, policyholders with no actual claims would have to make payments to those policyholders with claims. Moreover, policyholders with small expected losses would have to make payments to other policyholders with larger expected losses. Since the ex ante expected losses on all policies are generally not observable, policyholders have no basis for validating the share of the shortfall imposed on them.

Mahul and Wright (2004) note that while an *ex ante* expected loss allocation rule may lead to optimal risk sharing among policyholders, in practice the required cross-

payments between policyholders will be difficult to enforce.[2] Instead, in our model as explained below, we apply an ex post pro rata allocation rule in which policyholders share the default shortfall in proportion to their actual claims. The result is that, when the insurer defaults, policyholders without claims make no payments, and claimants always receive some payments from the insurer, in amounts proportional to their actual claim.

How to allocate capital within a multiline insurer is another important question. In the PCA model, where insurance is sold in informationally efficient, competitive, insurance markets and without costs of internal capital, insurance premiums are determined without the need to allocate capital. In our case, however, the frictional costs of holding capital make it imperative to allocate capital in order to determine the appropriate insurance premiums across lines. Two studies propose solutions to this capital allocation problem, Merton and Perold (1993) and Myers and Read (2001). In both cases, the amount of capital to be allocated to an insurance line is determined by an experiment in which the size of each line is changed and a computation is made of the required change in capital if the insurers risk of default is to remain unchanged.

The Merton and Perold (MP) experiment, more precisely, entirely removes an insurance line from the multiline firm and then computes the appropriate reduction in the insurers total capital requirements. The resulting reduction in capital is interpreted as the capital amount to be allocated when that line is part of the multiline firm. This procedure is then repeated for each line that the insurer covers. The MP method has the attribute that the sum of the capital allocations across all the lines will be less than the total capital required of the firm when it offers all the lines. The reason is that the overall firm receives a benefit of diversification that cannot be allocated to any of the individual lines.

The Myers and Read (MR) model also uses a marginal method to compute the capital allocation, but instead of removing each entire line from the total, they change the coverage amount of each line only by small incremental amounts. MR demonstrate that the capital allocations determined by their incremental technique satisfy the adding constraint whereby the sum of the capital allocations exactly equals the to-

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[2]The proper rule for allocating the default shortfall is related to the question of optimal contract design when insurer default may occur. Mahul and Wright (2004), Schlesinger (2000) and Doherty and Schlesinger (1990) are key papers in the small literature that discusses optimal insurance contracts for allocating the shortfall when an insurer defaults. For example, this is one motivation for mutual and participating insurance contracts, although these contracts then raise other issues such as limiting the diversification benefits from risk sharing.

tal amount of capital to be allocated. Our model, as developed below, applies the same concept for determining capital allocations and takes advantage of the same adding up condition. Our results differ from MR, however, because the MR computations are based on the PCA model's *ex ante* allocation rule for insurer shortfalls, whereas our results are developed on the basis of our *ex post* allocation rule.[3]

3   A competitive multiline insurance market

We first study the case of only one insured risk class. Consider the following one-period model of a competitive insurance market. At $t = 0$, an *insurer* (i.e., an insurance company) in a competitive insurance market sells insurance against a risk, $\tilde{L} \geq 0$,[4] to an *insuree*.[5] The expected loss of the risk is $\mu_L = E[\tilde{L}]$, $\mu_L < \infty$.

The insurer has limited liability and holds assets A that are available to pay the realized claims at $t = 1$. The assets A equal the sum of the premiums paid at the beginning of the insurance period and the equity capital contributed by the insurer. At time $t = 1$, as long as $\tilde{L} \leq A$, the insurer satisfies all claims by paying $\tilde{L}$ to the insurees; however, if $\tilde{L} > A$, then the insurer pays A and defaults on the shortfall $(\tilde{L} - A)$. Thus, the payment is

$$Payment = \min(\tilde{L}, A) = \tilde{L} - \max(\tilde{L} - A, 0) = \tilde{L} - \tilde{Q}(A),$$

where $\tilde{Q}(A) = \max(\tilde{L} - A, 0)$, i.e., $\tilde{Q}(A)$ is the payoff to the option the insurer has to default.[6] We note, in passing, that the rule can also be adjusted to take into account a deductible, D, by redefining the risk: $\hat{L} = \max(\tilde{L} - D, 0)$. In Arrow (1974), it was shown that having such a deductible is optimal under some conditions.

The premium paid for the insurance is the price P. For simplicity, throughout the

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[3]Lakdawalla and Zanjani (2006) provides an alternative approach to premium setting and capital allocation when there is default risk by specifying that the demand for insurance depends in part on the "quality" of the insurer. Although this approach can provide general conclusions comparable to the PCA and MR models, it lacks the quantitative precision that is provided by the default option approach. For our paper, a precise quantitative approach is essential to develop the difference between the ex ante and ex post rules for allocating default costs across insurance lines.

[4]Throughout the paper we use the convention that losses take on positive values.

[5]It is natural to think of each risk as an insurance line.

[6]When obvious, we suppress the A dependence, e.g., writing $\tilde{Q}$ instead of $\tilde{Q}(A)$.

paper, the risk-free discount rate is assumed to be zero.[7] In line with PCA, we also assume that there is a friction-free complete market for risk, admitting no arbitrage. The price for $\tilde{L}$ risk in the market is

$$P_L = Price\left[\tilde{L}\right] = E^*[\tilde{L}] = E[\tilde{m} \times \tilde{L}],$$

where $Price$ is a linear pricing function, which can be represented by the risk-neutral expectations operator $E^*[\cdot]$, or with the state-price kernel, $\tilde{m}$, in the objective probability measure,[8] and we assume that $E^*[\tilde{L}] < \infty$. Similarly, the price of the option to default is

$$P_Q = Price\left[\tilde{Q}(A)\right] = E^*[\tilde{Q}(A)] = E[\tilde{m} \times \tilde{L}].$$

We assume that there are friction costs when holding capital within an insurer, including both taxation and liquidity costs. The cost is δ per unit of capital. This means that to ensure that A is available at $t = 1$, $(1 + \delta) \times A$ needs to be contributed at $t = 0$.

Since the market is competitive and the cost of holding capital is δA, the price charged for the insurance is

$$P = P_L - P_Q + \delta A. \tag{1}$$

In words, the insurance premium equals the price of the covered risk minus the value of the default option plus the cost of internal capital.

We assume, in line with practice, that premiums are paid upfront, and thus to ensure that A is available at $t = 1$, the additional amount of $A - P_L + P_Q$ needs to be obtained by the insurer as invested capital. The total market structure is summarized in Figure 1, which also shows how noarbitrage pricing in the market for risk determines the price for insurance in the competitive insurance market.

It is natural to ask why insurees, recognizing that insurers impose a cost of holding capital, would not instead purchase their coverage directly in the market for risk. The answer is that we assume that insurees do not have direct access to the market for risk, or that the costs of doing so exceed the costs of internal capital for the insurer.[9]

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[7]The results are qualitatively similar with a non-zero risk-free rate.

[8]See, e.g., Duffie (2001) for standard results on existence and uniqueness of pricing functions under these completeness and noarbitrage assumptions.

[9]For example, if we think of the market for risk as a reinsurance market, this may be a natural constraint. A similar assumption, which would lead to identical results, is if the insuree faces costs of participating in the market for risk that are equal to or higher than the costs faced by the insurer,
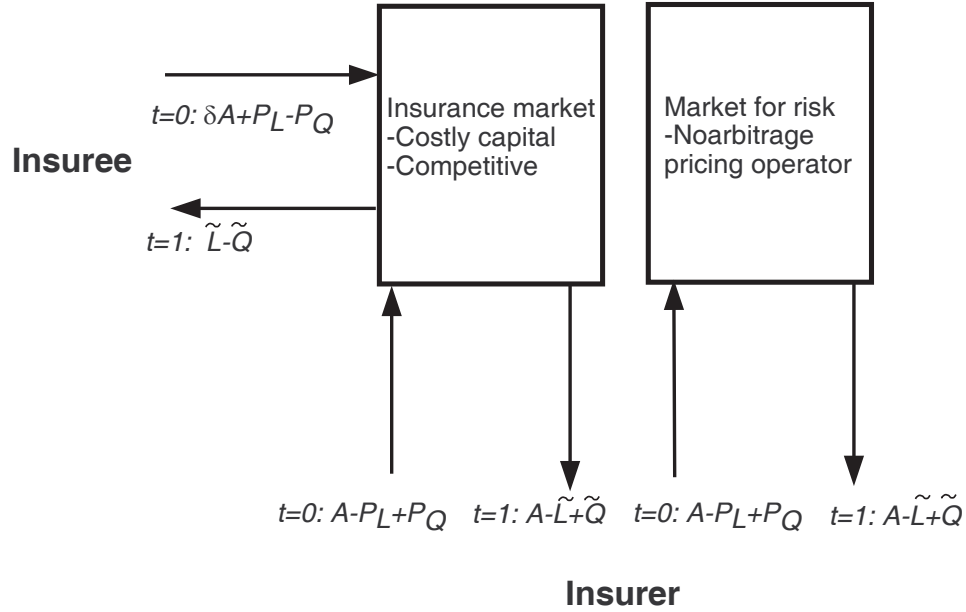
Insuree

t=0: $\delta A + P_L - P_Q$

t=1: $\widetilde{L} - \widetilde{Q}$

Insurance market
-Costly capital
-Competitive

Market for risk
-Noarbitrage
pricing operator

t=0: $A - P_L + P_Q$ t=1: $A - \widetilde{L} + \widetilde{Q}$ t=0: $A - P_L + P_Q$ t=1: $A - \widetilde{L} + \widetilde{Q}$

Insurer

Figure 1: *Structure of model. Insurers can invest in a market for risk or in a competitive insurance market. The cost of internal capital for a period is δ, so to ensure that A is available at $t = 1$, $(1 + \delta)A$ needs to be reserved at $t = 0$, as the sum of the premiums paid by the insurees and the capital invested in the insurer. Noarbitrage and competitive market conditions imply that the price for insurance is $P = \delta A + P_L - P_Q$, which is contributed by the insuree at $t = 0$ in return for which he receives his claim minus the default cost $(\tilde{L} - \tilde{Q})$ at $t = 1$. The amount $A - P_L + P_Q$ must therefore be contributed by the insurer's shareholders. The discount rate is normalized to zero.*

An assumption of this type is generally necessary, of course, to motivate the use of financial intermediaries compared to direct capital market access.

The generalization to the case when there are multiple risk classes is straightforward. Following PCA, we assume that claims on all the multilines are realized at the same time t = 1. If coverage against N risks is provided by one multiline insurer, the total payment made to all policyholders with claims, taking into account that the insurer may default, is

$$Total\ Payment = \tilde{L} + \max(\tilde{L} - A, 0) = \tilde{L} - \tilde{Q}(A), \tag{2}$$

where $Total\ Payment = \sum_i Payment_i$, $\tilde{L} = \sum_i \tilde{L}_i$ and $\tilde{Q}(A) = \max(\tilde{L} - A, 0)$. The total price for the risks is, $P \stackrel{\text{def}}{=} \sum_i P_i$, where P_i is the price for insurance against risk i, is once again on the form (1).

Our analysis so far is thus based on the following assumptions:

1. *Market completeness*: The market for risk is arbitrage-free and complete, such that there is a unique linear pricing operator.

2. *Limited liability*: Insurers have limited liability.

3. *Costly capital*: There is a cost for insurers to hold capital.

4. *Competitive insurance markets*: Prices for insurance are set competitively.

5. *Access to markets*: Insurees do not have direct access to the market for risk.

These assumptions completely determine the pricing of risks, as well as the capital and cost allocations across all lines, as we shall now show.

4   Pricing across lines

We now turn to our primary question, how premiums are set across insurance lines. From (1), given A and δ, we know the total premium, P, but not the individual

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in which case it will be optimal to buy from the insurer. Finally, if the market is incomplete and the insurer is risk-neutral, there may be no way to replicate the payoffs in the market for risk, leaving the insurance market as the sole market available to the insuree.

premiums, P_i. We first focus on the case when the cost of holding capital is zero, $\delta = 0$. We define line i's *share of default option value*, r_i as

$$r_i \overset{\text{def}}{=} \frac{P_{L_i} - P_i}{P_Q}. \tag{3}$$

This definition immediately implies that

$$P_i = P_{L_i} - r_i P_Q, \tag{4}$$

which motivates the terminology, since $r_i P_Q$ is the difference between the actual premium and the default-free premium. We note that, when $\delta > 0$, the premium will be higher than what is expressed in (4), since the insuree will also pay for costly capital. We discuss the general situation in Section 5.1, in which case the share of default option value in line i will be denoted by r_i^0, whereas r_i will still be defined through (3). Through (4), determining the premium in line i is essentially the same as determining r_i. This relies on the fact that the given amount of assets A determines the value of the overall default option P_Q (recall that the claims on all lines become due at the same $t = 1$, with the total claims determining whether the insurer defaults).

Obviously, (1,3) imply that $\sum_i r_i = 1$. We also define line i's *default option value per unit of liability*

$$z_i \overset{\text{def}}{=} \frac{r_i P_Q}{P_{L_i}}, \tag{5}$$

and, following Phillips, Cummins, and Allen (1998), the *premium-to-liability* ratio

$$\text{Premium-to-liability-ratio} \overset{\text{def}}{=} \frac{P_i}{P_{L_i}}. \tag{6}$$

From (4,5) it follows that

$$\frac{P_i}{P_{L_i}} = 1 - z_i = 1 - \frac{r_i P_Q}{P_{L_i}}, \tag{7}$$

We need a rule for how payments to the default option are shared between claimholders, to be able to calculate premiums. To define a general such sharing rule, we

specify N functions $F_i : \mathbb{R}_+^N \rightarrow \mathbb{R}_+$, $i = 1, \ldots, N$, such that

$$Payment_i = \tilde{L}_i - F_i(\tilde{L}_1, \ldots, \tilde{L}_N | A), \tag{8}$$

where (2) implies that

$$\sum_{i=1}^{N} F_i(\tilde{L}_1, \ldots, \tilde{L}_N | A) = \tilde{Q}(A) \tag{9}$$

in all states of the world. The sharing rule is then completely characterized by $\mathcal{F} \stackrel{\text{def}}{=}$ $(F_1, \ldots, F_N)$. From (4) and (8) it follows that

$$r_i = \frac{Price\,[F_i]}{P_Q}. \tag{10}$$

There are obviously many ways to construct such a payment rule, but a minimal set of requirements is

Condition 1 Sharing rule requirements:

1. *In case of no default: The payment to each insuree is exactly the amount claimed:* $\tilde{Q} = 0 \Rightarrow F_i = 0$, *for all* $i = 1, \ldots, N$.

2. *In case of default: The payment to each insuree is bounded above by the claim, and below by zero:* $0 \leq F_i \leq \tilde{L}_i$, *for all* $i = 1, \ldots, N$.

3. *No-claim policy: Insurees with no claims do not receive payments:* $\tilde{L}_i = 0 \Rightarrow$ $F_i = 0$.

In line with our discussion in previous sections, we focus on the *ex post* rule for

allocating the shortfall when the insurer defaults.[10] The rule specifies that

$$F_i = \frac{\tilde{L}_i \times \tilde{Q}(A)}{\tilde{L}},$$

so that

$$Payment_i = \frac{\tilde{L}_i}{\tilde{L}} \times Total\ Payment = \tilde{L}_i - \tilde{L}_i \times \frac{\tilde{Q}(A)}{\tilde{L}}, \tag{11}$$

i.e., in case of default the insurees share the total payments according to their fraction of total losses. It is easy to check that the *ex post* sharing rule satisfies the sharing rule requirements. Moreover, it corresponds well to the rules used in practice.[11]

In this case, using the pricing operator for (11) it follows that the market price for insurance in line i is

$$P_i = P_{L_i} - Price\left[\tilde{L}_i \times \frac{\tilde{Q}(A)}{\tilde{L}}\right]. \tag{12}$$

It then follows that line i's share of default option value, under the *ex post* sharing rule is

$$r_i = Price\left[\frac{\tilde{L}_i}{\tilde{L}} \times \frac{\tilde{Q}(A)}{P_Q}\right], \tag{13}$$

and that the default option value per unit of liability is

$$z_i = \frac{r_i P_Q}{P_{L_i}} = \frac{1}{P_{L_i}} \times Price\left[\frac{\tilde{L}_i}{\tilde{L}} \times \tilde{Q}(A)\right]. \tag{14}$$

Clearly, as indicated by (7,14), under the *ex post* sharing rule, we would, in general, expect the premium-to-liability ratio to vary with i.

An alternative expression for the premium-to-liability ratio, using the state price

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[10]We focus on contracts that are present in practice. If more general contracts are possible, it may for example be optimal to have ex post transfers from claimants who did not suffer any losses to claimants who did; such contracts, which would effectively turn the insuree into an insurer in some states of the world, do not seem to exist in common practice. It is beyond the scope of this paper to analyze why such contracts are rarely seen.

[11]For example, see National Association of Insurance Commissioners, Insurer Receivership Model Act, October 2007. An ex-post payout rule is also specified in the contracts offered by the California Earthquake Authority for circumstances in which the Authority has insufficient resources to pay all of its claims. See also (Mahul and Wright 2004) for a discussion of alternative payoff rules and the complications created by rules other than ex-post prorated payments.

kernel kernel, $\tilde{m}$, is

$$\frac{P_i}{P_{L_i}} = 1 - z_i = 1 - \frac{1}{P_{L_i}} \times E\left[\tilde{m} \times \frac{\tilde{L}_i}{\tilde{L}} \times \tilde{Q}(A)\right] = 1 - E\left[\frac{\tilde{Q}(A)}{\tilde{L}}\right] - \frac{cov\left[\tilde{m}\tilde{L}_i, \frac{\tilde{Q}(A)}{\tilde{L}}\right]}{E\left[\tilde{m}\tilde{L}_i\right]}. \quad (15)$$

If $\tilde{m}$ is independent of $\tilde{L}$ (and thereby to $\tilde{Q}(A)$ and $\tilde{L}_i$), then

$$\frac{cov\left[\tilde{m}\tilde{L}_i, \frac{\tilde{Q}(A)}{\tilde{L}}\right]}{E\left[\tilde{m}\tilde{L}_i\right]} = \frac{cov\left[\tilde{L}_i, \frac{\tilde{Q}(A)}{\tilde{L}}\right]}{E\left[\tilde{L}_i\right]},$$

so for an insurance company with risks that are independent of market risk, premium-to-liability ratios will be low in lines for which losses are high in the states of the world when the company defaults.

Implication 1 *For an insurer, insuring risk that is unrelated to market risk, define*
$\Phi = cov\left[\tilde{L}_i, \frac{\tilde{Q}(A)}{\tilde{L}}\right]$

- *Insurance lines, with losses that tend to be high in states of the world in which the insurer defaults (i.e. with high Φ) have low premium-to-liability ratios.*

- *Insurance lines, with losses that tend to be small in states of the world in which the insurer defaults (i.e. with low Φ) have high premium-to-liability ratios.*

The intuition here is that the premium will be lower on lines where claims are less likely to be paid in full as the result of insurer default.

It is difficult to draw general conclusions about premium-to-liability ratios in the general case. An interesting special case of (15) is when the insurer's risk is essentially independent of market risk, in the sense that $\tilde{m}$ is almost independent of $\frac{\tilde{Q}(A)}{\tilde{L}}$, even though there are some lines with risks that are correlated with market risk. An example of risk essentially independent of market risk arises when the firm insures a large number of lines, most of which are independent of market risk, but with a few lines with risks

13

that are correlated with market risks, adding up to a negligible part of the total $\tilde{L}$ risk. For such a company, if line i is one of the few lines with market correlated risk, e.g., $\tilde{L}_i = \alpha \tilde{m} + \beta \tilde{\xi}$ where $\tilde{\xi}$ is independent of $\tilde{m}$, it is the case that

$$
\frac{cov\left[\tilde{m}\tilde{L}_i, \frac{\check{Q}(A)}{\tilde{L}}\right]}{E\left[\tilde{m}\tilde{L}_i\right]} = \frac{cov\left[\tilde{m}(\alpha\tilde{m} + \beta\tilde{\xi}), \frac{\check{Q}(A)}{\tilde{L}}\right]}{E\left[\tilde{m}(\alpha\tilde{m} + \beta\tilde{\xi})\right]}
$$

$$
\approx \frac{\beta cov\left[\tilde{\xi}, \frac{\check{Q}(A)}{\tilde{L}}\right]}{E\left[\alpha\tilde{m}^2 + \beta\tilde{\xi}\right]}
$$

$$
= \frac{cov\left[\tilde{\xi}, \frac{\check{Q}(A)}{\tilde{L}}\right]}{\frac{\alpha}{\beta}E\left[\tilde{m}^2\right] + E\left[\tilde{\xi}\right]}.
$$

Assuming that $cov\left[\tilde{\xi}, \frac{\check{Q}(A)}{\tilde{L}}\right] > 0$, within this class of risks, given $\beta > 0$, risks that are more correlated with market risk (have higher α) will have higher premium-to-liability ratios. However, if $\tilde{\xi}$ is uncorrelated with $\frac{\check{Q}(A)}{\tilde{L}}$, the premium-to-liability ratio is the same, regardless of α and β. We summarize this in

Implication 2 *For an insurer, with a portfolio of risks, $\tilde{L}$, that is essentially unrelated to market risk, given a line i, with risk $\tilde{L}_i = \alpha\tilde{m} + \beta\tilde{\xi}$, where $\tilde{\xi}$ is independent of $\tilde{m}$, define $\Theta = cov[\tilde{\xi}, \frac{\check{Q}(A)}{\tilde{L}}]$*

- *If $\Theta > 0$, given $\beta > 0$, the premium-to-liability ratio is an increasing function of α.*

- *If $\Theta > 0$, given $\alpha > 0$, the premium-to-liability ratio is a decreasing function of β.*

- *If $\Theta < 0$, given $\beta > 0$, the premium-to-liability ratio is a decreasing function of α.*

14

- *If $\Theta < 0$, given $\alpha > 0$, the premium-to-liability ratio is a increasing function of β.*

- *If $\Theta = 0$, then the premium-to-liability ratio is independent of α and β.*

Thus, even in the special case when the insurer's portfolio is essentially unrelated to market risk and with a very special form of $\tilde{L}_i$, premium-to-liability ratios can be increasing or decreasing in market risk.

These implications are different than the predictions in Phillips, Cummins, and Allen (1998). According to PCA (Hypothesis 2, page 609), the default premium should be constant across lines.[12]

The source of these differences is that PCAs results are based on an ex ante sharing rule, with default costs allocated based on the expected claims at $t = 0$, whereas our results are based on the ex post sharing rule, with default costs allocated based on the actual claims at $t = 1$. The *ex ante* sharing rule is formally defined as

$$F_i = \frac{P_{L_i} \times \tilde{Q}(A)}{P_L},$$

which implies the payment rule

$$Payment_i = \tilde{L}_i - \frac{P_{L_i}}{P_L} \times \tilde{Q}(A). \tag{16}$$

By applying the pricing operator to (16), it follows that

$$r_i = \frac{P_{L_i}}{P_L}, \tag{17}$$

which, via (5,7), implies that

$$z_i = \frac{P_Q}{P_L}, \tag{18}$$

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[12]PCA assume that $\delta = 0$, an assumption that we will relax in subsequent analysis. Their model is also more general: It is dynamic and includes an inflation premium, as well as risky processes for the returns on A over time. Our model could be generalized along such lines, but the the differences between our model and that of PCA are not related to these factors. For this reason, we have kept our analysis as simple as possible.

and

$$\frac{P_i}{P_{L_i}} = 1 - \frac{P_Q}{P_L}. \tag{19}$$

Thus, under the *ex ante* sharing rule, the premium-to-liability ratio does not vary with i. Equation (19) corresponds to equation (18) in PCA, under the assumption that the discount rate and inflation are 0. In contrast, under our ex-post sharing rule, the premium-to-liability ratio will generally vary with i.

As we have argued, we believe that, in practice, only ex post sharing rules are generally used. Moreover, as shown in the example below, the ex ante rule will in general not satisfy the sharing rule requirements:

4.1 A pricing example

We provide a simple example that emphasizes the difference between the two sharing rules. Consider two independent risks, $\tilde{L}_1$ with 50% chance of being 40 and 50% chance of being zero, and $\tilde{L}_2$, with 50% chance of being 10 and 50% chance of being zero. The losses in the four states of the world are shown in Table 1.

Further, assume that the total liability is $A = 20$, and that risk-neutral probabilities coincide with physical probabilities. Then, the price of $\tilde{L}_1$ is $P_{L_1} = 20$, $P_{L_2} = 5$, and $P_Q = 25\% \times 20 + 25\% \times 30 = 12.5$. Now, the payout in the four states of the world can be derived by using (11), and are shown in Table 2.

The sharing rule for the case when both insurees have claims follows from the fraction of losses being $\tilde{L}_1/(\tilde{L}_1 + \tilde{L}_2) = 40/(40 + 10) = 80\%$, so 80% of the total capital of 20, i.e., 16 goes to insuree 1, and the remaining 4 to insuree 2. Thus, the value of the limited liability insurance against $\tilde{L}_1$ is $25\% \times 20 + 25\% \times 16 = 9$, and the value of the limited liability insurance against $\tilde{L}_2$ is $25\% \times 10 + 25\% \times 4 = 3.5$.

This in turn implies that the premium-to-liability ratios are

$$\frac{P_1}{P_{L_1}} = \frac{9}{20} = 0.45, \qquad \frac{P_2}{P_{L_2}} = \frac{3.5}{5} = 0.7,$$

which are not equal. This is a simple example of the general property that premium-to-liability ratios will generally vary across lines under the ex post sharing rule. Intuitively it is clear that the premium-to-liability ratio should be lower for the first risk, since it is more likely to have losses in the states of the world in which the insurer defaults.

State	Probability	$\tilde{L}_1$	$\tilde{L}_2$
1	25%	0	0
2	25%	40	0
3	25%	0	10
4	25%	40	10
Price of risk		20	5

Table 1: Losses for two risks in four equally probable states of the world.

State	Ex post rule		Ex ante rule	
	$Payment_1$	$Payment_2$	$Payment_1$	$Payment_2$
1	0	0	0	0
2	20	0	24	-4
3	0	10	0	10
4	16	4	16	4
Price of insurance	9	3.5	10	2.5
Value of default option	11	1.5	10	2.5
Premium-to-liability ratio	0.45	0.7	0.5	0.5

Table 2: Payment in to insurees in different states of the world under ex post and ex ante sharing rules. Total assets $A = 20$.

Therefore, the insurer tends to pay back less (relative to claims) to insuree 1, which leads to a lower premium-to-liability for risk 1. Also, the payments satisfy the sharing rule requirements (condition 1), as is always the case under the *ex post* sharing rule.

We now study the same example in a setup similar to PCA, i.e., under the *ex ante* sharing rule. The payouts in the four states of the world follow from (16), and that $P_{L_1}/(P_{L_1} + P_{L_2}) = 80\%$, and are shown in Table 2. Specifically, the rule indicates that insuree 2 should pay insuree 1 20% of the option payout of 20, i.e., 4, in the state of the world in which only insuree 1 realizes losses. This obviously violates both assumptions 2 and 3 of the sharing rule requirement.

Implication 3 *The* ex post *sharing rule satisfies the sharing rule requirements, whereas the* ex ante *sharing rule, in general, does not.*

Under the ex ante sharing rule, the premiums are $P_1 = 20 - 0.8 \times 12.5 = 10$ and

$P_2 = 5 - 0.2 \times 12.5 = 2.5$, so the premium-to-liability ratios are equal:

$$\frac{P_1}{P_{L_1}} = \frac{10}{20} = 0.5, \qquad \frac{P_2}{P_{L_2}} = \frac{2.5}{5} = 0.5,$$

in line with PCA's argument.

5 Capital allocation across lines

We now turn to the question of how capital, A, and costs, δA, should be allocated between business lines. Our analysis follows the lead of Myers and Read (2001). However, since they, like PCA, base their analysis on the *ex ante* sharing rule, and we apply the *ex post* sharing rule, our results will be different. For the time being, we continue to assume that $\delta = 0$, in order to focus on capital allocation. A complete derivation of the results in this section is given in the appendix.

In general, we wish to allocate the assets A among lines, $A = \sum_i A_i = \sum_i v_i A$, where v_i defines the *capital allocation rule*. As showed in Myers and Read (2001), any capital allocation rule, v_i, (such that $\sum_{i=1}^{N} v_i = 1$) implies a "summing up" relationship in the following sense: We study insurance portfolios

$$\sum_i q_i \tilde{L}_i,$$

(with weight, q_i reflecting the relative amounts of risk class i) where we have, so far, assumed that $q_1 = q_2 = \ldots = q_N = 1$, but we now assume that the insurer may (marginally) change his exposure to risk in line i, e.g., by selling more insurance. At the same time, we also assume that the insurer marginally increases the capital as following the capital allocation rule defined by the v_i's. Thus, if the insurer increases exposure in line i from q_i, to $q_i + \Delta q$, then assets are increased from A to $A + v_i \times \Delta q$. Then the payout of the default option for general q's is

$$\tilde{Q}_q = \max\left(\sum_i q_i(-\tilde{L}_i - v_i A_i), 0\right), \qquad (20)$$

18

which generalizes equation (2).

Now, (20) implies that

$$\frac{\partial \tilde{Q}_q}{\partial q_i} = (\tilde{L}_i - A_i)\tilde{V}(A), \tag{21}$$

where $\tilde{V}(A)$ is a digital default option: it pays a dollar in the states of the world in which the insurer defaults. The price of such an instrument is $P_V = Price[\tilde{V}(A)]$. We assume that the insurer defaults in at least one state of the world, so $P_V > 0$.

The summing up relationship now states that

$$\sum_i q_i \frac{\partial \tilde{Q}_q}{\partial q_i} \equiv \tilde{Q}_q \tag{22}$$

in all states of the world, for all $q_1, \ldots, q_N > 0$, which is seen by summing over i in (21). The summing up rule therefore suggests that a capital allocation rule is consistent with a decomposition of the value of the option to default, at the margin.

Applying the pricing operator to both sides of (22), immediately leads to

$$\sum_i q_i \frac{\partial P_{Q_q}}{\partial q_i} = P_{Q_q}.$$

However, this in turn implies that the marginal price of one extra unit of exposure to $\tilde{L}_i$ risk, at $q_1 = q_2 = \cdots = 1$, is

$$P_i = P_{L_i} - \frac{\partial P_{Q_q}}{\partial q_i}. \tag{23}$$

Now, if q_i increases marginally, representing a marginal increase in insurance sold to new insurees, then any other choice of $\frac{\partial P_{Q_q}}{\partial q_i}$ than $\frac{\partial P_{Q_q}}{\partial q_i} = r_i P_Q$ leads to a redistribution of value between current and new insurees, as long as the same payout rule is used for new and current insurees in line i. This intuitive fact is shown formally in the appendix. Any such redistribution would violate the original no-arbitrage assumption and, therefore, we conclude that the capital allocation rule must be consistent with the

19

following relation:

$$r_i = \frac{\partial P_{Q_q}/\partial q_i}{P_Q}. \tag{24}$$

It is shown in the appendix that there is bijection, $\{r_i\} \leftrightarrow \{v_i\}$, between relative asset allocations, v_i, satisfying $\sum_i v_i = 1$ and allocation of shares of default option value, r_i's, (also satisfying $\sum_i r_i = 1$), so that given r_i's, the v_i's are uniquely defined by

$$v_i = \frac{1}{A \times P_V} \times \left(Price\left[\tilde{L}_i \tilde{V}(A)\right] - r_i P_Q \right). \tag{25}$$

We note that v_i also determines the *surplus allocation* in line i, S_i.[13] In our notation, we have $S_i \stackrel{\text{def}}{=} v_i A - P_{L_i}$. Also, the *relative surplus allocation* is, $s_i \stackrel{\text{def}}{=} S_i/P_{L_i} = A\frac{v_i}{P_{L_i}} - 1$.

So far, our arguments have been made for general sharing rules, $\mathcal{F}$. Under the *ex post* sharing rule, the capital allocation rule reduces to

$$v_i = \frac{Price\left[\frac{\tilde{L}_i}{\tilde{L}} \tilde{V}(A)\right]}{P_V}. \tag{26}$$

Further, for the special case in which there is only one state of the world in which the insurer defaults, the rule is particularly simple: $v_i = r_i$, $i = 1, \dots, N$. We summarize these results in the following

Implication 4 *If an* ex post *sharing rule is used, then (26) is the only capital allocation rule that does not lead to redistribution between new and old insurees with marginal expansions of insurance lines. In the special case in which the insuree defaults in only one state of the world this capital allocation rule reduces to $v_i = r_i$, $i = 1, \dots, N$.*

This capital allocation rule differs from the ones derived in Myers and Read (2001), who suggest that the marginal contribution to default value, d_i — in our notation defined as $d_i \stackrel{\text{def}}{=} \frac{\partial P_Q}{\partial P_{L_i}}$, i.e., the increase in default option value for a one dollar increase in liability — should be chosen such that d_i is the same across lines (see page 554).

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[13]The term surplus allocation and surplus requirement is interchangeably used for S_i in Myers and Read (2001).

However, via (24), and the relationship $\frac{\partial P_Q}{\partial P_{L_i}} = \frac{\partial P_Q/\partial q_i}{P_{L_i}}$, we have

$$d_i = P_Q \times \frac{r_i}{P_{L_i}}, \tag{27}$$

which, under the *ex post* sharing rule, in general varies with i. Under the *ex ante* sharing rule, since $r_i = P_{L_i}/P_L$ we get $d_i = P_Q/P_L$, so the equal allocation rule proposed in Myers and Read (2001) is consistent with the *ex ante* sharing rule. We summarize the differences between the *ex post* and *ex ante* sharing rules in Table 3.

5.1 Pricing and cost allocation when $\delta > 0$

It is straightforward to generalize our analysis to the case when $\delta > 0$. In line with our previous allocation discussion, and following Myers and Read (2001), capital costs should be allocated according to a marginal cost pricing rule. The marginal increase in capital is $v_i A dq_i$, so the marginal cost is $\delta v_i A dq_i$, and under marginal cost pricing, $v_i \delta A$ is allocated to line i. Therefore, in the case of costly capital, $\delta > 0$, the pricing formula becomes:

$$P_i = P_{L_i} - r_i^0 P_Q + v_i \delta A, \tag{28}$$

where

$$r_i^0 \overset{\text{def}}{=} \frac{Price\,[F_i]}{P_Q},$$

is the share of the default option value, previously denoted by r_i.

This form of r_i^0 is in line with (10) but different from r_i defined in (4), since, when $\delta > 0$ the share of the default option value is no longer simply $(P_{L_i} - P_i)/P_Q$. The form of r_i^0 is obviously the same as in (13) and (17) under the *ex post* and *ex ante* sharing rule, respectively.

The formula for r_i^0 provides the r_i's we would get if $\delta = 0$. Through (4) and (28) it follows that $r_i = r_i^0 - \frac{v_i \delta A}{P_Q}$ when $\delta > 0$. Thus, when $\delta > 0$ two factors cause premium-to-liability ratios to vary across lines: (1) the ex-post sharing rule and (2) the capital cost factor δ.

We summarize our results in

Implication 5 *For all $\delta \geq 0$, if an* ex post *sharing rule is used, then capital $A_i = v_i A$*

should be allocated to line i, where v_i is defined in (26). Moreover, capital costs, δA, should be allocated so that $v_i \delta A$ is allocated to line i. Finally, the price for insurance in line i is given by (28), where $r_i^0 = Price\left[\frac{\tilde{L}_i}{\tilde{L}} \times \frac{\tilde{Q}(A)}{P_Q}\right]$.

We next show in an example how the capital allocation rule applies under the *ex post* sharing rule.

5.2   A capital allocation example

We show in a simple example that, under the *ex post* sharing rule, the only capital allocation rule that is consistent with no redistribution is $A_i = v_i A$. In our example, default only occurs in one state of the world, so $v_i = r_i$. We assume that $\delta = 0$.

We assume that there are two risks, $\tilde{L}_1$ and $\tilde{L}_2$, and three states of the world. The state prices are π_j, $j = 1, 2, 3$. The losses and state prices in the different states of the world are shown in Table 4 below. The default-free prices for insurance are thus, $P_{L_1} = 0.25 \times 10 + 0.25 \times 50 = 15$, and $P_{L_2} = 0.25 \times 10 + 0.25 \times 30 = 10$. We assume that the total assets $A = 40$. In this case, when the *ex post* sharing rule is used, the payout in different states of the world are summarized in Table 5.

Default thus only occurs in state 3, in which only 40 of the total losses of 80 is paid out, which in turn implies that the value of the default option is $P_Q = 0.25 \times 40 = 10$. Since the expected value of losses is $P_L = P_{L_1} + P_{L_2} = 15 + 10 = 25$, the total surplus is $S = A - P_L = 40 - 25 = 15$.

We first wish to understand how the total assets, $A = 40$, should be allocated between the two lines. We wish to find v_1 and v_2, such that $A = A_1 + A_2$, and $A_1 = v_1 A$, $A_2 = v_2 A$. It follows from our previous arguments that $v_1 = r_1$ and $v_2 = r_2$, which via (13) implies that $v_1 = 0.625$, $v_2 = 0.375$, $A_1 = 25$ and $A_2 = 15$.

The surplus allocation between the two lines is then, $S_1 = A_1 - P_{L_1} = 25 - 15 = 10$, $S_2 = A_2 - P_{L_2} = 15 - 10 = 5$. Finally, it is easy to check that the price of the cash-flows in Table 5 is $P_1 = 8.75$ and $P_2 = 6.25$.

The allocation implied by v_1 and v_2 is important in that it provides a rule for how assets need to change if more insurance in one line is sold. For example, assume that the insurer increases his exposure to risk 2 by $\Delta q = 10\%$. In this case, the allocation

22

Variable	Definition	Ex post sharing rule	Ex ante sharing rule
Default option payment, $\tilde{Q}(A)$	$\tilde{Q}(A) = \max(\tilde{L} - A, 0)$		
Digital default option, $\tilde{V}(A)$	$\tilde{V}(A) = I_{\{\tilde{L}_q - A_q > 0\}}$		
Price of default-free insurance, P_{L_i}			
Price of defaultable insurance, P_i			
Payment to insuree		$\tilde{L}_i - \dfrac{\tilde{L}_i}{L} \times \tilde{Q}(A)$	$\tilde{L}_i - \dfrac{P_{L_i}}{P_L} \times \tilde{Q}(A)$
Share of default option value	$r_i = \dfrac{P_{L_i} - P_i}{P_Q}$	$r_i = Price\left[\dfrac{\tilde{L}_i \times \tilde{Q}(A)}{L \times P_Q}\right]$	$r_i = \dfrac{P_{L_i}}{P_L}$
Default option value per unit liability	$z_i = \dfrac{r_i}{P_{L_i}} \times P_Q$	$z_i = Price\left[\dfrac{\tilde{L}_i \times \tilde{Q}(A)}{L \times P_{L_i}}\right]$	$z_i = \dfrac{P_Q}{P_L}$
Relative capital allocation,	$v_i = \dfrac{A_i}{A}$	$v_i = Price\left[\dfrac{\tilde{L}_i}{L} \dfrac{\tilde{V}(A)}{P_V}\right]$	$v_i = \dfrac{1}{A \times P_V} \times \left(Price\left[\tilde{L}_i \tilde{V}(A)\right] - \dfrac{P_{L_i} \times P_Q}{P_L}\right)$
Premium-to-liability ratio	$\dfrac{P_i}{P_{L_i}}$	$1 - z_i$	$1 - \dfrac{P_Q}{P_L}$
Marginal contribution to default value	$d_i = \dfrac{r_i}{P_{L_i}} \times P_Q$	$Price\left[\dfrac{\tilde{L}_i \times \tilde{Q}(A)}{L \times P_{L_i}}\right]$	$\dfrac{P_Q}{P_L}$
Line surplus	$S_i = v_i A - P_{L_i}$	$S_i = r_i A - P_{L_i}$	$S_i = v_i A - P_{L_i}$
Line surplus-to-liability ratio	$s_i = \dfrac{v_i}{P_{L_i}} \times A - 1$	$s_i = \dfrac{r_i}{P_{L_i}} \times A - 1$	$s_i = \dfrac{v_i}{P_{L_i}} \times A - 1$

Table 3: Relationship between variables under *ex post* and *ex ante* sharing rules, $\delta = 0$.

23

State, j	State price, π_j	$\tilde{L}_1$	$\tilde{L}_2$
1	0.5	0	0
2	0.25	10	10
3	0.25	50	30
Price of risk		15	10

Table 4: *Example with two risks and* ex post *sharing rule. State prices and losses.*

State, j	$Payment_1$	$Payment_2$
1	0	0
2	10	10
3	25	15
Price of insurance	8.75	6.25
Value of default option	6.25	3.75
Asset allocation	25	15
Surplus allocation	10	5

Table 5: *Payments under ex post sharing rule. Assets, $A = 40$.*

rule implies that the total assets need to increase by

$$v_2 \times \Delta q \times A = 0.375 \times 10\% \times 40 = 1.5.$$

The total assets then become $A = 41.5$, and $A_1 = 25$, $A_2 = 15 + 1.5 = 16.5$ and the expected loss of risk 2 is $P_{L_2} = 1.1 \times 10 = 11$. The new surplus allocation is therefore $S_1 = 25 - 15 = 10$, $S_2 = 16.5 - 11 = 5.5$.

Why is this the right capital allocation rule when the ex post sharing rule is used? Because it does not change the price or risk-structure of the payout to old insurees, and it prices the new risk correctly. In short, there are no value transfers between insurers when the new insurance is sold. With the new risk insured, the total losses are as shown in Table 6 and the payments are as shown in Table 7.

The calculations of the payments are identical to the previous ones, using the ex post sharing rule. For example, for $Payment_2$ when $j = 3$, we have $16.5 = 33/(50 + 33) \times 41.5$. We see in Table 7 that, in all states of the world, the payments to risk 1 are

State, j	State price, π_j	$\tilde{L}_1$	$\tilde{L}_2$
1	0.5	0	0
2	0.25	10	11
3	0.25	50	33

Table 6: *Losses when risk 2's exposure has increased by 10%.*

State, j	$Payment_1$	$Payment_2$
1	0	0
2	10	11
3	25	16.5

Table 7: *Payments when risk 2's exposure has increased by 10%, and $A = 41.5$.*

identical, and that the payments to risk 2 have increased by exactly 10%. Thus, the price per unit risk is the same as before, and 1.5 is the "correct" amount to increase assets with under the ex post sharing rule.

It is easy to check that this is the only choice of v_1 (and thereby of $v_2 = 1 - v_1$) that has this property under the ex post sharing rule. Specifically, any other choice of v_i's will lead to a price change for the already insured risks. Therefore, value would be transfered between new and old insurees when the company scales up one insurance line. For example, the rule proposed in Myers and Read (2001) leads to redistribution under the *ex post* sharing rule (although it does not lead to redistribution under the *ex ante* rule, which is assumed in their paper). Their proposed allocation is such that the option value of default ($P_Q = 10$) is shared such that $r_1 = P_{L_1}/P_L = 15/25 = 0.6$, $r_2 = P_{L_2}/P_L = 10/25 = 0.4$, which by the relationship

$$r_i P_Q = Price\left[(\tilde{L}_i - A_i)\tilde{V}(A)\right],$$

(see equation (41)) leads to $A_1 = 26$ and $A_2 = 14$, and thereby to $v_1 = 0.65$, $v_2 = 0.35$. Therefore, if insurance 2 is scaled up by 10%, total assets should increase by

$$v_2 \times \Delta q \times A = 0.35 \times 10\% \times 40 = 1.4,$$

25

so capital is now $A = 41.4$. The option value of default increases with $r_2 \times \Delta q \times P_Q = 0.4 \times 10\% \times 10 = 0.4$ to $P_Q = 10.4$.

The payments in different states of the world will be as in Table 8, using identical calculations as in Table 7, but with $A = 41.4$. However, this implies that the insurance

State, j	$Payment_1$	$Payment_2$
1	0	0
2	10	11
3	24.94	16.46

Table 8: *Payments when risk 2's exposure has increased by 10%, and $A = 41.4$.*

against $\tilde{L}_1$ has become less worth by the new investments in line 2, as has the value of insurance against $\tilde{L}_2$ risk for the original insurees. There has thus been a value transfer from the old insurees to the new ones.[14] The choice of sharing rule thus determines the r_i's and thereby the v_i's uniquely.

6   Concluding remarks

This paper developed a model of the insurance market under the assumptions of limited liability, perfect competition and costly capital. We focus on the determination of insurance premiums, as well as capital and cost allocations across the insurance lines.

The premium setting and capital allocations are based on the no-arbitrage, option-based, technique, first developed in the papers by Phillips, Cummins, and Allen (1998) and Myers and Read (2001). These papers, however, apply an ex ante rule for allocating the shortfall in claim paying capacity when the insurer defaults, which has the undesirable features that (i) it is based on the unobservable initially expected loss and that (ii) it may require that policyholders with small expected claims make payments to the policyholders facing large expected claims.

Instead, this paper develops the solution when the shortfall created by an insurer default is shared among the claimants under an ex post, pro rata, rule based on the actual realized claims. As summarized in Table 1, the implications of such a rule —

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[14]If the original insurees knew that such a transfer might take place, they would not pay the premium in the first place, since the correct price for the insurance that included transfer risk would be different.

which is common in practice — are quite different compared with what is implied by an ex ante sharing rule. Specifically, premium-to-liability ratios will vary across line in a nontrivial way, and capital will be allocated according to the lines' share of a digital default option on company default.

Appendix

Capital and cost allocation

To study the capital and cost allocation questions, we look at general portfolios of risks, $\tilde{L}_q = \sum_i q_i \tilde{L}_i$, with capital allocation $A_q = \sum_i q_i A_i$. For the time being, we focus on the case in which $\delta = 0$. Moreover, to avoid degenerate solutions, we restrict our attention to cases in which $A > 0$, and there is a positive probability for default, i.e., we assume that $\tilde{L}_q > A_q$ in at least one state of the world.

So far, we have studied the special case when $q_1 = q_2 = \cdots = q_N = 1$. In the general case the q's can take on other values. We assume, however, that $q_i > 0$ for all i, so that the number of lines do not change.

The division of A_q into $\sum_i q_i A_i$ is interpreted as there being a rule that for every unit of $\tilde{L}_i$ risk, A_i units of capital are obtained. We also define $v_i \stackrel{\text{def}}{=} \frac{A_i}{A_q}$.

The general total payment rule is

$$Total\ Payment_q = \tilde{L}_q - \max(\tilde{L}_q - A_q, 0) = \tilde{L}_q - \tilde{Q}_q, \tag{29}$$

where

$$\tilde{Q}_q \stackrel{\text{def}}{=} \max(\tilde{L}_q - A_q, 0) = \max\left(\sum_i q_i(\tilde{L}_i - A_i), 0\right). \tag{30}$$

This formula generalizes (2). If we apply the pricing operator to both the r.h.s. and l.h.s. of (29), we get

$$P_q = P_{L_q} - P_{Q_q}, \tag{31}$$

with the obvious definition of P_q, P_{L_q} and P_{Q_q}. The special case of P, P_L and P_Q in the main text arises when $q_1 = q_2 = \cdots = q_N = 1$. Obviously, for specific values of the q_i's, the payment rule does not depend on the specific A_i's, but only on the total capital, A_q. The A_i's, however, are needed for us to understand how the total capital will *change* when q_i changes.

We are especially interested in how a marginal change in q around $q_1 = q_2 = \cdots = q_N = 1$ will change the value of the insurance contracts. From (30), we have

$$\frac{\partial \tilde{Q}_q}{\partial q_i} = (\tilde{L}_i - A_i) I_{\{\tilde{L}_q - A_q > 0\}}. \tag{32}$$

Here, I is the indicator function on the set X, $I_X(x) = 1$ if $x \in X$ and $I_X(x) = 0$ otherwise. Equation (32) shows how the payout of the option to default changes if the exposure to $\tilde{L}_i$ risk is increased by dq_i and the capital at the same time is increased to by $A_i dq_i$.

Clearly,

$$\sum_i q_i \frac{\partial \tilde{Q}_q}{\partial q_i} = \sum_i q_i(\tilde{L}_i - A_i) I_{\{\tilde{L}_q - A_q > 0\}} = \max\left(\sum_i q_i(\tilde{L}_i - A_i), 0\right) = \tilde{Q}_q,$$

and by applying the pricing operator to both the l.h.s. and r.h.s., we get

$$\sum_i q_i \frac{\partial P_{Q_q}}{\partial q_i} = P_{Q_q},$$

which, at $q_1 = q_2 = \cdots = q_N = 1$ reduces to

$$\sum_i \frac{\partial P_Q}{\partial q_i} = P_Q, \tag{33}$$

Equation (33) is the "summing up" rule of Myers and Read (2001). It states that, given a capital allocation rule — defined by $v_1, \ldots, v_N$, such that $\sum_i v_i = 1$ — the sum of the marginal changes in the value of the option to default, by marginal changes of line size in each line, equals the option value. The rule therefore suggests that a capital allocation rule is consistent with a decomposition of the value of the option to default, at the margin.

The summing up rule is valid regardless of choice of v_i's (and thereby A_i's), and it does not depend on which sharing rule $\mathcal{F}$ is used. Given that the insurer, when marginally scaling up size of line i, wishes to keep the same payout for the new and old insurees in line i, the v_i's are uniquely defined. We call this the *same-sharing assumption*. The general form of (8) when the q's vary is

$$Payment_i = q_i \tilde{L}_i - F_i(q_1 \tilde{L}_1, \ldots, q_N \tilde{L}_N | A, q_1, \ldots, q_N), \tag{34}$$

and the generalization of (9) is

$$\sum_i F_i(q_1 \tilde{L}_1, \ldots, q_N \tilde{L}_N | A, q_1, \ldots, q_N) = \tilde{Q}(A), \tag{35}$$

for all q's, in all states of the world.

We first show that the same sharing assumption implies a unique form for v_i. Assume that, at $q_1 = q_2 = \cdots = q_N$, insurees have paid

$$P_{L_i} - r_i P_Q \tag{36}$$

for their insurances. Now, from (31), if dq_i of new insurance is sold in line i, to new insurees, these insurees will pay

$$\frac{\partial P_q}{\partial q_i} dq_i = \frac{\partial P_{L_q}}{\partial q_i} dq_i - \frac{\partial P_{Q_q}}{\partial q_i} dq_i = dq_i \times \left(P_{L_i} - \frac{\partial P_{Q_q}}{\partial q_i} \right). \tag{37}$$

Now, given that the new and old insurees in line i have the same rule, it is clear that the old insurees will collect:

$$\tilde{L}_i - \frac{F_i\left(\tilde{L}_1, \ldots, \tilde{L}_i(1 + dq_i), \ldots, \tilde{L}_N | A(1 + v_i dq_i), q_1, \ldots, q_i + dq_i, \ldots, q_N \right)}{1 + dq_i}, \tag{38}$$

29

whereas new insurees will collect

$$dq_i \times \left(\tilde{L}_i - \frac{F_i\left(\tilde{L}_1, \ldots, \tilde{L}_i(1 + dq_i), \ldots, \tilde{L}_N \mid A(1 + v_i dq_i), q_1, \ldots, q_i + dq_i, \ldots, q_N \right)}{1 + dq_i} \right). \tag{39}$$

Since the new insurees through (38,39) get the same compensation, per unit of risk insured, noarbitrage implies that they pay the same, which, via (36,37), is only the case if the v_i's are chosen so that $r_i P_Q = \frac{\partial P_{Q_q}}{\partial q_i}$, i.e.,

$$r_i = \frac{\partial P_{Q_q}/\partial q_i}{P_Q}. \tag{40}$$

If (40) is not satisfied, there must be a redistribution of wealth between new and old insurees. For example, if $r_i P_Q > \frac{\partial P_{Q_q}}{\partial q_i}$, then there is a redistribution from old to new insurees. We make a *no-redistribution* assumption, and our result can be formulated as

Implication 6 *Under the same-sharing and no-redistribution assumptions, the v_i's in the capital allocation rule must be defined such that (40) holds.*

Implication 6 provides the precise formulation of the first part Implication 4 in the main text. Going forward, we assume that the same-sharing and no-redistribution assumptions hold.

We use (32) to rewrite (40) as

$$r_i P_Q = Price\left[(\tilde{L}_i - v_i A)\tilde{V}(A) \right]. \tag{41}$$

Here, $\tilde{V}(A) \stackrel{\text{def}}{=} I_{\{\tilde{L}-A>0\}}$, can be interpreted to be a credit default contract, which pays one dollar if the insurer defaults. The price of such a contract is P_V, and since we assume that the insurer defaults in some states of the world, $P_V > 0$.

Given an r_i, there is a unique v_i, such that (41) is satisfied. This follows from the derivative of the r.h.s. of (41) with respect to v_i being equal to $-A \times P_V$, which is constant and strictly less than zero. Thus, there is a bijection between r_i's and v_i's. Moreover,

$$
\begin{aligned}
P_Q &= \sum_i r_i P_Q = Price\left(\sum_i (\tilde{L}_i - v_i A)\tilde{V}(A) \right) \\
&= Price\left(\left(\tilde{L} - \left(\sum_i v_i \right) A \right) \tilde{V}(A) \right) \\
&= P_Q + \left(1 - \sum_i v_i \right) \times A \times P_V,
\end{aligned}
$$

and since $P_V > 0$, this implies that $\sum_i r_i = 1$ if and only if $\sum_i v_i = 1$, so there is also a bijection between the $\{r\}_i$'s and $\{v\}_i$'s in this restricted subspace. The v_i's, in turn, immediately provides us

30

with
$$A_i = v_i A.$$

From (41), it follows that, in general,

$$v_i = \frac{1}{A \times P_V} \times \left[Price\left(\tilde{L}_i \tilde{V}(A) \right) - r_i P_Q \right]. \tag{42}$$

For general sharing rules, $\mathcal{F}$, there may be externalities across lines if the size of line is scaled up, since the risk structure of the whole company changes when the mix of risks change. However, it is always possible for lines to compensate each other, such that the price is fair, as long as v_i is chosen according to (42). We show this as follows: The change in payment in line i for an increase in scale of line i at $q_1 = \ldots = q_N = 1$, is:

$$\frac{\partial \, Payment_i}{\partial q_i} = \tilde{L}_i - \frac{\partial F_i(\tilde{L}_1, \ldots, q_i \tilde{L}_i, \ldots, \tilde{L}_N | A(1 + (q_i - 1)v_i), 1, \ldots, q_i, \ldots, 1)}{\partial q_i}.$$

Moreover, the change in line j of such a change in line i is

$$\frac{\partial \, Payment_j}{\partial q_i} = -\frac{\partial F_j(\tilde{L}_1, \ldots, q_i \tilde{L}_i, \ldots, \tilde{L}_N | A(1 + (q_i - 1)v_i), 1, \ldots, q_i, \ldots, 1)}{\partial q_i}.$$

Thus, the total change in payment, via (35) is

$$\begin{aligned}
\sum_j \frac{\partial \, Payment_j}{\partial q_i} &= \tilde{L}_i - \sum_j \frac{\partial F_j(\tilde{L}_1, \ldots, q_i \tilde{L}_i, \ldots, \tilde{L}_N | A(1 + (q_i - 1)v_i), 1, \ldots, q_i, \ldots, 1)}{\partial q_i} \\
&= \tilde{L}_i - \frac{\partial \tilde{Q}_q(A(1 + (q_i - 1)v_i))}{\partial q_i},
\end{aligned}$$

and the price of this change in total payment is then, from (32,42),

$$\begin{aligned}
P_{L_i} - \frac{\partial P_{Q_q}}{\partial q_i} &= P_{L_i} - Price\left[(\tilde{L}_i - v_i A)\tilde{V}(A) \right] \\
&= P_{L_i} - Price\left[\tilde{L}_i \tilde{V}(A) \right] + v_i A \times Price\left[\tilde{V}(A) \right] \\
&= P_{L_i} - Price\left[\tilde{L}_i \tilde{V}(A) \right] + \frac{1}{AP_V} \times \left(Price\left[\tilde{L}_i \tilde{V}(A) \right] - r_i P_Q \right) \times AP_V \\
&= P_{L_i} - r_i P_Q.
\end{aligned}$$

This is exactly the price paid by the new insuree for the insurance. The net contribution of the insuree is thus exactly what is needed for a transfer to take place such that all insurees are fairly compensated for any change in risk that occurs when line i is scaled up. This is the *transfer rule* implied by $\mathcal{F}$. We

31

note that the transfer can be incorporated into the definition $\mathcal{F}$, so that no transfer is needed when the up-scaling occurs.

The rule proposed in Myers and Read (2001) is to choose the v_i's, such that the *marginal contribution to default value* of line i, d_i, is the same for all i (see Myers and Read (2001), pages 554 and 559). In our notation

$$d_i \overset{\text{def}}{=} \frac{1}{P_{L_i}} \frac{\partial P_Q}{\partial q_i} = P_Q \frac{r_i}{P_{L_i}},$$

Since $\sum_i r_i = 1$, the only way of getting the $d_i = d_j$ for all i and j is to have $r_i = \frac{P_{L_i}}{P_L}$. This choice of r_i's is consistent with the *ex ante* sharing rule. From (42), this implies that under the *ex ante* sharing rule,

$$v_i = \frac{1}{A \times P_V} \times \left(Price \left[\tilde{L}_i \tilde{V}(A) \right] - \frac{P_{L_i} \times P_Q}{P_L} \right).$$

On the contrary, under the *ex post* sharing rule, it follows from (13), that

$$v_i = \frac{Price \left[\frac{\tilde{L}_i}{\tilde{L}} \tilde{V}(A) \right]}{P_V}.$$

The formula for v_i, thus, has the same form as the formula for r_i, but with $\tilde{V}(A)$ replacing $\tilde{Q}(A)$. It is also easy to check that the transfer rule between lines in case of up-scaling in line i, under the *ex post* sharing rule is that line j, $1 \leq j \leq N$, receives a transfer of

$$A \left(\frac{Price \left[\frac{\tilde{L}_i}{\tilde{L}} \tilde{V}(A) \right] \times Price \left[\frac{\tilde{L}_j}{\tilde{L}} \tilde{V}(A) \right]}{P_V} - Price \left[\frac{\tilde{L}_i \tilde{L}_j}{\tilde{L}^2} \tilde{V}(A) \right] \right) dq_i, \tag{43}$$

when risk in line i increases with dq_i.

For the special case in which default occurs in only one state of the world and the *ex post* sharing rule is used, $r_i = v_i$ for all i. This follows from the following argument: If we assume that the realization of $\tilde{L}_i$ is $\tilde{L}_i = L_i^*$ in the state in which the insurer defaults and define $L^* = \sum_i L_i^*$, we have

$$
\begin{aligned}
v_i &= \frac{Price \left[\frac{\tilde{L}_i}{\tilde{L}} \tilde{V}(A) \right]}{P_V} = \frac{\frac{L_i^*}{L^*} \times P_V}{P_V} = \frac{L_i^*}{L^*} \\
&= \frac{\frac{L_i^*}{L^*} \times P_Q}{P_Q} = \frac{\frac{L_i^*}{L^*} \times P_V(L^* - A)}{P_Q} = \frac{Price \left[\frac{\tilde{L}_i}{\tilde{L}} \tilde{Q}(A) \right]}{P_Q} = r_i,
\end{aligned}
$$

so v_i is indeed equal to r_i in this case. This show the second part of Implication 4 in the main text is valid.

Also, in the special case of default in only one state of the world, (43), is 0 for all i, and thus no transfers between lines are needed when the scale increases in one line. This is, e.g., the case in the example in the main text.

In our terminology, the *surplus-to-liability* ratio of line i, s_i, — defined in Myers and Read (2001) — satisfies $A_i = (1 + s_i)P_{L_i}$, so

$$s_i \stackrel{\text{def}}{=} \frac{A_i}{P_{L_i}} - 1.$$

The total surplus is $S = A - P_L$, and the surplus allocated to line i is

$$S_i = s_i P_{L_i}.$$

We stress that the interpretation of the relationship between the $\{v_i\}$'s and the $\{r_i\}$'s is that the $\{v_i\}$'s tell us how much the insurer must increase assets, if there is a marginal increase of insurance in one line. If the insurer currently sells insurance $\tilde{L}_i$, and then increases risk exposure in line i to $\tilde{L}_i(1 + \Delta q)$, assets will change from A to $A(1 + v_i \Delta q)$. With such a change, the value of the option to default increases (to a first order approximation) by $P_Q r_i \times \Delta q$, if the insurer wishes to keep the same price and premium-to-liability ratio in new insurance within line i. Any other choice will lead to a redistribution between old and new insurees and will therefore be inconsistent with noarbitrage, as shown in the redistribution example in the main text.

For the case when $\delta > 0$, our whole analysis goes through, with the difference that the insurees now have to pay an additional amount, δA in total, for insurance. At the margin, if the insurer scales up insurance in line i, the extra amount $v_i \delta A$ has to be contributed by the new insurees. As argued in Myers and Read (2001), and supported by our previous analysis, it is therefore natural to allocate the costs $v_i \delta A$ to line i. This rule then leads to the competitive price

$$P_{L_i} - r_i P_Q + v_i \delta A,$$

for insurance in line i. The price has a *marginal cost pricing* interpretation: $C = P_L - P_Q + \delta A$ is the total cost of producing the "goods" (insurances), and

$$\frac{\partial C}{\partial q_i} = P_{L_i} - r_i P_Q + v_i \delta A,$$

is the marginal cost of producing more "goods" in line i, which under marginal cost pricing equals the price charged for line i insurance.

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