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by

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ABSTRACT

A low-cost dominant firm will drive all competitive fringe firms out of the market if all firms have rational expectations; however, the dominant firm will not predate (price below marginal cost). Since a dominant firm will not drive out fringe firms if they have myopic expectations, it may be in the dominant firm's best interests to inform the fringe. The effects of governmental intervention on the optimal path and welfare are presented.

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1. Introduction

A low-cost dominant firm with no capacity constraint that can precommit to a price path is in an excellent position to drive constant average cost fringe firms from the market. Surprisingly, much of the existing literature on dynamic models [e.g., Gaskins (1970, 1971)] concludes that the dominant firm eventually limit prices with a positive number of fringe firms left in the market. This result in a dynamic model is especially disturbing since in a static model (the extreme case of a dynamic model where adjustment is instantaneous) all fringe firms will be driven from the market. We show that these earilier results are due to an unreasonable assumption about expectation formations.

Gaskins and others have assumed that fringe firms use myopic (adaptive) expectations when determining whether to enter or exit. We show that, when all of Gaskins' other assumptions are maintained, but firms use rational expectations, the dynamic model's steady state resembles that of the static model: All fringe firms are driven out by the time the dominant firm limit prices.

There are two basic strands of the literature on dynamic models of dominant firms. In one strand, the rate of entry of fringe firms depends on the current price.

Examples of this literature include Gaskins (1970, 1971) and Baron (1973). In Gaskins' model, if there are initially few fringe firms, a low-cost dominant firm without a capacity constraint initially sets a high price which the dominant firm gradually lowers until it reaches the limit price so that a constant, positive number of fringe firms persist forever. Since price never falls below the limit price, at no time does exit occur. This result is startling when one realizes that, in a static model (where entry is instantaneous), such a dominant firm would drive out the fringe and be a monopolist. Gaskins' (1970, 1971) results stem from his assumption that, while the dominant firm has rational expectations, the fringe firms use myopic expectations in their entry decisions: They only look at instantaneous profits.

Another strand of the literature uses a two-period and/or two-firm model. Examples of this type of model include Kamien and Schwartz (1971) and De Bondt (1976). In these models, period one is the interval until the arrival of one or more entrants (which is typically uncertain); period two is the remainder of time. These two-period models by their very nature cannot be used to study the complex pattern of entry and exit by fringe firms which we investigate here. Indeed, in Kamien and Schwartz (1971), the price in the preentry period is a constant.

The reason that the dynamic models of Gaskins and his followers come to a counterintuitive result is that they assume that fringe firms form expectations that are not rational. Three papers consider similar problems with rational expectations. Flaherty (1980) assumes that firms are rational, but there is a single entrant with a cost functional identical to that of the existing firm. Judd and Petersen (1986) assume that fringe firms retain earnings in order to invest, and the dominant firm and the fringe play a Nash open-loop game. The fringe firms are rational and take the dominant firm's price as given, while the dominant firm takes the fringe firms' percent of retained earnings as given. Karp (1987) also has fringe firms determine optimal investment, but uses a feedback model. Thus, these three papers change a number of assumptions in addition to that of rationality. To isolate the effects of expectations alone, we maintain Gaskins' other assumptions including the fringe's linear entry equation and the ability of the dominant firm to commit to a price path.

This new assumption leads to qualitatively and quantitatively different results from those of previous papers. We begin by presenting a general dynamic model of a dominant firm with competitive fringe firms entering as a function of expected profits. Next, we present our rational expectations (perfect foresight) model.

The dominant firm uses a dynamic, open-loop strategy to maximize its present value of profits, where it chooses its price path at time zero. That is, the dominant firm cannot change its plans after they have been made, although fringe firms may enter or exit the industry at any time.[1]

Next, Gaskins' (1970, 1971) model is shown to be a special case of the general model where expectations are adaptive, and we compare his results to ours. We show that, even if fringe firms used adaptive expectations initially, it may be in the dominant firm's best interest to announce its long-run policy so that the fringe firms' expectations become rational. That is, if the fringe's expectation formation is endogenously determined, nonrational behavior may not persist.

We next consider a more realistic policy where the government restricts the dominant firm's market share. We conclude the paper with a summary and a discussion of the policy implications.

2. A general dynamic model

The dominant firm chose a price path over time, $p(t)$, to maximize the functional

$$V = \int_0^\infty [p(t) - c]\,[f(p) - x(t)]e^{-rt}\,dt, \tag{1}$$

where V is the present value of the dominant firm's profit stream, c is the average cost of production for the dominant firm (constant over time), f(p) is the market demand curve, x(t) is the level of sales by the competitive fringe (where each fringe firm produces one unit of output), r is all firms' discount rate, and [f(p) - x(t)] is the residual demand facing the dominant firm.

This functional is maximized subject to the evolution of the fringe, $\dot{x}(t)$. The initial number of fringe firms is given:

$$x(0) = x_0. \tag{2}$$

Fringe firms then enter or exit the industry based on their expectations about the present discounted value of their future profits, y(t). Because the cost of entry depends upon the speed of entry, fringe firms do not enter instantaneously. The rate of entry is proportional to the expected present value of profits:

$$\dot{x}(t) = ky(t) \tag{3}$$

$$y(t) = \int_t^\infty \pi^e(s)\, e^{-r(s-t)}\, ds, \tag{4}$$

where $\pi^e(t)$ are expected profits by a fringe firm at time t, k is a constant response coefficient $\geqq 0$, and y is the expected present value of a fringe firm's profits. We follow the common convention used by Gaskins (1970, 1971) and others that k is a constant.[2] Because the number of fringe firms cannot be negative, the dominant firm also faces a state constraint,

$$x(t) \geqq 0. \tag{5}$$

3. Rational fringe firms

At least where the demand curve is linear, the solution to the dominant firm's profit maximization problem is to choose a price path that drives out all rival firms in finite time and then keeps them out by limit pricing thereafter. If fringe firms use rational expectations (perfect foresight), then expected profits, $\pi^e(t)$, equal actual profits, $\pi(t) = [p(t) - \bar{p}]$ for all t, where p(t) is the actual path of prices and $\bar{p}(> c)$ is the limit price (the average cost of a fringe firm). As a result, we can rewrite (4) as

$$y(t) = \int_t^\infty [p(s) - \bar{p}]\ e^{-r(s-t)}\ ds. \tag{6}$$

Before we can state the dominant firm's problem as one solvable by the maximum principle, we must reduce equation (6) to a form that does not include an integral by differentiating it with respect to time:

$$\dot{y} = ry + \bar{p} - p. \tag{7}$$

The dominant firm which faces rational fringe firms chooses a price path at time t = 0 so as to maximize (1) subject to (2), (3), (5), and (7). The necessary conditions for a solution to the problem of maximizing subject to a state constraint, (5), are given by Jacobson, Lele, and Speyer (1971).

The method of solution begins by forming the usual Hamiltonian,

$$H = (p - c)\ (f - x)\ e^{-rt} + zky + v(ry + \bar{p} - p), \tag{8}$$

where z and v are the costate variables corresponding to the state variables x and y and adjoining a multiplier, (w), and the state constraint $[x(t) \geq 0]$,

$$L = H + wx. \tag{9}$$

As with the usual Lagrangian methods, $wx = 0$ is necessary for optimality. The rest of the necessary conditions are (a) the equations of motion, (2), (3), and (7); (b) the adjoint equations, $\dot{z} = -L_x$ and $\dot{v} = -L_y$; (c) the transversality conditions, $\lim_{t=\infty} zx = \lim_{t=0} zx = \lim_{t=\infty} vy = \lim_{t=0} vy = 0$; (d) the maximum principle; and (e) the necessary conditions at the "jump time," τ (which, as shown below, is finite), at which x becomes zero. For our problem, there are three jump-time conditions, which we state here, but discuss in more detail below.

First, because y is not constrained, v is continuous

$$v(\tau^-) = v(\tau^+), \tag{10}$$

where $v(t^-)$ is shorthand for $\lim_{\substack{t \to \tau \\ t<\tau}} v(t)$.

Second, because x is constrained, its multiplier jumps:

$$z(\tau^-) = z(\tau^+) + \phi \tag{11}$$

where $\phi > 0$.

Lastly, the Hamiltonian is continuous at (τ):

$$H(\tau^-) = H(\tau^+). \tag{12}$$

Our construction of an optimal solution proceeds by (1) describing the solution when there are competitive firms (the "interior" solution); (2) describing the solution after these fringe firms are driven from the market (the "corner" solution); and, finally, (3) piecing together the two types of solutions.

3.1 Interior solution

Since H and L are identical when $x > 0$, the usual Hamiltonian methods suffice to construct an interior solution. The necessary conditions include (2), (3), (7), the adjoint equations,

$$\dot{z} = (p - c)\, e^{-rt}, \tag{13}$$

$$\dot{v} = -zk - vr, \tag{14}$$

and the maximum principle which implies

$$H_p = [(f - x) + (p - c)\, f']\, e^{-rt} - v = 0. \tag{15}$$

By appropriate substitutions, the necessary conditions can be reduced to a single second-order differential equation. First, we solve (15) for v and differentiate with respect to time to obtain

$$\dot{v} = -rv + e^{-rt}[2\, f'\dot{p} + (p - c)\, f''\dot{p} - k y]. \tag{16}$$

We then equate equations (14) and (16) to eliminate $\dot{v}$,

$$\dot{p} = \frac{k y - zke^{rt}}{[2f' + (p - c)\, f'']}, \tag{17}$$

and differentiate with respect to time to obtain

$$\dot{z} = -rz - \frac{e^{-rt}}{k}\{(3f'' + [p - c]\, f''')\, (\dot{p})^2 + (2\, f' + [p - c]\, f'')\, \ddot{p} - k\dot{y}\}. \tag{18}$$

Substituting into (18) for $\dot{z}$ from (13), for z from (17), and for $\dot{y}$ from (7), we find that

$$\frac{1}{k}\,[2f' + (p - c)\, f'']\,\ddot{p} + \frac{1}{k}\,[3f'' + (p - c)\, f''']\,(\dot{p})^2$$
$$- \frac{r}{k}\,[2f' + (p - c)\, f'']\,\dot{p} + 2p - c - \overline{p} = 0. \tag{19}$$

Where demand is linear, $f(p) = a - bp$, equation (18) is a second-order, ordinary differential equation,

$$\ddot{p} - r\,\dot{p} - \frac{k}{b}\,p + \frac{k(\overline{p} + c)}{2b} = 0, \tag{20}$$

whose general solution is

$$p = \alpha_1\, e^{\gamma_1 t} + \alpha_2\, e^{\gamma_2 t} + \frac{\overline{p} + c}{2}, \tag{21}$$

where

$$\gamma_1 = \frac{1}{2}\left[r - \sqrt{r^2 + 4k/b}\right]$$

$$\gamma_2 = \frac{1}{2}\left[r + \sqrt{r^2 + 4\,k/b}\right].$$

Equation (21) gives the solution to the dynamic limit-price problem when there are fringe firms in the market. It depends upon two unknown parameters, α_1 and α_2. These parameters are determined from the conditions joining interior and corner solutions, where $x(t) = 0$.

We can show that the corner is reached in finite time since internal solutions that last forever and meet the transversality conditions eventually have strictly negative values of x. Not all values of α_1 and α_2 are compatible with the transversality conditions. If α_2 (the constant associated with the positive root) is positive, price will grow without bound. Since $\dot{z} = (p - c)\, e^{-rt}$, z will also grow without bound. Equations (3) and (6) guarantee that x will grow without bound, but this violates the transversality condition that

$\lim_{t\to\infty} zx = 0$. A similar argument can be made for $\alpha_2 < 0$. Therefore, interior trajectories that last forever must have $\alpha_2 = 0$.

By substituting for p from (21) in (6), integrating to obtain y, substituting for y in (3), and then integrating again with respect to t, one can calculate x:

$$x(t) = x_0 + \alpha_1 b - \alpha_1 b e^{\gamma_1 t} + \frac{k(c - \overline{p})}{2r} t. \tag{22}$$

Because $\gamma_1 < 0$, eventually the last term on the right-hand side dominates; and, since $c - \overline{p} < 0$, $\lim_{t\to\infty} x < 0$. Thus, an optimal policy starting with $x > 0$ will eventually drive x to zero at some time, τ. As a result, interior trajectories that last forever are impossible. Hence, α_2 may be nonzero. Since the constraint is eventually reached, we now consider corner solutions.

3.2 Corner solution

When x is zero and remains zero for an open time interval, y will equal 0 and p will equal $\overline{p}$. As a result, once x becomes zero on an open interval, it will remain zero forever.

The proof of the proposition that, if $x = 0$ on an open interval, $p = \overline{p}$, follows from continuity. Let x be zero from τ_1 to τ_2 (since x is continuous, it is zero at τ_1 and τ_2 as well). Since for $\tau_1 \leq t \leq \tau_2$,

$$x(t) = x(\tau_1) + \int_{\tau_1}^{t} ky \, ds, \tag{23}$$

y is certainly zero. Similarly,

$$y(t) = y(\tau_1) + \int_{\tau_1}^{t} e^{-rs}(p - \overline{p}) \, ds, \tag{24}$$

so $y(t) = 0$ implies that $p(t) = \overline{p}$, $\tau_1 \leq t \leq \tau_2$.

The final claim that once a corner solution is reached it will continue indefinitely follows from the principle of optimality. If it is optimal to set $p(t) = \bar{p}$ when $x(t) = y(t) = 0$ at $t = \tau_1$, it will be optimal to set $p = \bar{p}$ at any other time when $x(t) = y(t) = 0$. Since $x(t) = y(t) = 0$ for a corner solution and that implies $p(t) = \bar{p}$, it follows that $x(t + \varepsilon) = y(t + \varepsilon) = 0$ for small ε, so the corner solution will last indefinitely. Thus, an optimal program that begins with competitive fringe firms will consist of one interior segment of finite length followed by a corner segment of infinite length, where $p(t) = \bar{p}$ and $x(t) = 0$.

3.3 Linking the interior and corner solutions

Matching the interior to the corner solution determines α_1, α_2, and τ. The definition of x, the continuity of the Hamiltonian, and the transversality condition at time zero give three equations to determine α_1, α_2, and τ. This section derives each in turn.

Substituting for p from equation (21) in the definition of $\dot{x}$ in (2) and integrating between 0 and τ gives our first condition:

$$\begin{aligned} x(\tau) = x_0 &+ \alpha_1 b\left(1 - e^{\gamma_1\tau}\right) + \frac{\alpha_1 k}{r\gamma_2}\left(e^{-\gamma_2\tau} - e^{\gamma_1\tau}\right) \\ &+ \alpha_2 b\left(1 - e^{\gamma_2\tau}\right) + \frac{\alpha_2 k}{r\gamma_1}\left(e^{-\gamma_1\tau} - e^{\gamma_2\tau}\right) \\ &+ \frac{k(\bar{p} - c)}{2r^2}\left(1 - e^{-r\tau} - r\tau\right) = 0, \end{aligned} \tag{25}$$

where the second equality follows because $x(\tau) = 0$ by the definition of τ.

The continuity of the Hamiltonian implies that $p(\tau) = \bar{p}$. Since H for given x, y, z, and v has a unique maximum in p, H is said to be regular.

Therefore, p is continuous at τ and $p(\tau) = \bar{p}$ [Jacobson, Lele, and Speyer (1971), p. 272]. This result also can be shown directly using (12). This second condition, $p(\tau) = \bar{p}$, may be written as

$$\alpha_1 e^{\gamma_1 \tau} + \alpha_2 e^{\gamma_2 \tau} + \frac{c + \bar{p}}{2} = \bar{p}. \tag{26}$$

The last condition comes from noting that $v(0) = 0$ because $y(0)$ is free and $v(0)\ y(0)$ must equal zero by a transversality condition. Since $H_p = 0$ by the maximum principle, $a - bp(0) - x_0 - bp(0) + bc = 0$. (Notice that $p(0)$ is the short-run profit maximization price.) Substituting for $p(0)$ from (21) and rearranging gives

$$\alpha_1 + \alpha_2 = \frac{a - x_0 - b\,\bar{p}}{2b}. \tag{27}$$

Equations (25), (26), and (27) determine α_1, α_2, and τ.

3.4 <u>The price path</u>

The price path depends on the parameters of the system. Heuristically, if x_0 is relatively small (given the other parameters), the price starts high, falls below $\bar{p}$, and then rises to $\bar{p}$. Alternatively, if x_0 is relatively large, the price starts below $\bar{p}$ and then rises to $\bar{p}$.

In either case, as $t \to \tau$, $p(t)$ approaches $\bar{p}$ from below. The price must be below $\bar{p}$ for the dominant firm to drive the fringe firms out of the market. After all fringe firms are driven out, price must remain at $\bar{p}$ or new entry would occur.

In the interior, the price path is described by equation (21). The path depends on α_1 and α_2. From equation (27) and the associated argument,

$\alpha_1 + \alpha_2 > 0$. Since price must approach $\bar{p}$ from below at t near τ, $\dot{p}(t) = \alpha_1 \gamma_1 e^{\gamma_1 t} + \alpha_2 \gamma_2 e^{\gamma_2 t} \geq 0$. It follows that $\alpha_2 \leq 0$ is impossible. Since α_1 and α_2 cannot be negative, were $\alpha_2 \leq 0$, α_1 would have to be positive; but that would imply that $\alpha_1 \gamma_1$ and $\alpha_2 \gamma_2$ would both be negative which violates $\dot{p} \geq 0$.

Thus, $\alpha_2 > 0$, $\alpha_1 + \alpha_2 > 0$, and $\alpha_1 \lesseqgtr 0$. There are two possible price paths as shown in fig. 1. As the figure shows, the dominant firm never predates in the sense that its price is always greater than c--its marginal and average cost. As the figure shows, in all cases price is above $(\bar{p} + c)/2$. Since $\bar{p} > c$, $(\bar{p} + c)/2 > c$.

4. A comparison of our model and Gaskins' model

In contrast to our model where fringe firms have rational expectations, Gaskins (1970, 1971) implicitly assumes that firms form their expectations myopically: Profits tomorrow will be the same as today. His entry condition is given as:

$$\dot{x}(t) = \kappa[p(t) - \bar{p}]. \tag{28}$$

Equation (28) says that the rate of entry of fringe firms is a constant, κ, times a fringe firm's instantaneous profits.

Equation (28) can be derived from (3) and (4) if we assume that the potential entrants form their expectations myopically: $\pi^e(s) = [p(t) - \bar{p}]$, $s \geq t$, so

$$y(t) = \frac{[p(t) - \bar{p}]}{r}. \tag{29}$$

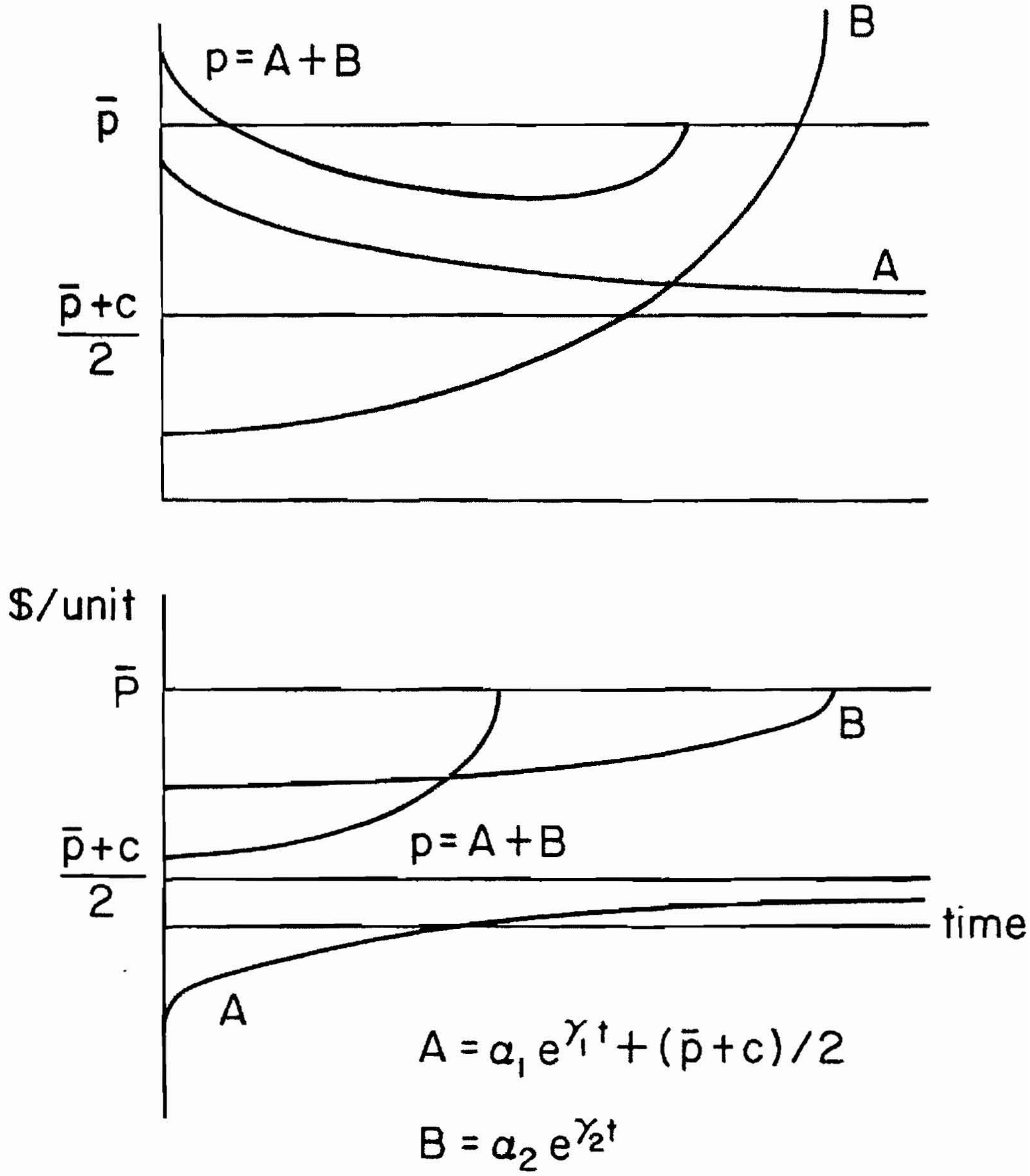

FIGURE I. Properties of the Price Path

Substituting for y from (29) into (3), we obtain Gaskins' (1970, 1971) equation (28), where $\kappa = k/r$.

Using the maximum principle, Gaskins (1970, 1971) derives the solution consisting of the differential equations in $\dot{x}(t)$, as given by (28) above, and

$$\dot{p}(t) = \frac{k(\overline{p} - c) + r[x - f(p) - (p - c)\, f'(p)]}{-2f'(p) - (p - c)\, f''(p)}. \tag{30}$$

Equations (28) and (30) generate a family of trajectories in the (p, x) plane.

The intersection of the $\dot{p} = 0$ and $\dot{x} = 0$ equations in the (p, x) plane determines the saddlepoint. Gaskins (1970, 1971) describes how to derive the unique trajectory meeting all of the necessary conditions.

The optimal paths of our model in (p, x) space and the path in Gaskins' (1970, 1971) model are shown in fig. 2 for a case in which x_0 is relatively small.[3] Notice that the path in Gaskins' (1970, 1971) model starts at a high price and falls to $\overline{p}$ where it remains, so a large finite number of fringe firms (50) produce in the limit. By contrast, the path in our model shows the price starting at a high price, falling below $\overline{p}$, and then rising to $\overline{p}$ where it remains and $x(t) = 0$, $t \geq \tau$. Indeed, in this example, our model implies there are always fewer than 21 fringe firms. The myopic path approaches within 2 percent of the steady-state number of fringe firms (49 firms) in 106 time periods. At 112 time periods, the rational path reaches the maximum number of firms. The rational path hits the corner in 298 time periods.

5. Endogenously determined expectations

In many cases, even if the fringe firms have myopic expectations, it is in the dominant firm's best interests to announce its price path so that the fringe firms' expectations become rational. For example, given the parameters

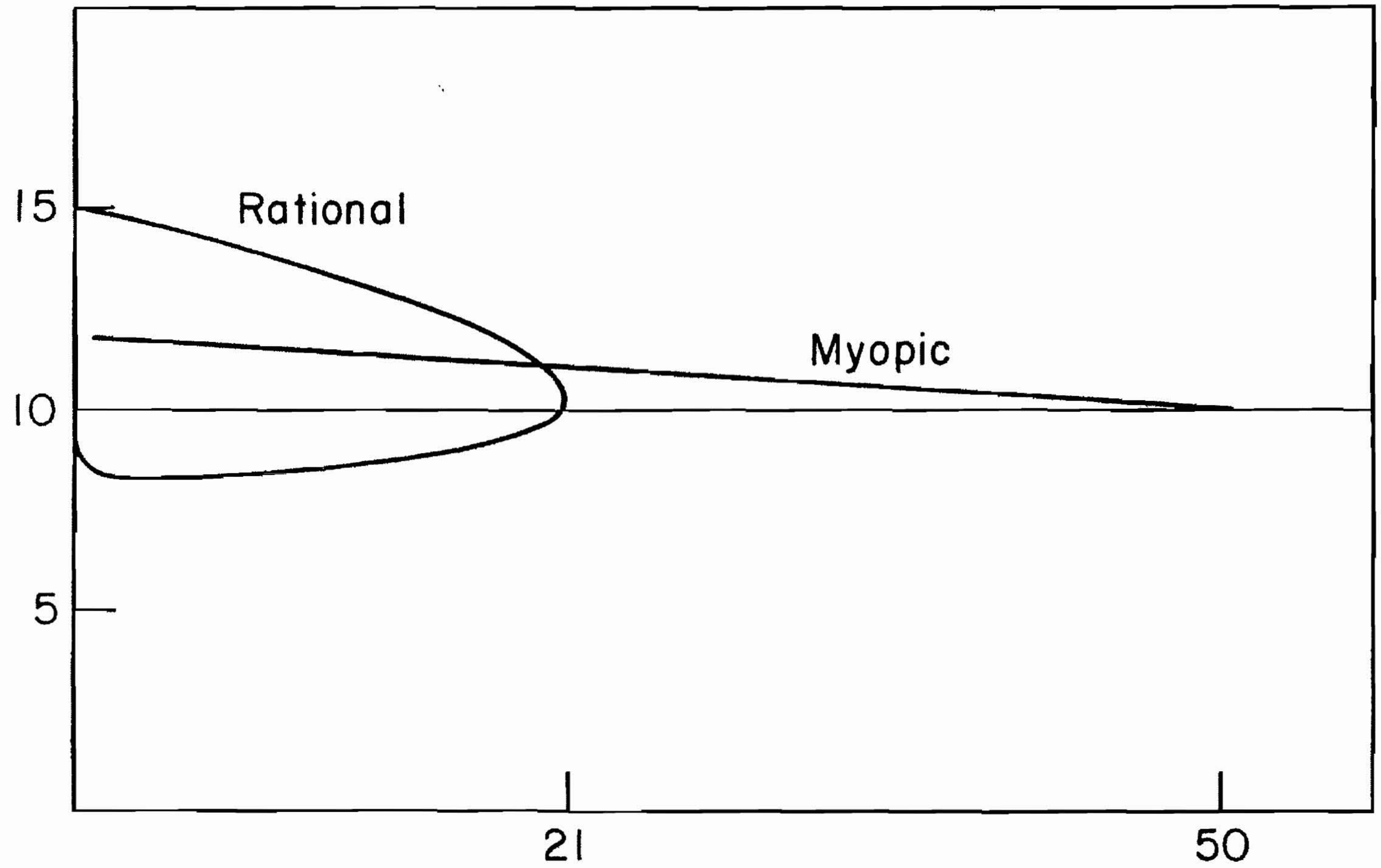

FIGURE 2. Comparison of Rational and Myopic Price Paths

used in fig. 1, when the fringe has myopic expectations, the present value of its profits is $141 and the present value of the dominant firm's profits is $7,354. In contrast, if the fringe's expectations are rational, the corresponding present values are $212 and $9,472. In other words it is in all firms' collective best interest for the fringe to use rational expectations.

In this example if the fringe becomes rational, consumer surplus would fall from $8,870 to $5,715. Thus, total welfare (here defined as consumer surplus plus combined profits) falls from $16,366 to $15,398. That is, the firms' gain does not offset consumers' losses.

The models are too complex for us to derive general conditions when it is in the dominant firm's best interest to correct fringe firm's myopic beliefs. We have found it difficult to find examples where it is not in the dominant firm's best interest to reveal its price path to the fringe; however, the welfare effect can go in either direction.[4]

6. Market share constraint

The possibility of antitrust action may partially explain why one does not observe dominant firms driving their competition completely out of business. Although neither statute nor case law sets absolute market shares that will support an antitrust conviction--a survey of recent cases shows findings of monopoly power were common above a 70 percent share and uncommon below a 50 percent share [Flynn (1981, p. 51)]--a dominant firm may form strongly held views as to the crucial share at which they would face prosecution. For example, some years ago, General Motors apparently felt that 60 percent was the relevant number. In this section we examine the optimal actions of a dominant firm subject to a constraint on its market share. The section begins with a formal statement of the problem as a control constrained optimization problem, which is fundamentally different from the state constrained problem of the earlier sections. Four theorems and three lemmas, all of which are technical in nature, narrow the set of possible optimal paths. Fig. 3 and the last paragraph of this section summarize these results.

The dominant firm's market share is defined as its sales divided by total sales:

$$\delta \equiv \frac{a - bp - x}{a - bp} . \tag{31}$$

Given the market share and the number of competitors, one can invert the share formula to express price as a function of x and δ:

$$p(x, \delta) = \frac{a(1 - \delta) - x}{b(1 - \delta)} . \tag{32}$$

Letting $\delta^* > 0$ be the maximum share that the dominant firm can attain without facing an antitrust suit, the control problem for that firm is to maximize the present discounted value of net revenues (profits):

$$\max_{\delta<\delta^*} \int R \, dt \tag{33}$$

subject to

$$\dot{x} = ky, \ \dot{y} = ry + \overline{p} - p(x, \delta), \ x(0) = x_0,$$

where

$$R = \{[a - bp(x, \delta) - x] \, [p(x, \delta) - c]\}e^{-rt}.$$

Since x is certainly bounded away from zero by δ being bounded away from ne, this problem is unlike the original problem. It involves only a constraint on the control, δ, and not a constraint on the state variable, x.

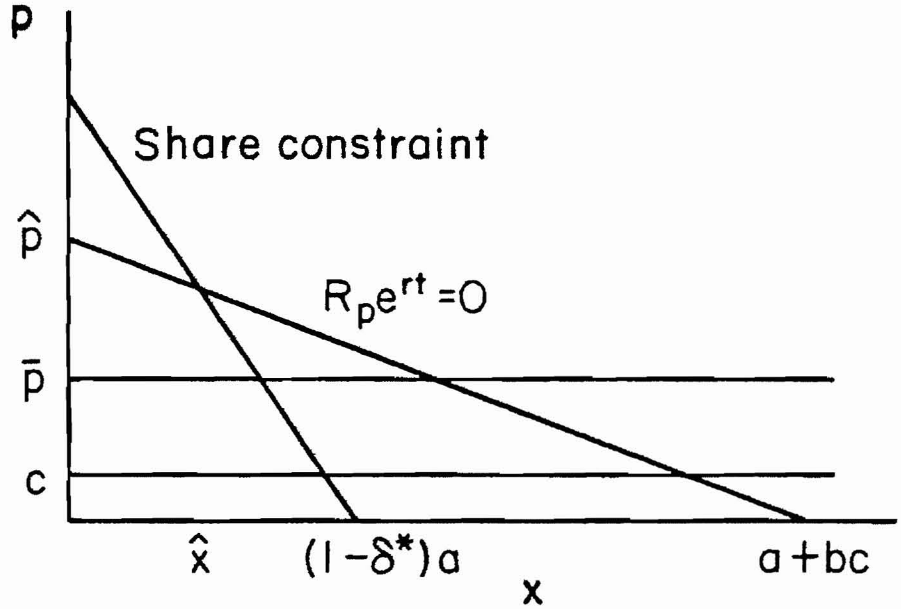

FIGURE 3. Phase Diagram

The Hamiltonian for this problem is:

$$H = R + zky + v(ry + \overline{p} - p). \tag{34}$$

The maximum principle yields (where subscripts denote partial derivatives):

$$R_p\, p_\delta - vp_\delta = 0 \text{ and } \delta < \delta^*$$

or (35)

$$R_p\, p_\delta - v\, p_\delta < 0 \text{ and } \delta = \delta^*.$$

Since $p_\delta \neq 0$ [see equation (32)], in the interior the maximum principle gives the same expression as before. The costate equations are

$$\dot{z} = -R_p p_x - R_x + v\, p_x \tag{36}$$

$$\dot{v} = -zk - rv. \tag{37}$$

These equations hold for all time. During those times for which δ is less than δ^* [$R_p = v$ from equation (25)], the equation for $\dot{z}$, (36), simplifies to give

$$\dot{z} = - R_x \tag{36'}$$

which is the same equation as for an interior solution in the original problem which does not have a share constraint.

The solution to this problem consists of two parts. First, in the interior solution, $\delta < \delta^*$, the maximum principle and costate equations are exactly those of an unconstrained interval for the original problem. Thus, the equations determining the price path in the interior solution are the same as in the original problem.

Second, when the share constraint binds ($\delta = \delta^*$), we rewrite the constraint as

$$x = (1 - \delta^*)(a - bp) \tag{38}$$

and take its time derivative and set it equal to ky [using (3)]:

$$\dot{x} = (\delta^* - 1)\, b\dot{p} = ky. \tag{39}$$

Differentiating (39) with respect to time and substituting $ry + \bar{p} - p$ for $\dot{y}$ using (7) and $[(\delta^* - 1)\, b\dot{p}]/k$ for y using (39), we obtain the second-order, ordinary differential equation

$$\ddot{p} - r\dot{p} + \frac{k(p - \bar{p})}{(\delta^* - 1)\, b} = 0. \tag{40}$$

The solution to this differential equation is

$$p(t) = \theta_1 e^{\mu_1 t} + \theta_2 e^{\mu_2 t} + \bar{p}, \tag{41}$$

where

$$\mu_1 = \frac{1}{2}\left(r - \sqrt{r^2 - 4k/(\delta^* b - b)} \right)$$

$$\mu_2 = \frac{1}{2}\left(r + \sqrt{r^2 - 4k/(\delta^* b - b)} \right).$$

There are many possibilities for an optimal path. Where $\delta < \delta^*$, optimal paths begin on the $R_p e^{rt} = 0$ line (the short-run, profit-maximizing condition) as interior segments. Eventually they reach the share constraint and equilibrium at $\bar{p}$, but there are many possible routes. We use a set of lemmas to restrict the set of possibilities.

Fig. 3 shows the relation of $\bar{p}$, $R_p = 0$, and the share constraint in $x - p$ space so that paths may be drawn in it as projections from (x, p, y) space into (x, p) space. The figure includes four lines--(A) the constraint (38); (B) a fringe firm's costs, p; (C) $R_p = 0$ which is $x = a - 2pb + bc$; and (D) the dominant firm's cost, c. The intersection of the constraint and $R_p = 0$ is labeled $(\hat{x}, \hat{p})$. Let (x, p)* denote the projection of an optimal path.

For small enough x, the optimal path follows the constraint:

Lemma 1: If a path is optimal and $x(t) < \hat{x}$, then (x(t), p(t))* lies on the share constraint.

Proof: Those points below the share constraint (see fig. 3) are not feasible. Thus, the optimal path must lie on or above the constraint. For $x(t) < \hat{x}$, points above the share constraint are also above the $R_p e^{rt} = 0$ line. Above the $R_p e^{rt} = 0$ line, increases in price decrease instantaneous profit. Since increases in price also encourage entry which decreases future profits, prices above the share constraint lead to less instantaneous and future revenues and cannot be optimal.

By the same reasoning:

Lemma 2: The interior portion of an optimal path does not lie above the $R_p e^{rt} = 0$ line at even a single point.

The next two lemmas and the theorem describe the general direction that interior optimal paths may follow. They show that the direction of travel is away from the $R_p e^{rt} = 0$ line.

Lemma 3: If, along an optimal interior arc above c, there is a time t for which d/dt $(R_p e^{rt})$ is negative, then d/dt $(R_p e^{rt})$ remains negative so long as (x, p)* remains interior and above c.

<u>Proof</u>: Since $(x, p)^*$ lies in the interior, from the maximum principle,

$$R_p = v. \tag{35'}$$

Multiplying both sides of equation (35') by e^{rt} and differentiating with respect to time gives

$$\frac{d}{dt} (R_p e^{rt}) = (\dot{v} + rv) e^{rt}.$$

The costate equation (37) requires

$$(\dot{v} + rv) = -zk.$$

Given the hypothesis of the lemma that $d/dt (R_p e^{rt})$ is negative, these last two equations show that z is positive. The costate equation for z is $\dot{z} = (p - c) e^{-rt}$, so z is increasing along $(x, p)^*$. Since z is positive and increasing, it must be that $d/dt (R_p e^{rt})$ is negative and decreasing which establishes the lemma.

A geometric interpretation of Lemma 3 uses a vector normal to $R_p = 0$: $N = (1, 2b)$ is the normal pointing in direction of decreasing R_p. The tangent vector to $(x, p)^*$, $(\dot{x}, \dot{p})^*$ points in the same half space as N--or $|(\dot{x}, \dot{p})^*, N| > 0$--when $d/dt (R_p e^{rt})$ is negative.

A final lemma tells us the direction of motion in the x plane:

<u>Lemma 4</u>: When $p > \bar{p}$ and $\dot{x} < 0$, then $\dot{x}$ remains negative as long as $(x, p)^*$ remains above $\bar{p}$. When $p < \bar{p}$ and $\dot{x} > 0$, then $\dot{x}$ remains positive as long as $(x, p)^*$ remains below $\bar{p}$.

<u>Proof</u>: Since $\dot{x} = ky$, if $\dot{x} < 0$, then $y < 0$. Further, since $\dot{y} = ry + \bar{p} - p$ if p is also greater than $\bar{p}$, $\dot{y}$ is negative. Thus, above $\bar{p}$, $\dot{x}$ must become more negative which establishes the first half of the lemma. The second assertion is established by the same steps with the signs reversed.

Corollary: An optimal path that begins with $x < \hat{x}$ travels along the share constraint toward $(\hat{x}, \hat{p})$.

Proof: Lemma 1 shows that the path travels along the share constraint. Lemma 4 shows that, if the optimal path starts moving away from $\hat{x}$ toward lower x, $\dot{x}$ must remain negative forever and p must remain above $\overline{p}$ forever. Such a path is impossible since, if p is always above $\overline{p}$, then y must be positive and hence $\dot{x}$ must be positive.

Theorem 1: Let $(x, p)^*$ be an optimal path, and let $(\dot{x}, \dot{p})^*$ be its tangent. When $\overline{p}$ is above p, $|(\dot{x}, \dot{p})^*, N| < 0$.

Proof: The proof proceeds by contradiction: Assume that at some time $t > 0$, $p > \overline{p}$, and $|(\dot{x}, \dot{p})^*, N| > 0$. Three steps are needed to establish the contradiction. First, the optimal path may not remain above $\overline{p}$. Second, if the optimal path passes below $\overline{p}$, it will recross the $\overline{p}$ line; when it recrosses, the optimal path will be closer to the $R_p\, e^{rt} = 0$ line than when it first crossed. Third, since there is no limit point on the $R_p\, e^{rt} = 0$ line and optimal paths cannot cross $R_p\, e^{rt} = 0$, an optimal path cannot become ever closer to $R_p\, e^{rt} = 0$ which is the contradiction that establishes the theorem.

It is impossible for an optimal interior path to remain within the triangle formed by the $R_p\, e^{rt} = 0$ line, the $p = \overline{p}$ line, and the constraint. Lemma 3 shows that once $(x, p)^*$ moves in the same half space as N it must move in the same half space as N as long as it remains in the interior. Therefore, it must eventually come to the boundary of the triangle. Lemma 2 establishes that an optimal path does not lie above the $R_p\, e^{rt} = 0$ line, so the optimal path must eventually hit one of the other two boundaries.

The optimal path, $(x, p)^*$, could join the constraint, but it would still have to move in the same direction as N which would mean that it either becomes interior again in the same direction as N or crosses $R_p e^{rt} = 0$ which is impossible. Fig. 4 shows $(x, p)^*$ joining the constraint while moving in the same half space as N. The vector N is the outward normal to $R_p e^{rt} = 0$, and it is drawn beginning at the point B, the point where the path $(x, p)^*$ begins traveling along the constraint. The diagram also shows a line through B parallel to $R_p e^{rt} = 0$ that is orthogonal to N. By hypothesis, the tangent to $(x, p)^*$ has positive inner product with N so it must lie between N and the line parallel to $R_p e^{rt} = 0$, and it cannot cross the constraint. If the optimal path travels along the constraint, $\dot{x} < 0$.

Using Lemma 4, when $\dot{x} < 0$ for p above $\bar{p}$, $\dot{x}$ cannot become positive. Since $\dot{x}$ cannot change signs, the direction of travel while on the constraint cannot reserve. When $\dot{x} < 0$ and $(\dot{x}, \dot{p})^*$ is on the constraint, then $|(\dot{x}, \dot{p})^*, N| > 0$. By continuity, if $(x, p)^*$ again becomes interior, $|(\dot{x}, \dot{p})^*, N|$ will still be positive. Thus, paths that move in the same direction as N and are above $\bar{p}$ must move in that half space and may not exit the triangle through either the constraint or the $R_p e^{rt} = 0$ line.

The only remaining possibility is that such a path eventually has $p < \bar{p}$. Again looking at fig. 4, the only way for a path to have $|(\dot{x}, \dot{p}), N| > 0$ and $|(\dot{x}, \dot{p}), (0, -1)| > 0$ (so the path crosses the $p = \bar{p}$ line) is for $\dot{x} > 0$.

Since $(x, p)^*$ must drop below $\bar{p}$ with $\dot{x} > 0$, all that remains is to describe the behavior of the path below $\bar{p}$. Lemma 4 shows that $\dot{x}$ cannot change signs while the path remains below $\bar{p}$. Since $\dot{x} > 0$ and firms only enter when present value of profits are positive $(y > 0)$, there must be some time when instantaneous profits are again positive so that $(x, p)^*$ must again cross $\bar{p}$.

The value of p is the same $(\bar{p})$ at both points where the optimal path crosses the $\bar{p}$ line, while the value of x is larger at the second point (since

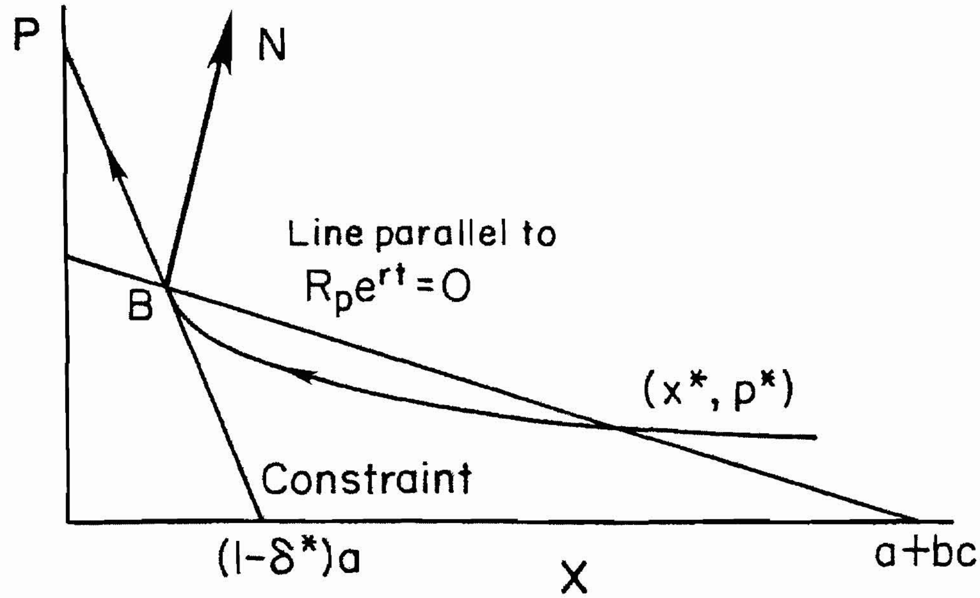

FIGURE 4. An Optimal Path Joining the Constraint

$\dot{x}$ was positive at all times in between) so, at the second point, $R_p e^{rt}$ must be smaller. Thus, $R_p e^{rt}$ must decrease regardless of whether or not the path is above or below $\bar{p}$. Since an optimal path cannot cross the $R_p e^{rt} = 0$ line, and there is no limit point on that line, it is not possible for $(x, p)^*$ to be constantly moving in the same half space as N, which establishes the theorem.

<u>Corollary</u>: When $(x, p)^*$ is optimal and $p < \bar{p}$, then $\dot{x} < 0$.

<u>Proof</u>: By contradiction, assume $\dot{x} > 0$. By the argument of the previous theorem, p must eventually exceed $\bar{p}$; and at that instant $\dot{p} > 0$. Since the direction of increasing x is the direction of decreasing p along the constraint--when p crosses $\bar{p}$--$(x, p)^*$ must be interior. An interior path that has increasing p and increasing x must have $|(\dot{x}, \dot{p})^*, N| > 0$ which is impossible by the theorem.

<u>Theorem 2</u>: An optimal path does not cross $\bar{p}$ from below.

<u>Proof</u>: The proof proceeds by contradiction: Assume that $(x, p)^*$ is optimal and crosses the $p = \bar{p}$ line from below. Lemma 4 and the corrollary show that along a path crossing $\bar{p}$ from below, $\dot{x} < 0$. Since $\dot{x} < 0$ at the time of crossing, there is some later time during which the fringe firms suffer losses. This implies that $(x, p)^*$ must recross $\bar{p}$. Since the optimal p is the sum of two real exponential functions with real coefficients, $\dot{p}$ can change signs, at most, once. Recrossing $\bar{p}$ requires a change of sign so, after the recrossing, $\dot{p} < 0$. Given decreasing p and x, $(x, p)^*$ must intersect the constraint. The direction of decreasing p on the constraint, however, is the direction of increasing x, so $(\dot{x}, \dot{p})^*$ could not be continuous at the point it meets the constraint which is a contradiction.

<u>Theorem 3</u>: Once $(x, p)^*$ lies on the constraint below $\bar{p}$, it will continue on the constraint until it reaches $\bar{p}$ where it will stop.

Proof: The proof proceeds by showing that the alternative--exiting to an interior arc--requires the tangent to the interior arc to point in a direction that is not "high" enough to leave the constraint. Along an interior arc, since $v = R_p$,

$$ve^{rt} = (a - 2bp + bc - x). \tag{42}$$

Differentiating (42) with respect to time gives

$$(\dot{v} + rv)\, e^{rt} = -2b\dot{p} - \dot{x},$$

which can be solved for $\dot{p}/\dot{x}$:

$$\left.\frac{dp}{dx}\right|_{\text{interior}} = \frac{\dot{p}}{\dot{x}} = \frac{\dfrac{-(\dot{v} + rv)\, e^{rt}}{\dot{x}} - 1}{2b} .$$

Along the constraint,

$$\left.\frac{dp}{dx}\right|_{\text{constraint}} = \frac{-1}{b(1 - \delta)} .$$

For an optimal path to leave the constraint, it must climb above it:

$$\left.\frac{dp}{dx}\right|_{\text{interior}} < \frac{dp}{dx}\,\text{constraint}$$

or

$$\frac{\dot{v} + rv}{e^{-rt}\,\dot{x}} > \frac{1 + \delta}{1 - \delta} . \tag{43}$$

Since leaving the constraint with $\dot{x} < 0$ means moving in the same direction as N, $(\dot{v} + rv)$ must be negative. Thus, the inequality asserts one positive number is greater than another. Returning to the constraint, however, requires the inequality to be reversed, which cannot happen. Since $(\dot{v} + rv) = -zk$ and z is growing along the path, the left-hand side numerator of equation (43) is growing in absolute value. The corresponding denominator, $e^{-rt}\dot{x}$, is shrinking in absolute value because

$$\frac{d}{dt}(e^{-rt}\dot{x}) = \frac{d}{dt}\left[\kappa\int_t^\infty e^{-rs}(p - \bar{p})\,ds\right] = \kappa e^{-rt}(\bar{p} - p) > 0,$$

and $\dot{x}$ was initially negative. Since the numerator grows and the denominator shrinks (in absolute value) and neither of the parts changes sign, the inequality in equation (43) can never be reversed, so an interior path beginning on the constraint below $\bar{p}$ can never return to the constraint. Thus, as with all other interior paths traveling in the same half space as N, this path cannot exist. The conclusion is that, once an optimal path joins the constraint below $\bar{p}$, it continues along the constraint.

We are now in a position to describe the optimal path that begins in the interior (not on the constraint). Fig. 3 shows the phase space for this problem. The arrows indicate what is known about the directions of travel. The path begins on the $R_p\, e^{rt} = 0$ line. By theorem (1), it travels in the same direction as -N, as indicated in fig. 3 by the arrow labelled -N. It could intersect the share constraint and travel down it and go from interior to corner solution any number of times. Sooner or later, it comes to rest at the intersection of the share constraint and $\bar{p}$ or it passes below $\bar{p}$. Once below $\bar{p}$, the corollary to theorem (1) shows that x must be decreasing, though the direction for p is unknown, as shown in fig. 3 by the arrow below $\bar{p}$. By theorem (2), the path cannot cross the $\bar{p}$ line from below. Thus, the only remaining posssibility is that of theorem (3): The path will join the constraint and move up it till it stops at the equilibrium.

7. Conclusions

When the fringe's expectations are rational and Gaskins' other assumptions are maintained, a low-cost dominant firm will eventually drive the fringe out of the industry. In doing so, the dominant firm will not predate (price below marginal cost).

In most, if not all, cases, it is in the dominant firm's best interest to reveal its intentions to the fringe--that is, the dominant firm makes it expectations rational.

It is generally not socially optimal nor in consumers' best interests for the government to set a minimum number of fringe firms. An antitrust policy which constrains the dominant firm's market share leads to a price path which is often qualitatively similar to the unconstrained path: falling and then rising. In both the constrained and unconstrained cases, the dominant firm limits prices and drives the fringe out or to the lowest market share allowable.

FOOTNOTES

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[1]See Flaherty (1980) for a defense of open-loop rather than closed-loop models in which firms incur adjustment costs and choose output rates.

[2]This Gaskins' (1970, 1971) assumption, which we maintain, is not innocuous. The entry condition makes sense where the cost of producing the necessary capital is quadratic in the rate of capital production (for now, the marginal cost of making capital is set equal to the present value of the profits that capital produces); but this equation also makes the decision to scrap capital stock a, the mirror image of the investment process. The model also does not allow temporary shutdowns.

[3]In this example, $x_0 = 1$, $\bar{p} = 10$, $c = 5$, $k = 0.01$, $r = 0.1$, $a = 10$, $b = 250$. As a result, in the rational expectations model, $\alpha_1 = 7.45$, $\alpha_2 = 1.46873 \times 10^{-14}$, $\tau = 298.2061$, $p(0) = 14.95$, and $y(0) = 43.2479$. Gaskins' (1970) myopic expectations model coefficients (see his papers) are $\theta = 50$ and $\lambda = -0.0366025$.

[4]For example, if $x_0 = 1$, $p = 10$, $c = 3$, $k = 0.01$, $r = 0.1$, $a = 250$, and $b = 10$, then switching to rational expectations raises the dominant firm's profits by \$3,806, raises the fringe's profits by \$119, and lowers consumer surplus by \$1,335 so total welfare rises by \$2,589.

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