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# On the Consistency of the Lucas Pricing Formula

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## Abstract

In order to find the real market value of an asset in an exchange economy, one would typically apply the formula appearing in Lucas (1978), developed in a discrete time framework. This theory has also been extended to continuous time models, in which case the same pricing formula has been universally applied.

While the discrete time theory is rather transparent, there has been some confusion regarding the continuous time analogue. In particular, the continuous time pricing formula must contain a certain type of a square covariance term that does not readily follow from the discrete time formulation. As a result, this term has sometimes been missing in situations where it should have been included.

In this paper we reformulate the discrete time theory in such a way that this covariance term does not come as a mystery in the continuous time version. It is shown, e.g., that this term is of importance also in the equivalent martingale measure approach to pricing.

In most real life situations dividends are paid out in lump sums, not in rates. This leads to a discontinuous model, and adding a continuous

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time framework, it appears that our framework is a most natural one in finance.

*KEYWORDS: Exchange economy, state price deflator, discrete time, continuous time, equivalent martingale measure, the Gordon growth model*

# 1 Introduction

## 1.1 The two time scales

In the Lucas (1978) exchange economy with one consumption good, the real market value  $S_t$  of a security at time  $t$  is given by the formula

$$S_t = \frac{1}{\pi_t} E_t \left\{ \sum_{s=t+1}^T \pi_s \delta_s \right\}, \quad (1)$$

where  $E_t$  provides the conditional expected value given the information  $\mathcal{F}_t$  at time  $t$ ,  $\pi$  is a state price deflator, also known as a pricing kernel or a marginal rate of substitution process, a strictly positive process, and  $\delta$  is the security's dividend process, measured in units of the consumption good.

Here  $S_t$  is the price of the security, ex dividend, at time  $t$ . That is, at each time  $t$  the security pays its dividend and is then available for trade at price  $S_t$ . The cum dividend price at time  $t$  is  $S_t + \delta_t$ .

In an equilibrium setting with a representative agent having the time additive utility representation given by

$$U(c) = E \left\{ \sum_{t=0}^T u(c_t, t) \right\}, \quad (2)$$

the state price deflator takes the form  $\pi_t = u'(c_t, t)$  under certain regularity conditions, where  $u'(\cdot, t)$  is the marginal utility index of the representative agent at time  $t$ , and  $c$  is interpreted as aggregate consumption in the market.

The formula (1) is equivalent to the condition of no arbitrage possibilities in the market. Consider the gains process  $G_t = S_t + \sum_{s=1}^t \delta_s$ , the price plus accumulated dividends. The deflated gains process  $G_t^\pi$  is defined as  $G_t^\pi = S_t \pi_t + \sum_{s=1}^t \delta_s \pi_s$ . Then there is no arbitrage if and only if there is a deflator  $\pi$  with the property that the deflated gains process is a martingale. Add the assumption that  $S_T = 0$  at the horizon  $T$ , and formula (1) follows directly.

The fact that the state price deflator  $\pi_t$  has the form  $\pi_t = u'(c_t, t)$  when  $U$  is given by (2), follows from the first order condition of agent optimality,

noticing that the directional derivative of  $U$  at the "point"  $c$  and in the "direction"  $\delta$  has the form  $\nabla U(c; \delta) = E\{\sum_{t=0}^T u'(c_t)\delta_t\}$  in this case. See Ross (1978) for the concept of state price in one period models, Duffie and Schaefer (1985) for the finite dimensional case in discrete time, and Harrison and Kreps (1979) for the general case.

Moving to continuous time, the pricing formula corresponding to (1) has been shown to have the form (see e.g., Aase (2002))

$$S_t = \frac{1}{\pi_t} E_t \left\{ \int_t^T (\pi_{s-} dD_s + d[\pi, D]_s) \right\}. \quad (3)$$

Here  $D_t$  is the accumulated dividends of the security by time  $t$ , and the term  $[\pi, D]$  is called the realized quadratic covariance between dividends and the deflator. Suppose  $\pi$  and  $D$  are semimartingales, the most general continuous time models for which integrals can be constructed in an intuitive manner, and consider a time grid  $0 = s_0 < s_1 < \dots < s_n = T$ , which is refined as  $n$  increases. Then

$$[\pi, D]_T = \lim_{\Delta s_{k-1} \rightarrow 0} \sum_{s_k \leq T} (\pi_{s_k} - \pi_{s_{k-1}})(D_{s_k} - D_{s_{k-1}}), \quad (4)$$

where  $\Delta s_{k-1} := (s_k - s_{k-1})$  and the convergence is in probability uniformly on  $[0, T]$ . Sometimes the notation  $\langle \pi, D \rangle_s^t = E_t[\pi, D]_s$  for  $t \leq s$  is used, in which case the formula (3) can be written

$$S_t = \frac{1}{\pi_t} \left( E_t \left\{ \int_t^T \pi_{s-} dD_s \right\} + \int_t^T d\langle \pi, D \rangle_s^t \right). \quad (5)$$

In neither of its forms can this quadratic covariance term be directly compared to an ordinary covariance, but comes fairly close in some cases. If, for example,  $\pi$  and  $D$  are both Itô-diffusions driven by the same Brownian motion, then  $d[\pi, D]_t = \sigma_\pi(t)\sigma_D(t)dt$ . In this case  $\frac{d}{ds}\text{cov}_t(\pi_s, D_s)|_{s=t^+} = \sigma_\pi(t) \cdot \sigma_D(t)$ , i.e., we have an instantaneous covariance interpretation at time  $t$ . There is a similar simple relationship for jump-diffusions driven by Poisson random measure and Brownian motion (see Section 6). The term may disappear under independence, but this assumption is certainly not enough, since it is easy to find even deterministic processes that jump at the same time points and have a nonzero quadratic covariance term.

The formula (3) is true for  $\pi$  and  $D$  appropriately integrable semimartingales, which includes both continuous Itô-processes, pure jump processes and the more general jump-diffusions. In Aase (2002) a proof of (3) was presented in the case of Itô-processes using the economic argument of numeraire invariance. We demonstrate in Section 4 that the same argument also works for more general semimartingales, containing, e.g., jumps.

In situations where this extra term has a covariance like flavor, it can be given an economic interpretation, as pointed out in Aase (2002). Suppose we consider the equilibrium setting with a decreasing marginal utility index. If the term  $[\pi, D]_s > 0$  for some  $s$  in the time interval  $(t, T]$  with positive probability, this means that the state price is positively correlated with the dividends of the security under consideration, which ought to have a positive effect on the market price of the security, as compared to the situation where this term is zero or negative. Such a security would simply possess the fortunate property paying out more dividends on the average, in units of consumption, in states where the consumption  $c$  tends to be low - in other words, in states where these dividends will be relatively valuable - a property of an asset that must be reflected in its market price.

## 1.2 Are the two formulas mutually consistent?

In various treatments the following continuous time analogue of (1) often appears

$$S_t = \frac{1}{\pi_t} E_t \left\{ \int_t^T \pi_{s-} dD_s \right\} \quad (6)$$

or close variants of this, but without the quadratic covariance term of (3), or (5). The formula (6) appears both in scientific papers published during the last couple of decades, e.g., Duffie and Zame (1989), and also in textbooks, even recent ones, e.g., Dana, R.-A., M. Jeanblanc-Picqué, and H. F. Koch (2003). Nielsen (2004) considers the case where adjusted price processes are Itô-processes, and gives an overview of some of the different uses and misuses of this formula.

In situations where  $\pi$  and  $D$  are independent processes, the formulas (3) and (6) may coincide, but mere independence is certainly not sufficient to get rid of the quadratic covariance term, as observed above. Moreover, in economically interesting situations  $\pi$  and  $D$  are certainly not independent. In continuous models like Itô-diffusions, however, if the aggregated dividends  $D$  are of bounded variation, then the formulas above are equal, since in this case  $d[\pi, D]_t = \sigma_\pi(t)\sigma_D(t)dt$ , and  $\sigma_D(t) = 0$  for all  $t$  if  $D$  is a finite variation process. It should be added that this is a fairly common assumption in the financial economics literature (see e.g., Merton (1973)).

However, in most real life situations dividends are paid out in lump sums, not in rates. This leads to a discontinuous model, and adding a continuous time framework, it follows from the subsequent discussion that our framework is the most natural one in finance.

The quadratic covariance term in (3) does not vanish for pure jump mod-

els, where all the changes in the quantities  $\pi$  and  $D$  take place at common, discrete, random time points  $0 < \tau_1 < \tau_2 < \dots < T$ . This would be a useful framework if there is one common source of risk. Since this situation is very close in spirit to the discrete time model, one may be led to wonder if there is some kind of inconsistency between the two formulas (1) and (3). The main difference between a pure random jump process and a discrete time process is, as far as we are concerned, that the time spacing between events, or observations, is not deterministic for the former model, but is so by construction for the latter.

Thus we ask the question: Can formulas (1) and (3) be reconciled? And if so, can it be done in an intuitive and transparent manner? In the next section we provide answers to these questions.

The paper is organized as follows: In Section 2 we clarify the puzzle mentioned above, in Section 3 we make the passage from discrete to continuous time, in Section 4 we prove the numeraire invariance theorem for semimartingales, in Section 5 we show how the realized quadratic covariance term enters under equivalent martingale measures, in Section 6 we develop the Gordon growth formula for continuous time models, and Section 7 concludes.

## 2 The appropriate informational constraints

In order to take a closer look at formula (1) in the discrete time setting, notice that the profit from time  $(t-1)$  to time  $t$  is  $(S_t + \delta_t - S_{t-1})$  for someone who bought the security at time  $(t-1)$ , after the dividend payment at time  $t$ . Thus the capital gain is  $(S_t - S_{t-1})$  and the dividend is  $\delta_t$  over this period, and the value of the security is given by (1) in the neoclassical world.

Using the notation  $\Delta S_{t-1} := (S_t - S_{t-1})$  and  $\Delta D_{t-1} := (D_t - D_{t-1})$ , note that  $\Delta D_{t-1} = \delta_t$  is the change in dividends over the period from  $(t-1)$  to  $t$ , or the dividends paid by the security at time  $t$ . Using this notation we may rewrite formula (1) as

$$S_t = \frac{1}{\pi_t} E_t \left\{ \sum_{s=t+1}^T \pi_s \Delta D_{s-1} \right\}. \quad (7)$$

To prepare for a continuous time sum, an integral, we must be careful in keeping track of when the various payments are being made. To this end it will be convenient to rewrite the formula (7) as follows:

$$S_t = \frac{1}{\pi_t} E_t \left\{ \sum_{s=t+1}^T (\pi_{s-1} \Delta D_{s-1} + (\pi_s - \pi_{s-1}) \Delta D_{s-1}) \right\}. \quad (8)$$

Here the essential part is that while the sum in (7) is not a candidate for a stochastic integral since the "integrand"  $\pi_s$  dates to the end of the interval over which the dividend  $(D_s - D_{s-1})$  is paid, the sum in (8) is such a candidate, since here the relevant informational constraints are satisfied. This suggests the continuous time analogue

$$S_t = \frac{1}{\pi_t} E_t \left\{ \int_t^T (\pi_{s-} dD_s + d[\pi, D]_s) \right\}, \quad (9)$$

where we have used the device  $\Delta[\pi, D]_t = \Delta\pi_t \Delta D_t$ . This is formula (3). Here it is also essential that the integrand  $\pi_{s-}$  dates to the beginning of the interval  $[s, s + ds)$  where the dividend over this interval  $dD_s$  is yet not known, which gives us the conventional stochastic integral in the limit, with the so called Itô's choice. Contrary to the case of ordinary Lebesgue-Stieltjes integrals, this latter distinction is crucial for *stochastic* integrals.

In other words, by paying attention to the fact that the dividend increment  $\Delta D_{s-1}$  is a *forward* difference, and that the prices are ex dividend, the rewriting of (7) to the form in (8) follows rather natural, in which case the continuous time analogue (9) does not appear as a big surprise.

In the next section we give sufficient conditions for the passage from discrete to continuous time.

### 3 From discrete to continuous time

In this section we indicate how the passage from the formula (1) to the formula (3) may be carried out by considering smaller and smaller time intervals  $\Delta s_{i-1}$ . Consider first any two semimartingales  $X$  and  $Y$  on a filtered probability space  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, P)$  satisfying the usual conditions. The processes are, by convention, right continuous having left hand limits. Consider the predictable version  $X_{t-}$  of the process  $X$ . Let  $\{X_t^n\}$  be a sequence of simple predictable, left continuous processes satisfying

$$\sup_{(t, \omega) \in [0, T] \times \Omega} |X_t^n(\omega) - X_{t-}(\omega)| \rightarrow 0. \quad (10)$$

Then

$$\int_0^T X_t^n dY_t \rightarrow \int_0^T X_{t-} dY_t, \quad (11)$$

where the convergence is in probability. The relations (10) and (11) may be viewed as a rather natural continuity property and is in fact a defining property of the semimartingale  $Y$  (e.g., Protter (2004)). If a continuity

property like this does not hold, the model may produce results that are difficult to interpret.

Now consider the case where  $\pi$  and  $D$  are semimartingales in  $L^2(P)$ , so that (10) and (11) hold for both these processes. Let us call  $\{\pi_{s_k}\}$  and  $\{D_{s_k}\}$ , respectively, the approximating sequences in (10) and assume that their variances are uniformly bounded. Then we have the following:

**Theorem 1**

$$S_0^{(n)} := \frac{1}{\pi_0} E \left\{ \sum_{k=1}^n \pi_{s_k} \Delta D_{s_{k-1}} \right\} \rightarrow$$

$$S_0 := \frac{1}{\pi_0} E \left\{ \int_0^T (\pi_{s-} dD_s + d[\pi, D]_s) \right\}$$

as  $n \rightarrow \infty$ , or  $\Delta s_{k-1} \rightarrow 0$ .

Proof: First we rewrite  $S_0^{(n)}$ , using equation (8) as

$$S_0^{(n)} = \frac{1}{\pi_0} E \left\{ \sum_{s_k \leq T} (\pi_{s_{k-1}} \Delta D_{s_{k-1}} + \Delta[\pi, D]_{s_{k-1}}) \right\}. \quad (12)$$

By our assumption that  $D$  is a semimartingale,

$$\sum_{s_k \leq T} \pi_{s_{k-1}} \Delta D_{s_{k-1}} \rightarrow \int_0^T \pi_{s-} dD_s$$

in probability. Also

$$E \left| \sum_{s_k \leq T} \pi_{s_{k-1}} \Delta D_{s_{k-1}} \right| \leq M_1 < \infty$$

by the Schwartz inequality, for some constant  $M_1$ , since the variances of  $\{\pi_{s_k}\}$  and  $\{D_{s_k}\}$  are uniformly bounded. Thus, by the dominated convergence theorem,

$$E \left( \sum_{s_k \leq T} \pi_{s_{k-1}} \Delta D_{s_{k-1}} \right) \rightarrow E \left( \int_0^T \pi_{s-} dD_s \right),$$

which takes care of the first term. Note now that the following identity holds for discrete sums

$$[\pi, D]_T := \sum_{s_k \leq T} (\pi_{s_k} - \pi_{s_{k-1}})(D_{s_k} - D_{s_{k-1}}) =$$

$$\pi_T D_T - \pi_0 D_0 - \sum_{s_k \leq T} \pi_{s_{k-1}} \Delta D_{s_{k-1}} - \sum_{s_k \leq T} D_{s_{k-1}} \Delta \pi_{s_{k-1}}. \quad (13)$$



As for the square covariance term we rewrite the expression for  $S_0^n$  in (12) using the identity (13):

$$\begin{aligned} S_0^{(n)} &= \frac{1}{\pi_0} E \left\{ \sum_{s_k \leq T} (\pi_{s_{k-1}} \Delta D_{s_{k-1}} + \Delta[\pi, D]_{s_{k-1}}) \right\} = \\ &= \frac{1}{\pi_0} E \left\{ \pi_T D_T - \pi_0 D_0 - \sum_{s_k \leq T} D_{s_{k-1}} \Delta \pi_{s_{k-1}} \right\}. \end{aligned} \quad (14)$$

By our assumption that  $\pi$  is a semimartingale, again

$$\sum_{s_k \leq T} D_{s_{k-1}} \Delta \pi_{s_{k-1}} \rightarrow \int_0^T D_{s-} d\pi_s$$

in probability. As above,

$$E \left| \sum_{s_k \leq T} D_{s_{k-1}} \Delta \pi_{s_{k-1}} \right| \leq M_2 < \infty$$

by the Schwartz inequality, for some constant  $M_2$ , since the variances of  $\{\pi_{s_k}\}$  and  $\{D_{s_k}\}$  are uniformly bounded. Thus, by the dominated convergence theorem, we have that

$$E \left( \sum_{s_k \leq T} D_{s_{k-1}} \Delta \pi_{s_{k-1}} \right) \rightarrow E \left( \int_0^T D_{s-} d\pi_s \right)$$

as well. The result of the theorem now follows since

$$\begin{aligned} S_0^{(n)} &= \frac{1}{\pi_0} E \left\{ \pi_T D_T - \pi_0 D_0 - \sum_{s_k \leq T} D_{s_{k-1}} \Delta \pi_{s_{k-1}} \right\} \rightarrow \\ &= \frac{1}{\pi_0} E \left\{ \pi_T D_T - \pi_0 D_0 - \int_0^T D_{s-} d\pi_s \right\} = \\ &= \frac{1}{\pi_0} E \left\{ \int_0^T (\pi_{s-} dD_s - d[\pi, D]_s) \right\}, \end{aligned}$$

where the latter equality follows from the integration by parts analogue to the formula (13), valid also for semimartingales.  $\square$

Notice from the above proof how stochastic integrals with respect to semimartingales behave just like ordinary sums. This is one reason why this class of processes appear to be a natural one to work with. Norberg and Steffensen (2004) discuss the solution to a certain stochastic differential equation, by finding the analogue solution to a corresponding difference equation in discrete time, demonstrating a similar connection between sums and integrals.

By some standard procedures of conditioning, the theorem can be shown valid also for an arbitrary time  $t \geq 0$ .

## 4 The numeraire invariance theorem

In this section we demonstrate that if there exists a state price deflator  $\pi$ , then prices are given by the formula (3) when price processes, gains processes and accumulated dividend processes are all semimartingales. To this end, consider a market of  $N$  assets having price processes  $S = (S_1, S_2, \dots, S_N)$  with an associated vector of dividends  $D$ , and of gains processes  $G = S + D$ . Maintaining our convention that prices are ex dividend, a portfolio  $\theta = (\theta_1, \theta_2, \dots, \theta_N)$  is *self-financing* if

$$\theta_t(S_t + \Delta D_t) = \theta_0 S_0 + \int_0^t \theta_{s-} dG_s = \theta_0 S_0 + \int_0^t \theta_{s-} dS_s + \int_0^t \theta_{s-} dD_s \quad (15)$$

for all  $t \leq T$ . Let  $X$  be any deflator, a strictly positive semimartingale, and consider the deflated price process  $S_t^X = S_t X_t$ . First we demonstrate the following:

**Theorem 2** *When prices  $S$ , adjusted prices  $G$ , and accumulated dividends processes  $D$  are all semimartingales, the deflated gains process  $G^X$  is given by*

$$G_t^X = S_t X_t + D_t^X, \quad \text{where} \quad dD_t^X = X_{t-} dD_t + d[X, D]_t \quad (16)$$

*is the deflated dividend process, for any of the risky assets in the market.*

Proof: The property that a portfolio is self-financing is invariant under a change of numeraire, so we have from (15)

$$\theta_t(S_t^X + \Delta D_t^X) = \theta_0 S_0^X + \int_0^t \theta_{s-} dS_s^X + \int_0^t \theta_{s-} dD_s^X. \quad (17)$$

Let us use the notation  $V_t^X := \theta_t(S_t^X + \Delta D_t^X) = \theta_t(S_t + \Delta D_t)X_t$ . By the product rule we get

$$\begin{aligned} d(\theta_t(S_t + \Delta D_t)X_t) &= X_{t-} d(\theta_t(S_t + \Delta D_t)) \\ &\quad + \theta_{t-}(S_{t-} + \Delta D_{t-})dX_t + d[\theta(S + \Delta D), X]_t. \end{aligned} \quad (18)$$

By the ex dividend convention,  $\Delta D_{t-} = 0$ . We now use the definition of a self-financing portfolio in (15), which can be written in differential form

$$d(\theta_t(S_t + \Delta D_t)) = \theta_{t-} dG_t = \theta_{t-} dS_t + \theta_{t-} dD_t.$$

From this it follows that  $d[\theta(S + \Delta D), X]_t = \theta_{t-}(d[S, X] + d[D, X]_t)$ , and thus we get

$$\begin{aligned} dV_t^X &= X_{t-}(\theta_{t-} dS_t + \theta_{t-} dD_t) \\ &\quad + (\theta_{t-} S_{t-})dX_t + \theta_{t-}(d[S, X] + d[D, X]_t). \end{aligned} \quad (19)$$

Furthermore, using the product rule once more, equation (19) can be written

$$dV_t^X = \theta_{t-}(d(S \cdot X)_t + X_{t-}dD_t + d[D, X]_t). \quad (20)$$

By comparing this to equation (17), which can alternatively be written

$$dV_t^X = \theta_{t-}(dS_t^X + dD_t^X),$$

we see that  $dD_t^X = X_{t-}dD_t + d[D, X]_t$ .  $\square$

If the pricing formula (3) holds, there is no arbitrage. This formula is true if there exists a state price deflator  $\pi$ , i.e., a deflator such that the deflated gains process is a martingale, assuming  $S_T = 0$  at the time horizon of the economy. Having solved the controversy of how any deflated gains process  $G^X$  looks like, we can now finally show the following

**Theorem 3** *Suppose there exists a state price deflator  $\pi$ . Then market prices of risky securities are given by (3).*

Proof: The requirement that  $G^\pi$  is a martingale means that  $G_t^\pi = E_t(G_T^\pi)$  for all  $t \leq T$ , which can be written

$$S_t\pi_t + D_t^\pi = E_t(S_T\pi_T + D_T^\pi), \quad \text{for all } t \leq T.$$

Using that  $S_T = 0$  this can be written

$$S_t = \frac{1}{\pi_t}E_t(D_T^\pi - D_t^\pi) = \frac{1}{\pi_t}E_t\left(\int_t^T dD_s^\pi\right),$$

for all  $t \leq T$ , which by virtue of (16) of Theorem 2 proves the assertion.  $\square$

## 5 The Equivalent Martingale Measures Approach to Pricing

By the popular approach in finance of pricing using an equivalent martingale measure, at first sight it seems like one can avoid the extra realized square covariance term. Let us assume that there is a locally riskless asset having dynamics  $dY_t = r_t Y_t dt$ , or  $Y_t = Y_0 e^{\int_0^t r_u du}$ , where  $r$  is interpreted as the short rate process. If this function is deterministic,  $Y_t^{-1}$  is the price of a zero coupon bond at time  $t = 0$ , maturing at time  $t$ . Consider again the price process  $G = S + D$  adjusted for dividends. If the discounted gains process  $G^{Y^{-1}}$  is a martingale under a probability measure  $Q$ , equivalent to the given one  $P$ , then the pricing formula can be written

$$S_t = E_t^Q \left[ \int_t^T e^{-\int_t^s r(u) du} dD_s \right]. \quad (21)$$

When our formula (3) is valid, the expression (21) follows by the "Bayes rule", where the probability measure  $Q$  is constructed via the Radon-Nikodym derivative  $dQ/dP = \xi_T$ , and the associated density process is given by  $\xi_T = (e^{\int_0^T r(u) du})\pi_T/\pi_0$  where  $\xi_t = E_t(\xi_T)$  is a  $P$ -martingale. This provides the connection between the state price deflator  $\pi$  in our presentation and the density process  $\xi$  associated to  $Q$  in this approach.

Since the square covariance term does not appear in (21), why bother? In reality the formula above could be written

$$S_t = Y_t E_t^Q \left[ \int_t^T (Y_s^{-1} dD_s + d[Y^{-1}, D]_s) \right], \quad (22)$$

but since the discount factor  $Y^{-1}$  is a continuous process of bounded variation, the square covariance term vanishes, since  $[Y^{-1}, D] = [Y^{-1}, D^c]$ , where  $D^c$  is the continuous part of  $D$ , and  $[Y^{-1}, D^c] = [Y^{-1}, D^c]^c = 0$  when  $Y^{-1}$  (or  $D^c$ ) is of bounded variation, so (21) results.

However, this is not the only way pricing is carried out using equivalent martingale measures. The essential property used above is that the price of a zero coupon bond is strictly positive. Consider any other strictly positive price process  $Z \in L^2(P)$ , and assume that there exists a probability measure  $Q^{Z^{-1}}$ , equivalent to  $P$ , such that the discounted gains process  $G^{Z^{-1}}$  is a  $Q^{Z^{-1}}$ -martingale. From theorems 2 and 3 we then conjecture that the discounted gains process  $G^{Z^{-1}}$  is of the form (under  $Q^{Z^{-1}}$ )

$$G_t^{Z^{-1}} = S_t Z_t^{-1} + D_t^{Z^{-1}} \quad \text{where} \quad dD_t^{Z^{-1}} = Z_{t-}^{-1} dD_t + d[Z^{-1}, D]_t. \quad (23)$$

If true, it follows from (23) that

$$S_t = Z_t E_t^{Q^{Z^{-1}}} \left\{ \int_t^T (Z_s^{-1} dD_s + d[Z^{-1}, D]_s) \right\}, \quad (24)$$

in which case the square covariance term does not vanish, even in the equivalent martingale measure approach. The connection to the pricing formula (3) is this time via the transformation  $\xi_t = Z_t(\pi_t/\pi_0 Z_0)$  for  $t \in [0, T]$ , where  $\xi_t$  is the density associated with the change of measure from  $P$  to  $Q^{Z^{-1}}$ , i.e.,  $\xi_T = dQ^{Z^{-1}}/dP$ . Since  $Z$  is assumed to be a price process in the market (with no dividends), the  $\xi$ -process defined this way becomes a  $P$ -martingale, and thus satisfies the requirements of being a density process.

Since the results in the previous section do not depend upon which probability measure is being employed, it appears that the pricing formula (24) must follow.

The deflation rule in (16) (and (23)) introduces a new kind of calculus, which we now illustrate by an example.

Example 1. Consider the case when the processes  $Z$ ,  $D$ ,  $S$ ,  $\xi$  and  $\pi$  are all Itô-diffusions. We now show by direct calculations that the formulas (3) and (24) are equivalent. We start with (24). By the Bayes rule

$$S_t = Z_t E_t^{Q^{Z^{-1}}} \left\{ \int_t^T (Z_s^{-1} dD_s + d[Z^{-1}, D]_s) \right\} = \\ Z_t \frac{1}{\xi_t} E_t \left\{ \xi_T \int_t^T (Z_s^{-1} dD_s + d[Z^{-1}, D]_s) \right\}.$$

By the fact that the density process  $\xi$  is a  $P$ -martingale, and the numeraire invariance rule (16) with  $X = \xi$ , we get by iterated expectations

$$S_t = Z_0 \pi_0 \frac{1}{\pi_t} E_t \left\{ \int_t^T (Z_s^{-1} (\xi_s dD_s + d[\xi, D]_s) + \xi_s d[Z^{-1}, D]_s) \right\}. \quad (25)$$

By Itô's lemma we have that

$$dZ_s^{-1} = \frac{1}{Z_0 \pi_0} d(\pi_s \cdot \xi_s^{-1}) = \frac{1}{Z_0 \pi_0} (\xi_s^{-1} d\pi_s + \pi_s d(\xi_s^{-1}) + (d(\xi_s^{-1}))(d\pi_s)).$$

From this it follows that

$$d[Z^{-1}, D]_s = \frac{1}{Z_0 \pi_0} (\xi_s^{-1} d[\pi, D]_s + \pi_s d[\xi^{-1}, D]_s),$$

since the last term in the above equation is of bounded variation, and thus does not contribute to the quadratic covariance term. Furthermore, by Itô's lemma

$$d\xi_s^{-1} = -\xi_s^{-2} d\xi_s + \xi_s^{-3} d[\xi, \xi]_s,$$

so  $d[\xi^{-1}, D]_s = -\xi_s^{-2} d[\xi, D]_s$  for the same reason as above. Also  $Z^{-1} d[\xi, D]_s = (1/Z_0 \pi_0) \pi_s \xi_s^{-1} d[\xi, D]_s$ , which follows from the functional relationship between  $Z$ ,  $\xi$  and  $\pi$ . Putting all this together, we have that the expectation in (25) can be written

$$E_t \left\{ \int_t^T (Z_s^{-1} \xi_s dD_s + Z_s^{-1} d[\xi, D]_s + \xi_s d[Z^{-1}, D]_s) \right\} \\ = \frac{1}{Z_0 \pi_0} E_t \left\{ \int_t^T (\pi_s dD_s + \pi_s \xi_s^{-1} d[\xi, D]_s + d[\pi, D]_s - \pi_s \xi_s^{-1} d[\xi, D]_s) \right\},$$

from which it follows that  $S_t$  is given by formula (3).

Starting with the latter, on the other hand, the argument goes as follows:

$$S_t = \frac{1}{\pi_t} E_t \left\{ \int_t^T (\pi_s dD_s + d[\pi, D]_s) \right\} = \frac{1}{\pi_t \xi_t^{-1}} E_t^{Q^{Z^{-1}}} \left\{ \xi_T^{-1} \int_t^T (\pi_s dD_s + d[\pi, D]_s) \right\} \\ = \frac{Z_t}{Z_0 \pi_0} E_t^{Q^{Z^{-1}}} \left\{ \int_t^T (\pi_s (\xi_s^{-1} dD_s + d[\xi^{-1}, D]_s) + \xi_s^{-1} d[\pi, D]_s) \right\}.$$

In the last equality above we have used that  $\xi^{-1}$  is a martingale under  $Q^{Z^{-1}}$ , iterated expectations, and the numeraire invariance rule (16) with  $X = \xi^{-1}$ . First notice that  $\pi_s \xi_s^{-1} dD_s = (Z_0 \pi_0) Z_s^{-1} dD_s$ . Second, we have similar to the above that  $\pi_s d[\xi^{-1}, D]_s = -(Z_0 \pi_0) Z_s^{-1} \xi_s^{-1} d[\xi, D]_s$ . The last term we get from Itô's lemma (the product rule):

$$d\pi_s = (Z_0 \pi_0) d(\xi_s Z_s^{-1}) = (Z_0 \pi_0) (Z_s^{-1} d\xi_s + \xi_s dZ_s^{-1} + d\xi_s dZ_s^{-1}).$$

From this the realized quadratic covariance term has the differential

$$d[\pi, D]_s = (Z_0 \pi_0) (Z_s^{-1} d[\xi, D]_s + \xi_s d[Z^{-1}, D]_s),$$

since the last term in the above is of bounded variation, and thus does not contribute to the quadratic covariance. Putting this together, we have that

$$S_t = Z_t E_t^{Q^{Z^{-1}}} \left\{ \int_t^T (Z_s^{-1} (dD_s - \xi_s^{-1} d[\xi, D]_s) + Z_s^{-1} \xi_s^{-1} d[\xi, D]_s + d[Z^{-1}, D]_s) \right\},$$

which is (24).  $\square$

In situations where the interest rate  $r$  is stochastic, the approach leading to (24) becomes important. Examples are the forward measure, where the numeraire is different from the one in (21), see e.g., Jamshidian (1989) and Geman, El Karoui and Rochet (1995), and also derivative pricing under stochastic interest rates require different, stochastic numeraires, see e.g., Amin and Jarrow (1993). Even the price of a European call option with exercise price  $K$  and short rate  $r$  equal to a constant (the standard Black and Scholes formula) can be written using different, stochastic numeraires, namely as  $S_t = V_t Q_t^{V^{-1}} (V_T \geq K) - (Y_t/Y_T) K Q_t^{Y^{-1}} (V_T \geq K)$ , where  $Q_t^{V^{-1}}$  is a conditional probability, given  $\mathcal{F}_t$ , using the equivalent martingale measure resulting from applying the underlying price process  $V$  as a numeraire, and similarly is  $Q_t^{Y^{-1}}$  derived from using the zero coupon bond as numeraire.

Consider the world of continuous Itô-diffusions. In this case the square covariance term in (24) would disappear if the accumulated dividends  $D$  are of bounded variation. An important class of financial instruments where this is not the case is futures contracts, where the futures price process is modeled as a non-trivial Itô-diffusion, and the futures price process is precisely the accumulated dividend process associated with the futures contract. Naturally, in the more general framework of semimartingales containing jumps, the mere assumption of accumulated dividends being of bounded variation is not enough for the square covariance term to vanish.

In conclusion, when different numeraires are being employed, it is important to take into account the realized, quadratic covariance term treated in this paper, when analyzing market prices and dividends.

## 6 The Gordon growth model

In the deterministic world of future cash discounting with an infinite horizon, the Gordon formula says something like  $S/D = \mu_D/(r - \mu_D)$ , where  $r$  is the discount rate and  $\mu_D$  is the dividend growth rate. Consider the following continuous model in continuous time:

$$dD_t = D_t(\mu_D dt + \sigma_{D,1} dB_1(t) + \sigma_{D,2} dB_2(t)), \quad (26)$$

and

$$d\pi_t = \pi_t(-r dt + \sigma_{\pi,1} dB_1(t) + \sigma_{\pi,2} dB_2(t)). \quad (27)$$

Here  $B_1$  and  $B_2$  are two independent standard Brownian motions,  $D$  and  $\pi$  have the same meanings as before,  $\mu_D$ ,  $\mu_\pi$ ,  $\sigma_{D,1}$ ,  $\sigma_{D,2}$ ,  $\sigma_{\pi,1}$ ,  $\sigma_{\pi,2}$  are all constants and  $r$  is the equilibrium risk free interest rate given by  $r = -\mu_\pi \pi_t / \pi_t$ , also a constant. In this case the pricing formula (3) can be written

$$\begin{aligned} S_t &= \frac{1}{\pi_t} E_t \left\{ \int_t^T (\pi_s dD_s + d[\pi, D]_s) \right\} \\ &= D_t E_t \left\{ \int_t^T \frac{\pi_s D_s}{\pi_t D_t} (\mu_D ds + \rho \sigma_\pi \sigma_D ds) \right\}. \end{aligned} \quad (28)$$

Here  $\sigma_\pi^2 := \sigma_{\pi,1}^2 + \sigma_{\pi,2}^2$ ,  $\sigma_D^2 := \sigma_{D,1}^2 + \sigma_{D,2}^2$ ,  $\sigma_{\pi,D} := \sigma_{\pi,1} \sigma_{D,1} + \sigma_{\pi,2} \sigma_{D,2}$ , and  $\rho := \frac{\sigma_{\pi,D}}{\sigma_\pi \sigma_D}$ , the latter parameter being the instantaneous correlation coefficient between the dividend growth and the change in the state price.

Using the fact that the product  $\pi_t D_t$  is lognormally distributed for any  $t$  in this model, we readily deduce the following version of the Gordon growth formula:

$$\frac{S_t}{D_t} = \frac{\mu_D + \rho \sigma_\pi \sigma_D}{r - \mu_D - \rho \sigma_\pi \sigma_D} (1 - e^{-(r - \mu_D - \rho \sigma_\pi \sigma_D)(T-t)}). \quad (29)$$

When  $T \rightarrow \infty$  we must require that  $r > \mu_D + \rho \sigma_\pi \sigma_D$ , in which case the formula reduces to

$$\frac{S_t}{D_t} = \frac{\mu_D + \rho \sigma_\pi \sigma_D}{r - \mu_D - \rho \sigma_\pi \sigma_D}. \quad (30)$$

We notice that the difference from the standard formula under certainty enters through the covariance rate  $\sigma_{\pi,D}$  appearing both in the numerator and in the denominator of formula (30), with different signs. Only the term in the numerator stems from the realized quadratic covariance term of formula (3). The effects of both terms point in the same direction, however, and is that of lowering the price of an asset relative to the case of no uncertainty if  $\rho < 0$ , and raising the price if  $\rho > 0$ . Typically the sign of  $\rho$  is negative, so the effect of uncertainty is to lower the price/dividend ratio for most assets.

We also notice that this ratio is a convex function in the parameter  $\mu_D$  in its most likely range of values. If we enlarge the model such that this parameter becomes a random variable  $\tilde{\mu}_D$ , by conditioning and using Jensen's inequality, it then follows that the price/dividend ratio is larger than the expression in (30). If there is uncertainty also about the covariance term, this effect may not be so clear anymore, since the Gordon formula is convex to the left, and concave to the right in the parameter  $\rho$ .

Perhaps more interestingly in this regard is to consider pure jump models, where dividends are paid by lump sums. We consider a pure jump model with two sources of jump risk as follows:

$$dD_t = D_{t-}(\mu_D dt + z_{D,1} d\tilde{N}_1(t) + z_{D,2} d\tilde{N}_2(t)), \quad (31)$$

and

$$d\pi_t = \pi_{t-}(-r dt + z_{\pi,1} d\tilde{N}_1(t) + z_{\pi,2} d\tilde{N}_2(t)). \quad (32)$$

Here  $\tilde{N}_i(t) = (N_i(t) - \lambda_i t)$ ,  $i = 1, 2$ , are two compensated Poisson processes, where the Poisson processes  $N_i(t)$  have frequencies  $\lambda_i$ ,  $i = 1, 2$ , respectively. The parameters  $z_{D,1}$ ,  $z_{D,2}$ ,  $z_{\pi,1}$  and  $z_{\pi,2}$  are all constants, signifying the respective jump sizes associated to the various sources of jump risk. If the dividends are paid out by positive lump sums, e.g., then  $z_{D,1}$  and  $z_{D,2}$  are both positive. Given that a dividend is paid out at some date  $t$ , it is of size  $z_{D,1}$  with probability  $\lambda_1/(\lambda_1 + \lambda_2)$  and of size  $z_{D,2}$  with probability  $\lambda_2/(\lambda_1 + \lambda_2)$ . A similar interpretation holds for the state price deflator. At least one of the jump sizes  $z_{\pi,1}$  and  $z_{\pi,2}$  would typically be negative, and the realized quadratic covariance term is here given by

$$[\pi, D]_t = (\lambda_1 z_{D,1} z_{\pi,1} + \lambda_2 z_{D,2} z_{\pi,2}) \int_0^t D_{s-} \pi_{s-} ds, \quad (33)$$

clearly indicating that this term does not vanish for pure jump type models. In this case the Gordon growth formula takes the form

$$\frac{S_t}{D_t} = \frac{\mu_D + (\lambda_1 z_{D,1} z_{\pi,1} + \lambda_2 z_{D,2} z_{\pi,2})}{r - \mu_D - (\lambda_1 z_{D,1} z_{\pi,1} + \lambda_2 z_{D,2} z_{\pi,2})} (1 - e^{-(r - \mu_D - \tilde{\sigma}_{\pi,D})(T-t)}),$$

where  $\tilde{\sigma}_{\pi,D} := (\lambda_1 z_{D,1} z_{\pi,1} + \lambda_2 z_{D,2} z_{\pi,2})$ . A similar simplification results as in (30) if the transversality condition is met. Here the instantaneous correlation coefficient  $\tilde{\rho}$  is given by

$$\tilde{\rho} := \frac{\lambda_1 z_{D,1} z_{\pi,1} + \lambda_2 z_{D,2} z_{\pi,2}}{\sqrt{\lambda_1 z_{D,1}^2 + \lambda_2 z_{D,2}^2} \sqrt{\lambda_1 z_{\pi,1}^2 + \lambda_2 z_{\pi,2}^2}}.$$



Finally, let us consider a time homogeneous jump-diffusion model with an arbitrary jump size distribution:

$$dD_t = D_{t-}(\mu_D dt + \sigma_D dB(t) + \int_R \gamma_D(z) \tilde{N}(dt, dz)), \quad (34)$$

and

$$d\pi_t = \pi_{t-}(-r dt + \sigma_\pi dB(t) + \int_R \gamma_\pi(z) \tilde{N}(dt, dz)). \quad (35)$$

Here  $B$  is a standard Brownian motion,  $\tilde{N}(dt, dz) = N(dt, dz) - \nu(dz)dt$  is a compensated Poisson random measure, where  $N(t, U)$  is the number of jumps which occur before or at time  $t$  with sizes in the set  $U$  of real numbers. The process  $N(t, U)$  is called the Poisson random measure of the underlying Lévy process. The functions  $\gamma_D(z)$  and  $\gamma_\pi(z)$  give the jump sizes in the processes  $D$  and  $\pi$  respectively, as a function of the random jump size  $Z(\omega)$  of the underlying jump source of the Lévy process ( $\omega$  signify a state of the economy). The deterministic functions  $\gamma_D(z)$  and  $\gamma_\pi(z)$  satisfy  $\gamma_D \geq -1$  and  $\gamma_\pi \geq -1$ , for all  $z \in R$ , and if only positive lump sum dividends are paid out, then  $\gamma_D > 0$  and  $\sigma_D = 0$ . The Lévy measure is denoted by  $\nu(U) = E[N(1, U)]$ . If we assume that this measure can be decomposed into  $\nu(dz) = \lambda F(dz)$ , where  $\lambda$  is the frequency of the jumps and  $F(dz)$  is the probability distribution function of the jump sizes  $Z(\omega)$ , this gives us a finite Lévy measure, and the jump part becomes a geometric compound Poisson process.

We have here two different sources of risk, one continuous and one jump type. In this case the realized quadratic covariance term can be written

$$[\pi, D]_t = \int_0^t (\pi_{s-} D_{s-}) (\sigma_\pi \sigma_D) ds + \int_0^t (D_{s-} \pi_{s-}) \int_R \gamma_D(z) \gamma_\pi(z) N(ds, dz),$$

and assuming that  $\pi$  and  $D$  are both in  $L^2$ , the Gordon growth formula is as follows:

$$\frac{S_t}{D_t} = \frac{\mu_D + (\sigma_\pi \sigma_D + \int_R \gamma_D(z) \gamma_\pi(z) \nu(dz))}{r - \mu_D - (\sigma_\pi \sigma_D + \int_R \gamma_D(z) \gamma_\pi(z) \nu(dz))} (1 - e^{-(r - \mu_D - \sigma_{\pi, D})(T-t)}),$$

where  $\sigma_{\pi, D} := (\sigma_\pi \sigma_D + \int_R \gamma_D(z) \gamma_\pi(z) \nu(dz))$ , and the respective instantaneous correlation coefficients are both equal to one. By adding one source of risk of each type, for example, these correlation coefficients will again be of the form indicated in the above two examples. We notice that the qualitative conclusions reached in the simpler cases also carry over to this more complex model.

## 7 Conclusions

In this paper we made the observation that since dividends are being paid out in lump sums, not in rates as assumed in most of the extant literature, we need a pricing theory that takes account of this fact in a continuous-time framework.

To this end we started out by trying to demystify the realized quadratic covariance term appearing in the pricing formula of risky assets, in the continuous time version of an exchange economy. We pointed out that when dividends have only jumps, this square covariance term must still be present even if there are no continuous parts in the dynamics of the relevant price processes and deflators in the market. Thus, this extra term should indeed show up also in the discrete time formulation.

By making the convention that prices are observed ex dividend, a closer inspection revealed that the analogous discrete time pricing formula (1) can indeed be written with an appropriate additional term, quantified in equation (8) in the paper, which was our starting point to explain this puzzle. We then proceeded by making the passage from discrete to continuous time in a more formal manner, where the quadratic covariance term appeared quite naturally.

We presented a proof of the general pricing formula, by using the principle that a self-financing portfolio is still self-financing after a change of numeraire.

We introduced equivalent martingale measures, and outlined the relevant pricing result in this setting. Although the square covariance term does not appear in the simplest version, where the short term interest rate is a continuous, deterministic process, this is no longer true when the discount factor is allowed to be a bit more general. If this is the case, there is also an additional quadratic covariance term in the equivalent martingale measure approach to pricing, and we point out some simple examples where this term must be taken into account.

The classical Gordon growth formula takes on a fairly simple form for most continuous-time financial models, indicated in Section 6, where the effect of the realized quadratic covariance term is further explored. Here we also indicate how this term looks like for jump-diffusions.

Our results are of course not depending upon the ex-dividend interpretation of prices, since this is only a convention. If another convention is preferred, we simply obtain additional terms in the pricing formulas. These terms could even be made to disappear if we also used another, appropriately chosen convention for stochastic integration, different from Itô's choice. However, we do not recommend this approach because of the rich integration theory developed for semimartingales using this standard. In particular when

it comes to portfolio theory, the current convention is a rather natural one in economics and finance, as can, e.g., be seen from the flow of information inherent in the definition of a self-financing portfolio.

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