Mathematics and Computers in Simulation **63** (2003) 73–78

A WAVELET "TIME-SHIFT-DETAIL" DECOMPOSITION

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Abstract. We show that, with respect to an orthonormal wavelet $\psi(\cdot) \in \mathcal{L}^2(\mathbb{R})$ any $f(\cdot) \in \mathcal{L}^2(\mathbb{R})$ is, on the one hand, the sum of its "layers of details" over all time-shifts, and on the other hand, the sum of its layers of details over all scales. The latter is well known and is a consequence of a wandering subspace decomposition of $\mathcal{L}^2(\mathbb{R})$ which, in turn, resulted from a wavelet Multiresolution Analysis (MRA). The former has not been discussed before. We show that it is a consequence of a decomposition of $\mathcal{L}^2(\mathbb{R})$ in terms of reducing subspaces of the dilation-by-2 shift operator.

1. Introduction

An element $\psi(\cdot) \in \mathcal{L}^2(\mathbb{R})$—with the usual inner product $[\cdot\,,\cdot]$ and norm $||\cdot||$—is called an orthonormal wavelet if the functions

$$\psi_{m,n}(\cdot) := 2^{\frac{m}{2}}\psi(2^m(\cdot) - n), \quad m, n \in \mathbb{Z} \tag{1.1}$$

—called wavelet orthonormal functions, form a basis for $\mathcal{L}^2(\mathbb{R})$ [5]. Therefore, corresponding to a given wavelet $\psi(\cdot)$, any $f(\cdot) \in \mathcal{L}^2(\mathbb{R})$ admits the orthogonal decomposition

$$f(\cdot) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} [f(\cdot), \psi_{m,n}(\cdot)]\,\psi_{m,n}(\cdot). \tag{1.2}$$

Now, for each $m, n \in \mathbb{Z}$, the projection of $f(\cdot)$ onto $\psi_{m,n}(\cdot)$ is

$$[f(\cdot), \psi_{m,n}(\cdot)]\,\psi_{m,n}(\cdot) = [f(\cdot), 2^{\frac{m}{2}}\psi(2^m(\cdot) - n)]\,2^{\frac{m}{2}}\psi(2^m(\cdot) - n). \tag{1.3}$$

This can be considered as a *detail variation* of $f(\cdot)$—at scale 2^m and at time-shift n. For each $m \in \mathbb{Z}$, the projection of $f(\cdot)$ onto the *scale-detail subspace* $W_m(\psi)$ defined by

$$W_m(\psi) := \bigvee_{n\in\mathbb{Z}}\{\psi_{m,n}\} = \overline{\operatorname{span}}\{\psi_{m,n}\}_{n\in\mathbb{Z}} \tag{1.4}$$

is the partial sum on the RHS of (1.2)

$$\sum_{n=-\infty}^{\infty} [f(\cdot), \psi_{m,n}]\,\psi_{m,n}(\cdot). \tag{1.5}$$

This, in turn, can be regarded as a *"layer of details"* (LOD) of $f(\cdot)$—at scale 2^m [5]. Consequently, any $f(\cdot) \in \mathcal{L}^2(\mathbb{R})$ is the sum of all its LOD over all scales.

Date: January 21, 2003.

2000 *Mathematics Subject Classification.* 42C05, 47A15.

Key words and phrases. Wavelet; scale and time-shift details; shift-wandering subspace decomposition; shift reducing subspaces decomposition.

The second author's work was supported in part by Brazilian National Research Council (CNPq).

Now, instead of LOD at scale 2^m, we define the LOD—at time-shift n—of $f(\cdot)$ as the sum

$$\sum_{m=-\infty}^{\infty} [f(\cdot), \psi_{m,n}(\cdot)]\, \psi_{m,n}(\cdot). \tag{1.6}$$

Then, is it true that any $f(\cdot) \in \mathcal{L}^2(\mathbb{R})$ is the sum of its LOD—over all time-shifts? In other words, is it true that

$$f(\cdot) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} [f(\cdot), \psi_{m,n}]\, \psi_{m,n}(\cdot)\,? \tag{1.7}$$

The answer is affirmative as we shall show. This implies that a wavelet approximation can be carried out either by scale-details summation or by time-shift-details summation.

We begin by recalling basic facts of Hilbert space bilateral shifts. The main result is given in Theorem 1.

2. Main Results

Let H be a separable Hilbert space with inner product $[\cdot\,,\cdot]_H$ and norm $||\cdot||_H$. Let $U\colon H \to H$ be a bounded linear operator. A closed subspace W is a wandering subspace for U if it is orthogonal to all its images under positive powers of U [3],

$$W \perp U^n W, \quad \forall\, n > 0. \tag{2.1}$$

In addition, if the subspaces $U^n W$, $n \geq 0$, span H, then W is a generating wandering subspace for U.

A bounded linear operator $U : H \to H$ is a bilateral shift, or simply a shift, if it is unitary and it admits a generating wandering subspace W. In other words, a unitary operator U on H is a shift if and only if H admits the "wandering subspace" decomposition

$$H = \bigoplus_{m=-\infty}^{\infty} U^m W. \tag{2.2}$$

The dimension of W is called multiplicity of the shift.

We must note that if S is a completely nonunitary isometry on H then $\ker S^*$ is the unique generating wandering subspace and

$$H = \bigoplus_{m=0}^{\infty} S^m W. \tag{2.3}$$

Therefore, S is now a unilateral shift.

To proceed, let $D\colon \mathcal{L}^2(\mathbb{R}) \to \mathcal{L}^2(\mathbb{R})$ be the dilation-by-2 (or dyadic scaling) operator defined by

$$Df = g, \quad g(t) = \sqrt{2}\, f(2t), \tag{2.4}$$

and $T\colon \mathcal{L}^2(\mathbb{R}) \to \mathcal{L}^2(\mathbb{R})$ be the translation-by-1 operator defined by

$$Tf = g, \quad g(t) = f(t-1). \tag{2.5}$$

It is easy to see that both D and T are shifts of infinite multiplicity. Moreover, the wavelet orthonormal functions $\psi_{m,n}(\cdot)$ generated from an orthonormal wavelet $\psi(\cdot)$ can now be written as

$$\psi_{m,n}(\cdot) = 2^{\frac{m}{2}}\psi(2^m(\cdot) - n) = D^m T^n \psi(\cdot), \quad m,\, n \in \mathbb{Z}. \tag{2.6}$$

Define the closed subspace

$$W(\psi) := \bigvee_{n\in\mathbb{Z}} \{T^n\psi\} = \overline{\operatorname{span}}\{T^n\psi\}_{n\in\mathbb{Z}}. \tag{2.7}$$

Then, since $\{\psi_{m,n}\}_{m,n\in\mathbb{Z}}$ is an orthonormal basis for $\mathcal{L}^2(\mathbb{R})$, [1],

$$\mathcal{L}^2(\mathbb{R}) = \bigoplus_{m=-\infty}^{\infty} D^m W(\psi) = \bigoplus_{m=-\infty}^{\infty} D^m \bigvee_{n\in\mathbb{Z}} \{T^n\psi\} = \bigoplus_{m=-\infty}^{\infty} W_m(\psi), \tag{2.8}$$

where

$$W_m(\psi) = D^m W(\psi), \quad m \in \mathbb{Z}. \tag{2.9}$$

We note that the subspaces $W_m(\psi)$ are neither D-invariant nor D^*-invariant.

Remark 1. The subspaces $W_m(\psi)$ were defined in (1.4). This and (2.6) yield

$$W_m(\psi) = \bigvee_{n\in\mathbb{Z}} \{D^m T^n\psi\}.$$

They were redefined in (2.9), which together with (2.7) yields

$$W_m(\psi) = D^m \bigvee_{n\in\mathbb{Z}} \{T^n\psi\}.$$

The next proposition shows that there is no ambiguity here; both expressions for $W_m(\psi)$ coincide.

Proposition 1. *The following identity holds for every integer $m \in \mathbb{Z}$.*

$$D^m \bigvee_{n\in\mathbb{Z}} \{T^n\psi\} = \bigvee_{n\in\mathbb{Z}} \{D^m T^n\psi\}. \tag{2.10}$$

Proof. Since $D\colon \mathcal{L}^2(\mathbb{R}) \to \mathcal{L}^2(\mathbb{R})$ is linear, continuous and invertible, it follows (by the Banach Continuous Inverse Theorem) that D^m is linear and continuous for every $m \in \mathbb{Z}$. Recall that

$$D^m \operatorname{span}\{T^n\psi\}_{n\in\mathbb{Z}} = \operatorname{span}\{D^m T^n\psi\}_{n\in\mathbb{Z}} \tag{2.11}$$

and

$$D^m \overline{\operatorname{span}}\{T^n\psi\}_{n\in\mathbb{Z}} \subseteq \overline{D^m \operatorname{span}\{T^n\psi\}_{n\in\mathbb{Z}}}, \tag{2.12}$$

since D^m is continuous [4, Problem 3.46]. Moreover,

$$\overline{D^m \overline{\operatorname{span}}\{T^n\psi\}_{n\in\mathbb{Z}}} = D^m \overline{\operatorname{span}}\{T^n\psi\}_{n\in\mathbb{Z}} \tag{2.13}$$

by the fact that D^{-m} is also continuous. Therefore, by (2.12) and (2.13),

$$\begin{aligned} D^m \overline{\operatorname{span}}\{T^n\psi\}_{n\in\mathbb{Z}} &\subseteq \overline{D^m \operatorname{span}\{T^n\psi\}_{n\in\mathbb{Z}}} \\ &\subseteq \overline{D^m \overline{\operatorname{span}}\{T^n\psi\}_{n\in\mathbb{Z}}} = D^m \overline{\operatorname{span}}\{T^n\psi\}_{n\in\mathbb{Z}}. \end{aligned}$$

Hence

$$D^m \overline{\operatorname{span}}\{T^n\psi\}_{n\in\mathbb{Z}} = \overline{D^m \operatorname{span}\{T^n\psi\}_{n\in\mathbb{Z}}}.$$

It follows from this and from (2.11) that

$$D^m\overline{\text{span}}\{T^n\psi\}_{n\in\mathbb{Z}} \,=\, \overline{D^m\text{span}\{T^n\psi\}_{n\in\mathbb{Z}}} \,=\, \overline{\text{span}}\{D^mT^n\psi\}_{n\in\mathbb{Z}}.$$

This proves (2.10). □

We now derive a second wavelet decomposition of $\mathcal{L}^2(\mathbb{R})$ into orthogonal sum of reducing subspaces for the shift D. For this we begin by defining, for each $n\in\mathbb{Z}$, the subspace

$$H_n(\psi) \,:=\, \bigvee_{m\in\mathbb{Z}}\{\psi_{m,n}\} \,=\, \bigvee_{m\in\mathbb{Z}}\{D^mT^n\psi\} \,=\, \overline{\mathit{span}}\{D^mT^n\psi\}_{m\in\mathbb{Z}} \tag{2.14}$$

—called *time-shift detail* subspace, which is invariant for every power of D. Since $D^* = D^{-1}$, and since m runs over $\mathbb{Z}$, it follows that $H_n(\psi)$ is also D^*-invariant. Hence $H_n(\psi)$ reduces D. Moreover, since $\psi_{m,n}(\cdot)\perp\psi_{m,p}(\cdot)$ whenever $n\neq p$,

$$H_n(\psi)\perp H_p(\psi) \quad\text{for}\quad n\neq p.$$

We now show.

Theorem 1. *Let $\psi(\cdot)\in\mathcal{L}^2(\mathbb{R})$ be an orthonormal wavelet. Then the space $\mathcal{L}^2(\mathbb{R})$ admits the orthogonal decomposition*

$$\mathcal{L}^2(\mathbb{R}) \,=\, \bigoplus_{n=-\infty}^{\infty} H_n(\psi), \tag{2.15}$$

where the subspaces

$$H_n(\psi) \,:=\, \bigvee_{m\in\mathbb{Z}}\{\psi_{m,n}\}, \quad n\in\mathbb{Z}$$

are reducing for D.

Proof. Recall that, since $H_n(\psi)\perp H_p(\psi)$ for $n\neq p$,

$$\bigoplus_{n=-\infty}^{\infty}\bigvee_{m\in\mathbb{Z}}\{D^mT^n\psi\} \,\cong\, \overline{\Big(\sum_{n=-\infty}^{\infty}\bigvee_{m\in\mathbb{Z}}\{D^mT^n\psi\}\Big)} \,=\, \bigvee_{n\in\mathbb{Z}}\bigvee_{m\in\mathbb{Z}}\{D^mT^n\psi\}, \tag{2.16}$$

where $\cong$ means *unitarily equivalent.* Similarly, as $W_m(\psi)\perp W_p(\psi)$ for $m\neq p$ according to the orthogonal direct sum in (2.8), we get from (2.10) that

$$\bigoplus_{m=-\infty}^{\infty}\bigvee_{n\in\mathbb{Z}}\{D^mT^n\psi\} \,\cong\, \bigvee_{m\in\mathbb{Z}}\bigvee_{n\in\mathbb{Z}}\{D^mT^n\psi\}. \tag{2.17}$$

Since $\{\psi_{m,n}\}_{m,n\in\mathbb{Z}}$ is an orthonormal basis for $\mathcal{L}^2(\mathbb{R})$, it follows from (2.6) that

$$\mathcal{L}^2(\mathbb{R}) \,=\, \bigvee_{m,n\in\mathbb{Z}}\{\psi_{m,n}\} \,=\, \bigvee_{m,n\in\mathbb{Z}}\{D^mT^n\psi\}.$$

Thus, by unconditional convergence of the Fourier Series,

$$\bigvee_{m\in\mathbb{Z}}\bigvee_{n\in\mathbb{Z}}\{D^mT^n\psi\} \,=\, \bigvee_{m,n\in\mathbb{Z}}\{D^mT^n\psi\} \,=\bigvee_{n\in\mathbb{Z}}\bigvee_{m\in\mathbb{Z}}\{D^mT^n\psi\}, \tag{2.18}$$

and therefore, according to (2.14),

$$\mathcal{L}^2(\mathbb{R}) \,\cong\, \bigoplus_{n=-\infty}^{\infty}\bigvee_{m\in\mathbb{Z}}\{D^mT^n\psi\} \,=\, \bigoplus_{n=-\infty}^{\infty} H_n(\psi). \tag{2.19}$$

This completes the proof of the Theorem by writing $=$ for $\cong$, as usual. □

We conclude from the above that.

Proposition 2. *With respect to an orthonormal wavelet* $\psi(\cdot) \in \mathcal{L}^2(\mathbb{R})$*, any* $f(\cdot)$ *in* $\mathcal{L}^2(\mathbb{R})$ *admits the "scale-detail" decomposition*

$$f(\cdot) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} [f(\cdot), \psi_{m,n}(\cdot)]\, \psi_{m,n}(\cdot), \tag{2.20}$$

as well as the "time-shift-detail" decomposition

$$f(\cdot) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} [f(\cdot), \psi_{m,n}(\cdot)]\, \psi_{m,n}(\cdot). \tag{2.21}$$

We must note that the decomposition (2.8) is a wavelet wandering subspace decomposition of $\mathcal{L}^2(\mathbb{R})$ which gives rise to the decomposition (1.2) of $f(\cdot)$ in $\mathcal{L}^2(\mathbb{R})$. Moreover, it is also consequence of a wavelet MRA which is defined as follows [5].

A sequence of subspaces $\{V_m(\phi),\ m \in \mathbb{Z}\}$ of $\mathcal{L}^2(\mathbb{R})$ is a wavelet MRA, with *scaling function* $\phi(\cdot)$, if the following conditions hold:

(i) $V_m(\phi) \subset V_{m+1}(\phi), \quad m \in \mathbb{Z}$,
(ii) $\bigcap_{m=-\infty}^{\infty} V_m(\phi) = \{0\}$,
(iii) $\overline{\bigcup_{m=-\infty}^{\infty}} V_m(\phi) = \mathcal{L}^2(\mathbb{R})$,
(iv) $v(\cdot) \in V_m(\phi) \iff v(2\,(\cdot)) \in V_{m+1}(\phi), \quad m \in \mathbb{Z}$,
(v) $\{\phi((\cdot) - n),\ n \in \mathbb{Z}\}$ is an orthonormal basis of the subspace $V_0(\phi)$.

Condition (v) is "native" only to wavelet, while conditions (i)-(iv), on the one hand, define the shift operator D [2], and on the other hand define, in general, an incoming subspace V_0 for the shift operator D—"à la" Lax-Phillips Scattering Theory. Then with condition (v), V_0 depends on $\phi(\cdot)$, hence it is written as $V_0(\phi)$. We refer to the work of Antoniou and Gustafson [1] for these and other interesting connections between wavelet MRA and various parts of Mathematics.

What is interesting, from invariant subspace view point, is that by conditions (i) and (iv), each $V_m(\phi)$ is a D^*-invariant subspace. Moreover, it is also irreducible, i.e., it does not contain any nontrivial reducing subspace of D. This is due to the fact that $V_m(\phi)$ can be expressed in terms of the subspaces $W_k(\psi),\ -\infty < k \leq m-1$, as [5]

$$V_m(\phi) = \bigoplus_{k=-\infty}^{m-1} W_k(\psi), \tag{2.22}$$

and we have noted above that $W_k(\psi)$ are neither D-invariant nor D^*-invariant.

The decomposition (2.15), on the contrary, cannot be derived from a wavelet MRA since the subspaces $H_n(\psi)$ are reducing subspaces for D. Reducing subspaces of shifts are well understood. Thus the decomposition (2.15) of $\mathcal{L}^2(\mathbb{R})$—can be called a wavelet reducing subspaces decomposition—provides further understanding of wavelets as well as their relationships to shift operators.

We close by noting that, for each $m \in \mathbb{Z}$ and each $n \in \mathbb{Z}$,

$$W_m(\psi) \cap H_n(\psi) = D^m T^n \psi = \psi_{m,n}, \tag{2.23}$$

which is simply the detail at scale-2^m and at time-shift-n. Therefore, the projection of $f(\cdot) \in \mathcal{L}^2(\mathbb{R})$ onto $\psi_{m,n}$

$$P_{\psi_{m,n}} f(\cdot) = [f(\cdot), \psi_{m,n}(\cdot)]\, \psi_{m,n}(\cdot) \tag{2.24}$$

is the detail variation at scale-2^m and at time-shift-n. Then, since the orthogonal complements of $\{\psi_{m,n}\}$ in $W_m(\psi)$ and in $H_n(\psi)$, respectively, are orthogonal, we also have

$$P_{\psi_{m,n}} f(\cdot) = P_{W_m(\psi)} P_{H_n(\psi)} f(\cdot) = P_{H_n(\psi)} P_{W_m(\psi)} f(\cdot). \tag{2.25}$$

This again explains why the scale-detail decomposition (2.20) and the time-shift-detail decomposition (2.21) are decompositions of the same $f(\cdot) \in \mathcal{L}^2(\mathbb{R})$. Consequently, (2.2) and (2.15) are two orthogonal decompositions of the same space $\mathcal{L}^2(\mathbb{R})$.

Acknowledgment

We wish to thank Professor K. Gustafson for his interesting comments.

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