The Interaction of Convection & Internal Waves: A Natural Co-Dimension-Three Dynamics

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Abstract

The interactive dynamics involving convective motions in a layer heated from below and internal wave motions in a contiguous, stably stratified layer is analyzed in the weakly nonlinear limit. A reduced dynamical system consisting of three coupled, nonlinear amplitude equations describing the intensity of convection co-existing with bi-directionally propagating long internal waves is derived. This set is analyzed for various states of motion, and the results for special cases are discussed. In particular, we identify critical conditions when both the long-wave mode and the short-wave mode become unstable, allowing a "resonant" coupling of unstable modes. The amplitude equation model also possesses a third-order phase dynamics which, at least in the weakly nonlinear limit, can describe episodic mixing events.

1 Introduction

The phenomenon of thermal convection occurring in a horizontal layer heated from below has been studied extensively, both because of its relevance in physical contexts and its utility as a model for pattern forming instabilities and transition to (weak) turbulence. In the present study, onset of thermal instability is considered together with its coupling with motion in an adjacent layer supporting internal waves. The physical model is intended as one with potential for providing insight into, for example, the case of surface cooling of a mixed layer in a lake and a consequent, stimulated internal wave field generated in and across the metalimnion. Alternatively, the formulation may be considered as a model for the convective dynamics arising from surface heating at the base of the troposphere, and consequent driven internal waves in and across the tropopause.

In either of the physical paradigms noted, convective dynamics in a finite layer occurs in the presence of an adjoining layer of stably stratified media. As a consequence, convective motions may, depending on the static stability of the interface between the contiguous layers, couple strongly with internal waves, leading in such cases to enhanced transport, especially in cases where the underlying system is Galilean invariant. The Galilean invariance of the system is an important factor as convective cells are then readily displaced horizontally by propagating long waves impressed on the convecting layer.

On the other hand, the action of waves in a region adjacent to a convective layer will modulate the depth of the convecting layer and, therewith, the local Rayleigh number and the intensity of convective motion. Further, when the modulation timescale is sufficiently short, slower convective activity may even be inhibited. It appears, therefore, that offsetting affects exist in the nonlinear coupling between convection and internal waves, and the interaction is likely to lead to nontrivial dynamics which may have important practical implications. It is the exploration of these dynamics that is pursued in the present work.

2 Model Definition

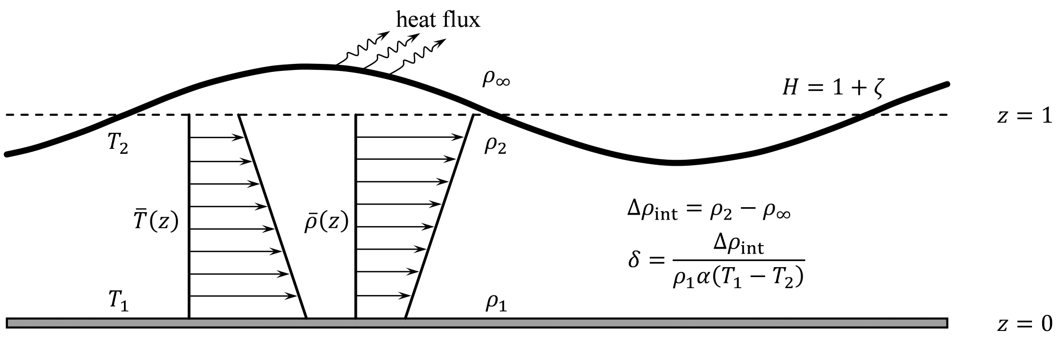

Figure 1: Schematic definition of the problem.

The idealized model examined in this study is depicted in Figure 1. The base state, as shown, consists of a horizontal layer of fluid with a non-deformable, impermeable, stress-free, isothermal lower boundary. The convecting layer is capped by an immiscible, deformable boundary separating it from a contiguous, stationary and stably stratified region. For model simplicity and analytic convenience, we consider a particular limit where the upper stratified layer is "collapsed" to a bounding, immiscible, deformable interface across which a finite, stable density jump exists. As such, the model may be viewed as a "one and one-half layer" configuration, yet with the capacity to support propagating waves of only the lowest internal wave mode of a more realistic, two-layer system. This model is chosen for analysis as it contains the minimum structure wherein the spontaneous onset of convection can readily couple with freely propagating internal waves impressed by a contiguous layer.

Employing this restricted physical model, we consider only plane motion and invoke the Boussinesq approximation. Important aspects of the formulation of this problem, where the focus was directed exclusively on the short-wave mode of convective instability, can be found by reference to Pavithran and Rdedekopp (1994). As in that earlier work, the disturbance state of motion, relative to the specified base state, is described by the following set of field equations:

$$\nabla \cdot \mathbf{V} = 0, \tag{1}$$

$$\frac{1}{\mathrm{Pr}} \frac{D\mathbf{V}}{Dt} = -\nabla p + \theta \hat{e}_z + \nabla^2 \mathbf{V}, \tag{2}$$

$$\frac{D\theta}{Dt} = \mathrm{Ra}\mathbf{V} \times \hat{e}_z + \nabla^2 \theta. \tag{3}$$

In these equations the velocity vector has two components (u, w) corresponding to motion in the coordinate directions (x, z), and all the variables are non-dimensional based on the scales $(d; d^2/\kappa; \kappa/d; \rho_1 \nu\kappa/d^2; \nu\kappa/\alpha g d^3)$ for length, time, velocity, pressure, and temperature, respectively. The two parameters appearing in the above set of field equations are the Prandtl number, Pr, and the Rayleigh number, Ra.

$$\mathrm{Pr} = \nu/\kappa; \quad \mathrm{Ra} = \left\{\alpha g (T_1 - T_2) d^3/(\nu\kappa)\right\}.$$

For purposes of providing only a concise definition of the present model, and emphasizing essential differences relative to the previous work of Pavithran and Rdedekopp (1994), we present the analytic form of just two of the boundary conditions. The two conditions in view are both applicable to the upper, deformable surface of the convecting layer, and they reveal the existence of two additional parameters that are pivotal to characterizing the interaction of convection and waves. First, the normal stress matching condition applied at the non-equilibrium location $z = 1 + \zeta(x,t)$, takes the form:

$$\mathrm{Ra}\left\{\delta + \frac{1}{2}\zeta\right\}\zeta + p + \frac{2}{1+\zeta_x^2}\left\{w_x - \zeta_x(u_z + w_x) + u_x\zeta_x^2\right\}. \tag{4}$$

Second, by use of a mixed thermal flux condition at the interface, allowing thereby for either a convective or radiative heat flux into the ambient above the deformable surface, one obtains the following condition:

$$\theta_z - \zeta_x\theta_x = -\mathrm{Bi}(\theta - \mathrm{Ra}\zeta)\sqrt{1+\zeta_x^2} - \mathrm{Ra}\left(\sqrt{1+\zeta_x^2} - 1\right) \quad \text{at} \quad z = 1 + \zeta(x,t). \tag{5}$$

The two additional parameters entering through these boundary conditions are the interfacial density parameter δ and the Biot number Bi. These important parameters have the definitions

$$\delta = \frac{\delta_{\mathrm{int}}}{\delta_{\mathrm{conv}}} \quad \text{and} \quad \mathrm{Bi} = \frac{q_{\mathrm{rad}}}{q_{\mathrm{cond}}}. \tag{6}$$

The terms in the expression for the first parameter, namely δ_{int} and δ_{conv}, represent, respectively, the (stable) density change across the deformable interface and the unstable density change across the convecting layer in the base state condition. It is a ratio of the stiffness of the upper interface compared to the driving potential for motion within the layer. One should note that, since the Boussinesq approximation is employed, there is an implicit restriction on the magnitude of these density changes. However, in the limit of increasing δ, the deformable interface becomes increasingly stiff, and the problem for convection between two plane boundaries is formally recovered as $\delta \to \infty$.

In a corresponding way, the terms q_{rad} and q_{cond} in the definition of the Biot number Bi specify the radiative heat flux from the deformable interface and the conductive heat flux across the convecting layer in the stationary, base state condition, respectively. When the Biot number becomes large, corresponding to very efficient heat transfer from the interface, the thermal condition at the interface approaches that of an isothermal surface. Alternatively, when the Biot number vanishes, the condition of an insulated boundary is recovered. The subsequent analysis reveals that the Biot number serves to distinguish the dominance of two different modes of convective instability, and it allows for identification of a "resonant" state wherein both a long-wave and a short-wave mode of convection share the same threshold condition for release of potential energy. In such a condition, the two modes can interact nonlinearly through the intrinsic coupling afforded with long gravity waves on the interface. This intrinsic dynamics can be understood from a deeper fundamental perspective via couplings arising from the simultaneous breaking of underlying invariance properties.

3 The Long-Wave Mode

The long-wave mode of instability can be readily assessed analytically by use of an asymptotic expansion involving use of scaled (slow) coordinates $(\zeta, \tau) = \epsilon(x, t)$, where ϵ is a small amplitude parameter. A consistent approximation is obtained in the case of stress-free surfaces provided the dependent variables are expanded as:

$$\zeta(x,t) = \epsilon^2 H^{(1)}(\zeta,\tau) + \epsilon^3 H^{(2)}(\zeta,\tau) + \ldots, \tag{7a}$$

$$(u, w, p, \theta) = \epsilon^2 \left\{U^{(1)}, \epsilon W^{(1)}, P^{(1)}, \vartheta^{(1)}\right\} + \epsilon^3 \{\ldots\} + \ldots. \tag{7b}$$

Each amplitude function in (7b) depends on the independent variables (ζ, z, τ). A similar long-wave analysis of convection in an altered physical model has been considered by Benguria and Depassier (1987).

The expansion noted in (7a,b) can be carried forward for several orders, yielding a hierarchical pair of amplitude equations characterizing the evolution of long-wave disturbances. For example, the leading order evolution is given simply by the DAlembert, linear wave equation pair

$$H^{(1)}_\tau = -U^{(1)}_x \quad \text{and} \quad U^{(1)}_\tau = -c_0^2 H^{(1)}_x. \tag{8}$$

The parameter c_0 is the long-wave phase speed for interfacial waves in this model, and is defined by

$$c_0^2 = \delta - \frac{1}{3}\left(\frac{\mathrm{Bi}}{\mathrm{Bi}+1}\right). \tag{9}$$

For propagating waves we require $c_0^2 > 0$, which yields a restriction on allowable values of the parameter space (δ, Bi). When this quantity is negative, the amplitude evolution requires special consideration that will not be discussed here.

Carrying the analysis to second order, and transforming back to laboratory coordinates, the long-wave evolution is defined by the pair

$$\zeta_t = -V_x + \alpha_0 \zeta_{xx} \quad \text{and} \quad V_t = -c_0^2 \zeta_x + \beta_0 V_{xx}. \tag{10}$$

A straightforward stability analysis yields the onset condition for the long-wave mode of convection. This condition is given by

$$\alpha_0 + \beta_0 = -4\mathrm{Pr}\frac{\mathrm{Ra} - \mathrm{Ra_c}}{\mathrm{Ra_c}} = -4\mathrm{Pr}\Delta, \quad \text{where} \quad \mathrm{Ra_c} = 180\frac{(\mathrm{Bi}+1)^2}{\mathrm{Bi}+6}. \tag{11}$$

One observes that the critical Rayleigh number for this mode is independent of the interface density parameter δ, although the acceptable range of this parameter is constrained by (9). Further, one observes that onset condition for this mode tends to $\mathrm{Ra_c} \to \infty$ as $\mathrm{Bi} \to \infty$, showing that the long-wave mode is linearly stable in the limit of an isothermal interface. Hence, the flux condition across the deformable interface exerts a pivotal role in the relative competition between the two possible modes of convection.

Carrying the long-wave analysis to higher orders, and using lower-order sets to eliminate time derivatives appearing in higher-order sets in favor of space derivatives, one obtains an extended dispersive-dissipative, nonlinear system for the long-wave dynamics. This system, when used to consider waves propagating in only one direction (say, to the right), can be cast into an extended KdV equation for the interface displacement:

$$\zeta_t + c_0\zeta_x + \alpha\zeta\zeta_x + \beta\zeta_{xxx} = -2\mathrm{Pr}\Delta\zeta_{xx} + \gamma_4\zeta_{xxxx} + \beta_2\left(\zeta\zeta_x\right)_x . \tag{12}$$

This equation possesses equilibrium, solitary wave states sustained against dissipation by the release of potential energy through the long-wave instability operating when $\Delta > 0$. Linear growth rate curves for several values of the criticality parameter Δ are shown in Figure 2.

4 The Short-Wave Mode

Analysis of the short-wave mode of convection follows closely the earlier work of Pavithran and Rdedekopp (1994), albeit here the effect of the mixed flux condition (5) at the deformable interface must be considered. The onset conditions for this mode have been explored in some detail, and in Figure 3 we show parameter values that lead to identical critical Rayleigh numbers for onset of convection for both the long-wave and the short-wave modes. The intersection of the dotted curve with any solid curve defines a condition for spontaneous release of potential energy for both modes, and for possible long-wave/short-wave resonant interaction via nonlinear coupling.

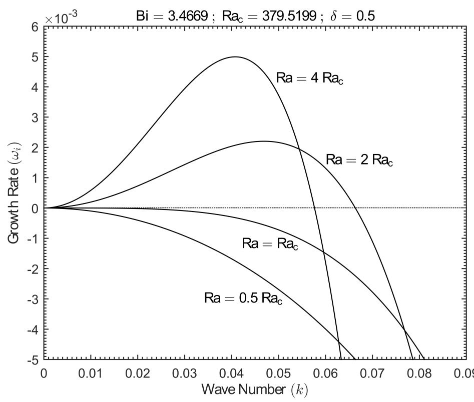

Figure 2: Linear growth rate for the long-wave mode at the indicated degree of super-criticality.

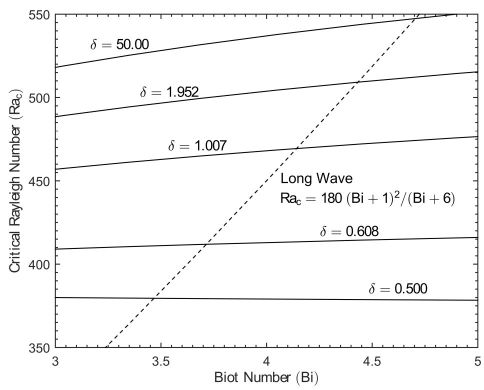

Figure 3: The critical Rayleigh number for a 'resonant' onset of convection involving both the long-wave and the shortwave modes.

A standard, multiple scale analysis for the weakly nonlinear dynamics ensuing via the stationary bifurcation in the vicinity of the critical condition (Ra, k_c) leads to the following (scaled) evolution system:

$$A_t = -ik_cUA + \Delta A - \gamma HA + A_{xx} - |A|^2 A, \tag{13}$$

$$H_t = -U_x + \alpha_0 H_{xx} - \nu_0\left(|A|^2\right)_x , \tag{14}$$

$$U_t = -c_0^2 H_x + \sigma_0 \left(|A|^2\right)_x + \lambda_0 U_{xx}. \quad (15)$$

In this system $A(x, t)$ is the complex modulation amplitude of, say, the vertical velocity of the disturbance field associated with convective motions with spatial periodicity $2\pi/k_c$, and Δ is the linear instability parameter proportional to $(\mathrm{Ra} - \mathrm{Ra_c})$, where $\mathrm{Ra_c}$ in this case is the threshold Rayleigh number for instability. The variables $H(x, t)$ and $U(x, t)$ describe the long-wave component of the interface distortion and the associated drift velocity.

The first amplitude equation is consistent with that obtained by Newell and Whitehead (1969) in the case where $H = U = 0$. The drift effect (Doppler shift) under the restrictive condition $H = 0$, wherein both boundaries are non-deformable, was shown by Zippelius and Siggia (1983) and by Coullet and Huerre (1986) to have an important dynamical role in pattern forming instabilities and pattern selection. They demonstrated that the marginal vertical vorticity mode, in cases with underlying Galilean invariance, can couple effectively with convection through spontaneous onset of drift currents in two-space-dimensional evolution. In the present case, a strong dynamical coupling is possible even for one-space-dimensional evolution, and even when the vertical vorticity mode is restricted because of the present limitation of plane flow. One observes that, when convection is not active (*i.e.*, $A = 0$), the latter two amplitude equations describe a linear, damped and non-dispersive, long-wave motion. The degree of freedom associated with the compressional mode (the presence of a stable, deformable interface) gives rise to a non-trivial coupling between convection and wave motion even in the case of planar motion.

The system (13-15) possesses the stationary, spatially periodic solution

$$A(x, t) = Qe^{(qx+\phi)} , \quad H(x, t) = U(x, t) = 0. \quad (16)$$

The quantity ϕ is an arbitrary phase constant and Q, the amplitude, is given by:

$$Q^2 = \begin{cases} 0, & \text{when } \Delta < 0 \\ \Delta - q^2, & \text{when } \Delta > 0. \end{cases} \quad (17)$$

Another family of stationary solutions exist in which H and U are arbitrary constants, but these are isomorphic to the set in (16) and have identical stability properties.

The stability of the family of stationary solutions given in (16-17) can be analyzed by adding a small perturbation to the amplitude set (Q, H, U), and allowing for a general, evolving phase $\phi(x, t)$. It emerges that the amplitude perturbation to Q is strongly damped, leading in particular to a third-order in time dynamics for the phase ϕ. The evolution of the phase is found to couple strongly with disturbances associated with presence of a compressional mode; that is, with disturbances in the height and drift fields H and U. Focusing on long-wave disturbances, the linear evolution equation for the phase takes the form

$$\phi_{ttt} = a\phi_{xxtt} + b\phi_{xxt} + m\phi_{xxxx} + n\phi_{xxxxxx}. \quad (18)$$

This dynamical model is the only known example of this co-dimension-three instability, and it arises naturally through the breaking of the three fundamental invariances implicit to the structure of the underlying physical and mathematical model. Equation (18) contains two well-known limits of phase instability analysis in the problem of convection, each of which arises generically from breaking of particular invariances. For example, when one considers the case of convection between plane, non-deformable surfaces with no-slip boundaries, the stability of the finite-amplitude (roll) state near onset occurs via the Eckhaus instability. This instability, associated with the breaking of the underlying translational invariance, corresponds with the balance $\phi_{ttt} \sim \phi_{xxtt}$ – a diffusive phase instability. Also, when one considers convection between plane, non-deformable, stress-free surfaces, the base system satisfies both translational and Galilean invariance. Then, the onset of convection couples with a two-dimensional drift field (via the marginal vertical vorticity mode), leading to a phase which is propagative a balance captured in the present 1-D case by the balance $\phi_{ttt} \sim \phi_{xxt}$ (see Coullet and Fauve (1985) and Coullet and Huerre (1986)).

In the case of the physical problem considered here, the underlying system satisfies not only translational and Galilean invariance, but also Newtonian invariance. As a consequence, the onset of convection gives rise naturally to a phase dynamics that is third-order in time. The linear equation in (18) has been analyzed in detail to characterize the nature of the third-order fixed point as a function of the parameters of the problem. The nonlinear evolution of this phase dynamics has been derived and is being studied through numerical simulation.

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