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Abstract: We extend the set of full rank nominal and deflated income demand systems to rational demand systems of any rank and present a unifying expression for the indirect preferences of all full rank demand models.

Key Words: Aggregation, functional form, integrability, rank, rational demand systems

JEL CLASSIFICATION: **D12, E21**

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FULL RANK RATIONAL DEMAND SYSTEMS

Common reasons for the choice of functional form for demand analysis include parsimony, ease of estimation and interpretation, generality, flexibility, in- and out-of-sample fit, aggregation, and consistency with economic theory (Cooper and McLaren 1992, 1996). However, at least since the path-breaking papers of Gorman (1953, 1961, 1981), Muellbauer (1975, 1976), and Deaton and Muellbauer (1980), flexibility under aggregability has guided much of the development and application of functional forms. Thus, the rank of Engel curves became a central feature to nuance flexibility while maintaining desirable properties of aggregation (Lewbel 1987a, 1990). Recent emphasis on micro-micro data sets, empirical models and methods have perhaps decreased the emphasis on aggregation in empirical work but allow for increased parametric flexibility due to relatively large data sets. This flexibility is increasingly less costly to implement using modern computation and storage capacities. In either case, it is relatively routine to impose any or all of the theoretical constructs of non-negativity, Slutsky symmetry, and homogeneity including adding up.

In this paper, we present an extremely flexible structural model of micro-level consumer or producer behavior that encompasses nearly every existing model of demand or supply as a special case. This form, developed using group theory, begins with the insights initiated by Gorman (1981), Russell (1983, 1996), and Russell and Farris (1993, 1998) and ends with a flexible form with useful applications consistent with aggregation as a special case, extending to empirical analyses of the micro variety where there may be little interest in aggregation. We attain this flexibility by extending the deflated income systems of Lewbel (1989a) to a class of rational demand systems that are associated with a projective transformation group.

In addition to all Gorman polar forms, including the normalized quadratic expenditure function (Diewert and Wales 1988) and translog, generalized Leontief, and generalized McFadden cost functions (Diewert and Wales 1987), we show that the general translog (Christenson, Jorgenson, and Lau 1975), modified almost ideal demand system and general exponential form (Cooper and McLaren 1992, 1996), fractional and rational rank four demand systems (Lewbel 1987b, 2003, 2004), reciprocal generalized Leontief and minflex Laurent demand systems (Barnett and Lee 1985; Barnett, Lee, and Wolfe 1985), and reciprocal indirect normalized quadratic (Diewert and Wales 1988) are members this new class of demand systems. As a consequence, nearly all commonly used demand models are encompassed by the class of rational demand systems derived and presented here. Thus, we extend and synthesize a large literature on functional forms in demand and production analysis.

A Brief Summary of Aggregation Theory

Gorman (1953) derived the necessary and sufficient conditions for the existence of a representative consumer and then found the indirect preference function, commonly called the Gorman Polar Form (Gorman 1961). Muellbauer (1975, 1976) extended this to one nonlinear function of income, obtaining the price independent generalized linear (PIGL) and price independent generalized logarithmic (PIGLOG) models. Gorman (1981) extended this further to a complete system that is a finite sum of functions of income, each multiplied by a vector of price functions. *Gorman systems* are the foundation of an important literature in demand theory (Deaton and Muellbauer 1980; Gorman 1953, 1961, 1981; Jerison 1993; Lewbel 1987a 1988, 1989a, 1990; Muellbauer 1975, 1976; Russell 1983, 1996; Russell and Farris 1993, 1998; van Daal and Merkies 1989).

An important fact is that many – even most – existing empirical models of consumer and producer behavior can be represented as a Gorman system. This includes, *inter alia*, quadratic utility and many extensions, linear expenditure system, quadratic expenditure system, almost ideal demand system (AIDS), quadratic AIDS, Rotterdam model, generalized Leontief, normalized quadratic, translog, and trigonometric demand models.

There are three main insights in the literature on aggregation theory. Gorman's (1981) seminal work, as clarified and extended by Lewbel (1987a 1988, 1989a, 1990) and van Daal and Merkies (1989), tells us that the maximum number of linearly independent vectors of price functions – called the *rank* of the system – is no more than three. A Gorman system is said to have *full rank* if the rank of the matrix of price functions is equal to the number of its columns, which in turn equals the number of unique income functions (Lewbel 1990). As a result of the research program of Lewbel, the indirect preferences of all full rank nominal income Gorman systems are known.

The second insight was originally presented in Russell (1983), and clarified and extended by Jerison (1993), Russell (1996), and Russell and Farris (1993, 1998). All full rank Gorman systems are projective transformation groups of differential geometry. This provides an opportunity – as yet unfulfilled – for a unifying expression for all full rank Gorman systems in a single framework. We provide this representation below.

The third (group of) insight(s) comes from Lewbel (1989a). Deflating income prior to considering the implications of symmetry (hereafter called deflating *ex ante*) overcomes the functional form restrictions due to homogeneity, deflated income Gorman systems can have a maximum rank of four, and deflated income Gorman systems include all of the nominal income Gorman systems as special cases.

This paper combines the above results with at least two additional insights to expand the class of full rank models to any rank – up to the number of goods in the demand system – while maintaining the strict equivalence between the number of independent price indices and the rank of the demand system in all nominal or deflated income Gorman systems. As noted in the introduction, this extension encompasses many well-known alternatives to Gorman systems, including the general exponential form, or GEF (Cooper and McLaren 1996), modified AIDS, or MAIDS (Cooper and McLaren 1992), translog (Christensen, Jorgenson, and Lau 1975; Jorgenson 1990; Jorgenson, Lau and Stoker 1980, 1981, 1982; Jorgenson and Slesnick 1984, 1987), fractional demand systems (Lewbel 1987b), rational rank four demand system (Lewbel 2003, 2004), reciprocal indirect utility generalized Leontief and minflex Laurent demand systems (Barnett and Lee 1985; Barnett, Lee, and Wolfe 1985), and reciprocal indirect normalized quadratic demand system (Diewert and Wales 1988).

First, we find a unifying expression for all full rank nominal and deflated income Gorman systems in terms of a projective group transformation, which also provides a comprehensive answer to the question raised by Lewbel (1989a, 1990). Second, we overcome the restriction to rank three of a projective group transformation – which appears on the right-hand side of the group theory representation we find for all nominal and deflated income Gorman systems – by generalizing the left-hand side in a comprehensive, yet natural and intuitively appealing way. This produces a class of rational full rank demand systems that can have any rank, up to the total number of goods in the demand system.

We proceed by first briefly reviewing the properties of projective group transformations. Next, we identify an important common characteristic of full rank three nominal income Gorman systems. Third, we present the group theory representation for all full rank nominal and deflated income Gorman systems. Fourth, this class of models is extended to a large class of rational demand systems in which any rank can be achieved and all but one of the income functions is flexible. This class is shown to encompass almost all well-known and commonly implemented empirical models of demand behavior. The final section summarizes and concludes. Proofs are in the Appendix.

Projective Transformation Groups

A *group* is a set of elements that is closed under a binary operator, called *multiplication* regardless of what the operator truly is, an inverse operator, and a well-defined identity operator. The *projective transformation group* is equivalent to the *special linear group two*, $\mathfrak{sl}(2)$. The latter group is defined by the set of all 2×2 matrices,

$$\boldsymbol{A} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}, \tag{1}$$

with a unit determinant, $\alpha\delta - \beta\gamma = 1$ (Olver 1993). The inverse of any $\boldsymbol{A} \in \mathfrak{sl}(2)$,

$$\boldsymbol{A}^{-1} = \begin{bmatrix} \delta & -\beta \\ -\gamma & \alpha \end{bmatrix}, \tag{2}$$

satisfies $\boldsymbol{A}^{-1} \in \mathfrak{sl}(2)$, since $|\boldsymbol{A}^{-1}| = 1/|\boldsymbol{A}| = 1$. It is clear that the 2×2 identity matrix belongs to $\mathfrak{sl}(2)$, while $\boldsymbol{I}_2\boldsymbol{A} = \boldsymbol{A}$ for all $\boldsymbol{A} \in \mathfrak{sl}(2)$. Last, for all $\boldsymbol{A}, \boldsymbol{B} \in \mathfrak{sl}(2)$, the matrix product, $\boldsymbol{AB} \in \mathfrak{sl}(2)$, since $|\boldsymbol{AB}| = |\boldsymbol{A}| \times |\boldsymbol{B}| = 1$. Thus, matrix multiplication is the multiplicative operator for this group, matrix inversion is the inverse operator, and multiplication by the 2×2 identity matrix is the identity operator. The restriction to a unit determinant is a normalization that gives three independent parameters in the group.

The matrix $\boldsymbol{A} \in \mathfrak{sl}(2)$ in (1) is associated with the projective group transformation $y(x) = (\alpha x + \beta)/(\gamma x + \delta)$ for all $x \in \mathbb{R}$ such that $|y(x)| < \infty$, a rational map with linear functions in both the numerator and the denominator. The inverse projective group transformation, $x(y) = (\delta y - \beta)/(-\gamma y + \alpha)$, is associated with $\boldsymbol{A}^{-1} \in \mathfrak{sl}(2)$. $\boldsymbol{I}_2$ defines the identity map $y(x) = x$. Let $\boldsymbol{A}, \boldsymbol{B} \in \mathfrak{sl}(2)$ define a pair of projective group transformations by $a(x) = (a_{11}x + a_{12})/(a_{21}x + a_{22})$ and $b(x) = (b_{11}x + b_{12})/(b_{21}x + b_{22})$. The matrix product $\boldsymbol{BA}$ is associated with the composition of the projective group transformations,

$$b(a(x)) = \frac{(b_{11}a_{11} + b_{12}a_{21})x + (b_{11}a_{12} + b_{12}a_{22})}{(b_{21}a_{11} + b_{22}a_{21})x + (b_{21}a_{12} + b_{22}a_{22})}. \tag{3}$$

This can be verified by computing the individual elements of $\boldsymbol{BA}$ and rearranging terms in the composite function $b(a(x))$. That is, *multiplication* is the composition of transformations in the group and this is one-to-one and onto with matrix multiplication in $\mathfrak{sl}(2)$. A simple inductive argument implies that any sequence of compositions of projective group transformations is a projective group transformation.

As noted above, the determinant restriction implies that there are three parameters in this group. Sophus Lie (1880; English translation with commentary in Hermann 1975) proved that this is the most general group transformation on the real line. In general, the projective transformation group is defined only up to arbitrary monotonic, smooth transformations, say $z(y)$, $z'(y) \neq 0$. The group $\mathfrak{sl}(2)$ also is closed for matrices with complex elements. Consequently, the projective transformation group extends from $\mathbb{R}$ to $\mathbb{C} = \{x + \imath y : x, y \in \mathbb{R}\}$, where $\imath = \sqrt{-1}$. These two facts are essential and advantageous in what follows.

In the following sections, the identity matrix will be associated with rank one Gorman systems. A second special case of interest is the set of upper triangular 2×2 matrices of the form,

$$\boldsymbol{A} = \begin{bmatrix} 1 & \beta \\ 0 & 1 \end{bmatrix}, \tag{4}$$

which is associated with full rank two Gorman systems via the map $y(x) = x + \beta$. A third case of interest is the set of diagonal matrices of the form

$$\boldsymbol{A} = \begin{bmatrix} \alpha & 0 \\ 0 & 1/\alpha \end{bmatrix}, \tag{5}$$

which also is associated with full rank two systems. Upper triangular matrices of the form

$$\boldsymbol{A} = \begin{bmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{bmatrix} \tag{6}$$

are associated with full rank three deflated income systems, while matrices satisfying $\alpha\delta - \beta\gamma = 1$ with no other restriction are associated with full rank three nominal income systems and full rank four deflated income systems.

A Characteristic of Full Rank Three Nominal Income Gorman Systems

Let $\boldsymbol{p} \in \mathcal{P} \subset \mathbb{R}_+^n$ be a vector of market prices, let $\boldsymbol{q} \in \mathcal{Q} \subset \mathbb{R}_+^n$ be a vector of goods, let $m \in \mathcal{M} \subset \mathbb{R}_+$ be total expenditure, and let utility be $u(\boldsymbol{q})$, where $u : \mathcal{Q} \to \mathcal{U} \subset \mathbb{R}$ is smooth ($u \in \mathcal{C}^\infty$), increasing, and quasi-concave on $\mathcal{Q}$. We use the sobriquet *income* to denote m throughout. Define the expenditure function by

$$e(\boldsymbol{p}, u) \equiv \min\left\{\boldsymbol{p}^\top \boldsymbol{q} : u(\boldsymbol{q}) \geq u\right\}. \tag{7}$$

Assume $e : \mathcal{P} \times \mathcal{U} \to \mathcal{M}$ is smooth ($e \in \mathcal{C}^\infty$), increasing, 1° homogeneous, and concave in $\boldsymbol{p}$, and increasing in u. Assume an interior solution for $\boldsymbol{q}$. Any nominal income Gorman system can be written as

$$\partial e(\boldsymbol{p},u)/\partial \boldsymbol{p} = \sum_{k=1}^{K} \boldsymbol{\alpha}_k(\boldsymbol{p}) h_k(e(\boldsymbol{p},u)), \tag{8}$$

where $\boldsymbol{\alpha}_k : \mathcal{P} \to \mathbb{R}^n$, $\boldsymbol{\alpha}_k \in \mathcal{C}^\infty$, $h_k : \mathcal{M} \to \mathbb{R}$, $h_k \in \mathcal{C}^\infty$, $k = 1,\cdots,K$, are smooth functions of prices and income, respectively. Finally, we also assume that the $\{\boldsymbol{\alpha}_1,\cdots,\boldsymbol{\alpha}_K\}$ are linearly independent across the *K*–dimensional constants and that the $\{h_1,\cdots,h_K\}$ are linearly independent across the *K*–dimensional constants, to ensure a unique representation of the demand system (see, e.g., Gorman (1981) or the Appendix of Russell and Farris (1998) by Robert Bryant).

The Appendix includes the algebraic steps that will put each of the extended PIGL, PIGLOG, and QES full rank three nominal income Gorman systems in the form,

$$\frac{\partial}{\partial \boldsymbol{p}}\left(\frac{f(e(\boldsymbol{p},u)) - \beta_1(\boldsymbol{p})}{\beta_2(\boldsymbol{p})}\right) = \left[\theta(\eta(\boldsymbol{p})) + \left(\frac{f(e(\boldsymbol{p},u)) - \beta_1(\boldsymbol{p})}{\beta_2(\boldsymbol{p})}\right)^2\right]\frac{\partial \eta(\boldsymbol{p})}{\partial \boldsymbol{p}}, \tag{9}$$

where $f(m) \in \{\ln m, m^\kappa\}$, $\beta_1, \beta_2, \eta : \mathcal{P} \to \mathbb{R}$, $\theta : \mathbb{R} \to \mathbb{R}$, and $\beta_1, \beta_2, \eta, \theta \in \mathcal{C}^\infty$. With a change of variables to $z(\boldsymbol{p},u) = \left[f(e(\boldsymbol{p},u)) - \beta_1(\boldsymbol{p})\right]/\beta_2(\boldsymbol{p})$, this simplifies to

$$\partial z(\boldsymbol{p},u)/\partial \boldsymbol{p} = \left[\theta(\eta(\boldsymbol{p})) + z(\boldsymbol{p},u)^2\right]\partial \eta(\boldsymbol{p})/\partial \boldsymbol{p}. \tag{10}$$

Therefore, all full rank three QES, generalized PIGL, and generalized PIGLOG models are characterized by the following property (A proof is contained in the Appendix).

***Lemma* 1:** *If* $z : \mathcal{P} \times \mathcal{U} \to \mathbb{R}$, $\theta : \mathbb{R} \to \mathbb{R}$, $\eta : \mathcal{P} \to \mathbb{R}$, $z, \theta, \eta \in \mathcal{C}^\infty$, *satisfy* (10) *and* $\partial \eta(\boldsymbol{p})/\partial \boldsymbol{p} \neq \boldsymbol{0}$, *then* $z(\boldsymbol{p},u) \equiv w(\eta(\boldsymbol{p}),u)$, $\partial w(x,u)/\partial x = \theta(x) + w(x,u)^2$.

The function $w : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $w \in \mathcal{C}^\infty$, can be defined formally by

$$w(\eta(\boldsymbol{p}),u) = \int_0^{\eta(\boldsymbol{p})} \left[\theta(x) + w(x,u)^2\right] dx, \tag{11}$$

with initial conditions[1] $w(0,u) = u$ and $\partial w(0,u)/\partial x = \theta(0) + u^2$. In the Appendix, as part

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[1]Make the change of variables $w(x,u) = -\partial v(x,u)/\partial x / v(x,u)$ to convert the Riccati partial differential equation in *w* to the linear second-order partial differential equation $\partial^2 v(x,u)/\partial x^2 + \theta(x) v(x,u) = 0$, which requires two initial conditions. The two chosen here normalize the utility index and guarantee smoothness at *x*=0 for all *u*. In general, linear second-order differential equations with non-constant coefficients do not have solutions that are expressible in terms of elementary functions, although solutions in terms of conver-

of the proof of proposition 1 below, it is shown that when $\theta(x) \equiv \lambda$ solutions for $w(x,u)$ can be expressed in terms of elementary functions. We use (11) to obtain the projective group transformations for this class of demand systems. The special cases that can be expressed in terms of elementary functions illustrate the nature of these transformations.

Lewbel (1988, 1990) finds the indirect preferences for the trigonometric full rank three nominal income Gorman system,

$$v(\boldsymbol{p},m) = \beta_2(\boldsymbol{p}) + \frac{\beta_3(\boldsymbol{p})\cos\left(\tau \ln(m/\beta_1(\boldsymbol{p}))\right)}{\left[1-\sin\left(\tau \ln(m/\beta_1(\boldsymbol{p}))\right)\right]}. \tag{12}$$

We use this expression to find the projective group transformation for this system.

Full Rank Gorman Systems as Projective Group Transformations

With this background, we present a complete characterization of all full rank nominal and deflated income Gorman systems in terms of projective group transformations.

Proposition 1: *Let* $\{w,\eta,\theta\}$ *satisfy lemma 1; let* $\pi : \mathcal{P} \to \mathbb{R}$, $\pi \in \mathcal{C}^\infty$, *be* 1° *homogeneous; let* $\alpha,\beta,\gamma,\delta : \mathcal{P} \to \mathbb{C}$, $\alpha,\beta,\gamma,\delta \in \mathcal{C}^\infty$, *be* 0° *homogeneous and satisfy* $\alpha(\boldsymbol{p})\delta(\boldsymbol{p}) - \beta(\boldsymbol{p})\gamma(\boldsymbol{p}) \equiv 1$; *let* $\lambda \in \mathbb{R}$; *and let* $f : \mathbb{R} \to \mathbb{C}$, $f \in \mathcal{C}^\infty$, $f' \neq 0$. *Then the expenditure function of any full rank Gorman system can be written as:*

Rank 1 $\dfrac{e(\boldsymbol{p},u)}{\pi(\boldsymbol{p})} = u$;

Rank 2 $f\left(\dfrac{e(\boldsymbol{p},u)}{\pi(\boldsymbol{p})}\right) = u + \beta(\boldsymbol{p})$ *if* $f(x) \neq \ln x$, *or*

$f\left(\dfrac{e(\boldsymbol{p},u)}{\pi(\boldsymbol{p})}\right) = \alpha(\boldsymbol{p})u$ *if* $f(x) = \ln x$;

Rank 3 $f\left(\dfrac{e(\boldsymbol{p},u)}{\pi(\boldsymbol{p})}\right) = \alpha(\boldsymbol{p})u + \beta(\boldsymbol{p})$ *if* $f(x) \notin \{\ln x, x^\kappa, x^{\iota\tau}\}$,

$f\left(\dfrac{e(\boldsymbol{p},u)}{\pi(\boldsymbol{p})}\right) = \dfrac{\alpha(\boldsymbol{p})u + \beta(\boldsymbol{p})}{\gamma(\boldsymbol{p})u + \delta(\boldsymbol{p})}$ *if* $f(x) \in \{\ln x, x^\kappa\}$ *and* $\theta(\eta) \equiv \lambda$,

or if $f(x) = x^{\iota\tau}$, *or*

---

gent infinite series can often be found (e.g., Boyce and DiPrima 1977, chapter 4).

$$f\left(\frac{e(\boldsymbol{p},u)}{\pi(\boldsymbol{p})}\right)=\frac{\alpha(\boldsymbol{p})w(\eta(\boldsymbol{p}),u)+\beta(\boldsymbol{p})}{\gamma(\boldsymbol{p})w(\eta(\boldsymbol{p}),u)+\delta(\boldsymbol{p})},\ if\ \ f(x)\in\{\ln x, x^{\kappa}\}\ \ and\ \ \theta'(\eta)\neq 0\,;$$

Rank 4 $$f\left(\frac{e(\boldsymbol{p},u)}{\pi(\boldsymbol{p})}\right)=\frac{\alpha(\boldsymbol{p})u+\beta(\boldsymbol{p})}{\gamma(\boldsymbol{p})u+\delta(\boldsymbol{p})}\ \ if\ \theta(\eta)\equiv\lambda\,,\ or$$

$$f\left(\frac{e(\boldsymbol{p},u)}{\pi(\boldsymbol{p})}\right)=\frac{\alpha(\boldsymbol{p})w(\eta(\boldsymbol{p}),u)+\beta(\boldsymbol{p})}{\gamma(\boldsymbol{p})w(\eta(\boldsymbol{p}),u)+\delta(\boldsymbol{p})}\ \ if\ \theta'(\eta)\neq 0\,,$$

with $f(x)\notin\{\ln x, x^{\kappa}, x^{\iota\tau}\}$ *in both of these two cases.*

The discussion in the previous section shows that the projective transformation group is closed over any sequence of compositions of transformations in the group. This fact illuminates the content of proposition 1. Define $\{\alpha,\beta,\gamma,\delta,\theta,\eta,\pi,f\}$ as in the proposition and extend the definition of the function $w:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ to

$$w(\eta(\boldsymbol{p}),u)=\begin{cases} u, & if\ \ \theta(\eta(\boldsymbol{p}))\equiv\lambda, \\ \int_0^{\eta(\boldsymbol{p})}\left[\theta(x)+w(x,u)^2\right]dx, & if\ \ \theta'(\eta(\boldsymbol{p}))\neq 0, \end{cases} \tag{13}$$

with initial conditions $w(0,u)=u$ and $\partial w(0,u)/\partial x=\theta(0)+u^2$ in the second case. Then

$$\partial w(\eta(\boldsymbol{p}),u)/\partial\boldsymbol{p}=\begin{cases} \boldsymbol{0}, & if\ \ \theta(\eta(\boldsymbol{p}))\equiv\lambda, \\ \left[\theta(\eta(\boldsymbol{p}))+w(\eta(\boldsymbol{p}),u)^2\right]\partial\eta(\boldsymbol{p})/\partial\boldsymbol{p}, & if\ \ \theta'(\eta(\boldsymbol{p}))\neq 0. \end{cases} \tag{14}$$

Define the projective group transformation from $w(\eta(\boldsymbol{p}),u)$ to $f(e(\boldsymbol{p},u)/\pi(\boldsymbol{p}))$ by

$$f\left(\frac{e(\boldsymbol{p},u)}{\pi(\boldsymbol{p})}\right)=\frac{\alpha(\boldsymbol{p})w(\eta(\boldsymbol{p}),u)+\beta(\boldsymbol{p})}{\gamma(\boldsymbol{p})w(\eta(\boldsymbol{p}),u)+\delta(\boldsymbol{p})}. \tag{15}$$

The inverse transformation is

$$w(\eta(\boldsymbol{p}),u)=\frac{\delta(\boldsymbol{p})f(e(\boldsymbol{p},u)/\pi(\boldsymbol{p}))-\beta(\boldsymbol{p})}{-\gamma(\boldsymbol{p})f(e(\boldsymbol{p},u)/\pi(\boldsymbol{p}))+\alpha(\boldsymbol{p})}, \tag{16}$$

and some straightforward algebra yields

$$\gamma(\boldsymbol{p})w(\eta(\boldsymbol{p}),u)+\delta(\boldsymbol{p})=\frac{1}{-\gamma(\boldsymbol{p})f(e(\boldsymbol{p},u)/\pi(\boldsymbol{p}))+\alpha(\boldsymbol{p})}. \tag{17}$$

Let subscripts denote partial derivatives, suppress the arguments of all functions for compactness, and make the substitution $m=e(\boldsymbol{p},u)$ throughout, to yield

$$
\begin{aligned}
f'\times\left(\frac{\boldsymbol{q}}{\pi}-\frac{m}{\pi^2}\pi_{\boldsymbol{p}}\right) &= \frac{(\alpha_{\boldsymbol{p}}w+\alpha w_{\boldsymbol{p}}+\beta_{\boldsymbol{p}})}{(\gamma w+\delta)}-\frac{(\alpha w+\beta)(\gamma_{\boldsymbol{p}}w+\gamma w_{\boldsymbol{p}}+\delta_{\boldsymbol{p}})}{(\gamma w+\delta)^2} \\
&= \frac{(\gamma\alpha_{\boldsymbol{p}}-\alpha\gamma_{\boldsymbol{p}})w^2+(\gamma\beta_{\boldsymbol{p}}-\beta\gamma_{\boldsymbol{p}}+\delta\alpha_{\boldsymbol{p}}-\alpha\delta_{\boldsymbol{p}})w+(\delta\beta_{\boldsymbol{p}}-\beta\delta_{\boldsymbol{p}})}{(\gamma w+\delta)^2} \\
&\quad + \frac{\left[(\alpha\gamma-\alpha\gamma)w+(\alpha\delta-\beta\gamma)\right]w_{\boldsymbol{p}}}{(\gamma w+\delta)^2}.
\end{aligned} \tag{18}
$$

The first term in parentheses inside of the square brackets on the last line of (18) vanishes identically. The second term in parentheses is identically one. Substitute (14), (16), and (17) into the right-hand side of (18) and rearrange terms to obtain:

$$
\boldsymbol{q}=\frac{m}{\pi}\pi_{\boldsymbol{p}}+\frac{\pi}{f'}\left[(\alpha\beta_{\boldsymbol{p}}-\beta\alpha_{\boldsymbol{p}})+(\beta\gamma_{\boldsymbol{p}}-\gamma\beta_{\boldsymbol{p}}+\delta\alpha_{\boldsymbol{p}}-\alpha\delta_{\boldsymbol{p}})f+(\gamma\delta_{\boldsymbol{p}}-\delta\gamma_{\boldsymbol{p}})f^2\right], \tag{19}
$$

if $\theta(\eta)\equiv\lambda$; and

$$
\begin{aligned}
\boldsymbol{q}=\frac{m}{\pi}\pi_{\boldsymbol{p}}&+\frac{\pi}{f'}\left[(\alpha\beta_{\boldsymbol{p}}-\beta\alpha_{\boldsymbol{p}})+(\beta\gamma_{\boldsymbol{p}}-\gamma\beta_{\boldsymbol{p}}+\delta\alpha_{\boldsymbol{p}}-\alpha\delta_{\boldsymbol{p}})f+(\gamma\delta_{\boldsymbol{p}}-\delta\gamma_{\boldsymbol{p}})f^2\right] \\
&+\frac{\pi}{f'}\eta_{\boldsymbol{p}}\left[(\delta f-\beta)^2+\theta(-\gamma f+\alpha)^2\right],
\end{aligned} \tag{20}
$$

if $\theta'(\eta)\neq 0$.

In both cases, the right-hand side is quadratic in *f*. Note that if $f(x)\in\{\ln x, x^{\kappa}, x^{\iota\tau}\}$, then $f(x)/f'(x)$ or $1/f'(x)$ is proportional to *x*. The first term on the right in (19) and (20) already is proportional to $x=m/\pi$, so that such a choice for *f* reduces the number of income terms by one. In other words, a deflated income system has a greater rank than an otherwise identical nominal income system if and only if *f* is *not* one of the functional forms found by Gorman. Moreover, every full rank deflated income Gorman system has the same group structure for indirect preferences as a nominal income Gorman system.

Rational Demand Systems of Any Rank

These developments show that the main difference between full rank deflated and nominal income Gorman systems is one of functional form, while Lie's theory of transformation groups limits the rank of the demand system generated by the projective group transformation on the right-hand side for all choices of *f* on the left-hand side. No added flexibility – in terms of the rank of the demand system – can be achieved by further manipulating the right-hand side. But nothing precludes further manipulating the left-hand side.

As discussed in previous sections of this paper, a common alternative to Gorman systems of Engel curves are rational demand systems, including the translog, MAIDS, GEF, fractional and rational rank four demand systems, and the reciprocal indirect generalized Leontief, minflex Laurent, and normalized quadratic demand systems. So far, these alternatives have at best achieved rank four, most are rank two, a few are rank three, and even fewer are full rank. From the known algebraic results on rational polynomials (e.g., Jorgenson 1966), one would expect that the flexibility of demand models would increase dramatically with the use of rational systems. Moreover, it appears desirable to nest and test for the functional form, rank, and aggregation properties of an empirical demand model, both for aggregate and micro-level data sets. The following allows one to take this to any level that may be desired. It also illustrates a tradeoff when seeking to increase the flexibility and rank of a demand system beyond rank four.

Let $\varphi_j : \mathbb{R} \to \mathbb{R}$, $\varphi_j \in \mathcal{C}^\infty$, $\varphi_j' \neq 0$, $j = 1, \cdots, J$, let $\pi_j : \mathcal{P} \to \mathbb{R}_+$, $\pi_j \in \mathcal{C}^\infty$, $j = 1, \cdots, J$, be 1° homogeneous, let $f : \mathbb{R} \to \mathbb{R}$, $f \in \mathcal{C}^\infty$, $f' \neq 0$, $f'' \neq 0$, and define[2]

$$y(\boldsymbol{p}, m) = f\left(\sum_{j=1}^{J} \varphi_j\left(m / \pi_j(\boldsymbol{p})\right)\right). \tag{21}$$

Define the expenditure function implicitly by the real projective group transformation,[3]

$$y(\boldsymbol{p}, e(\boldsymbol{p}, u)) = \frac{\alpha(\boldsymbol{p})u + \beta(\boldsymbol{p})}{\gamma(\boldsymbol{p})u + \delta(\boldsymbol{p})}, \tag{22}$$

with $\alpha, \beta, \gamma, \delta : \mathcal{P} \to \mathbb{R}$, but otherwise the same as in proposition 1.[4] Hotelling's Lemma

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[2] If $J \geq 2$, we can also let $\varphi_J(\pi_J(\boldsymbol{p}))$ be independent of *m*, where $\pi_J(\boldsymbol{p})$ is 0° homogeneous, although this may not result in a full rank system (see the discussion below for details). We restrict our attention to real projective group transformations. However, systems defined by a complex-valued *f* and appropriately restricted complex-valued $\alpha, \beta, \gamma, \delta : \mathbb{R} \to \mathbb{C}$ also are possible. For example, such systems could be used to generate rational Fourier series demand systems (Chalfant 1987; Chalfant and Gallant 1985; Elbadawi, Gallant, and Souza 1983; Gallant 1981, 1982, 1984; Piggott 2003).

[3] Equivalently, define the indirect utility function explicitly by the inverse projective transformation group,

$$v(\boldsymbol{p}, m) = \frac{\delta(\boldsymbol{p})y(\boldsymbol{p}, m) - \beta(\boldsymbol{p})}{-\gamma(\boldsymbol{p})y(\boldsymbol{p}, m) + \alpha(\boldsymbol{p})}.$$

One can impose properties on *f* and φ_j, $j = 1, \cdots, J$, so that *y* is increasing and convex in *m*. Monotonicity then implies $v_m = y_m \big/ (-\gamma y + \alpha)^2 > 0$. However, $v_{mm} = y_{mm} \big/ (-\gamma y + \alpha)^2 + 2\gamma y_m^2 \big/ (-\gamma y + \alpha)^3$, and there can be increasing, constant, or decreasing marginal utility of money even if *y* is increasing and convex in *m*. Such properties affect the region of economic regularity of the demand equations, but are not related to the rank of the system.

yields

$$\boldsymbol{q} = \frac{m\sum_{j=1}^{J}\varphi_j'\,\pi_{j\boldsymbol{p}}\big/\pi_j^2}{\sum_{j=1}^{J}\varphi_j'\big/\pi_j} + \frac{\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2 y + \boldsymbol{\alpha}_3 y^2}{f' \times \sum_{j=1}^{J}\varphi_j'\big/\pi_j}, \tag{23}$$

with $\boldsymbol{\alpha}_1 = \alpha\beta_{\boldsymbol{p}} - \beta\alpha_{\boldsymbol{p}}$, $\boldsymbol{\alpha}_2 = \beta\gamma_{\boldsymbol{p}} - \gamma\beta_{\boldsymbol{p}} + \delta\alpha_{\boldsymbol{p}} - \alpha\delta_{\boldsymbol{p}}$, and $\boldsymbol{\alpha}_3 = \gamma\delta_{\boldsymbol{p}} - \delta\gamma_{\boldsymbol{p}}$.

This demand system can attain full rank J+3 ≤ n. To see this, set

$$\varphi_j\big(m/\pi_j(\boldsymbol{p})\big) = (\phi_j\big/ j) \times \big(m/\pi_j(\boldsymbol{p})\big)^j, \quad j = 1, \cdots, J, \tag{24}$$

and $y = f(x) = \mathrm{e}^x$. Then (23) becomes

$$\boldsymbol{q} = m\left\{\frac{\sum_{j=1}^{J}\left[\phi_j (m/\pi_j)^j\,\pi_{j\boldsymbol{p}}/\pi_j\right] + \boldsymbol{\alpha}_1 y^{-1} + \boldsymbol{\alpha}_2 + \boldsymbol{\alpha}_3 y}{\sum_{j=1}^{J}\phi_j (m/\pi_j)^j}\right\}. \tag{25}$$

which is a J^{th}–order rational polynomial in deflated income plus the sum of three linearly independent terms in $\{y^{-1}, 1, y\}$. Other definitions of f yield similar results, as long as there is nonzero curvature. For example, $f(x) = x^{1/\kappa}$ and (24) yields

$$\boldsymbol{q} = m\left\{\frac{\sum_{j=1}^{J}\left[\phi_j (m/\pi_j)^j\,\pi_{j\boldsymbol{p}}/\pi_j\right] + \kappa y^{\kappa-1}\left(\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2 y + \boldsymbol{\alpha}_3 y^2\right)}{\sum_{j=1}^{J}\phi_j (m/\pi_j)^j}\right\}. \tag{26}$$

The tradeoff associated with increasing the rank of demand systems beyond four is that y becomes increasingly complex with each added term. Full rank also restricts the functional forms that can be chosen for $f(x)$. For example, the identity map (which is WLOG for a linear $f(x)$ since the multiplicative constant can be absorbed into the φ_j's) combined with the power terms (24) results in a rational polynomial in m with a reduced rank of at most J+2. This is so because (23) then reduces to

[4] We could extend the definition of rational full rank demand systems to include $w(\eta(\boldsymbol{p}), u)$ as defined in (14) and a non-constant $\theta(\eta)$. Lewbel (1987a, p. 1454) states, "… this complicates the demand equations while adding nothing to either income or price flexibility, so demands with [$\theta(\eta) \neq \lambda$] are not likely to be of much practical interest." He repeats this sentiment in Lewbel (1990, p.292) and we agree with his assessment. Nevertheless, the complete set of rational demand systems in this class would include this case.

$$\boldsymbol{q} = m\left\{\frac{\boldsymbol{\alpha}_1 + \sum_{j=1}^{J}\phi_j\left[\left(\pi_{j\boldsymbol{p}}/\pi_j\right) + \left(\boldsymbol{\alpha}_2/j\right)\right](m/\pi_j)^j}{\sum_{j=1}^{J}\phi_j(m/\pi_j)^j} + \frac{\boldsymbol{\alpha}_3\sum_{j=1}^{J}\sum_{k=1}^{J}(\phi_j\phi_k/jk)\,m^{j+k}/\pi_j^j\pi_k^k}{\sum_{j=1}^{J}\phi_j(m/\pi_j)^j}\right\}. \tag{27}$$

The polynomial in the numerator on the first line of (27) has J+1 linearly independent price vectors, but the polynomial in the numerator on the second line has only one, $\boldsymbol{\alpha}_3$, with ½J(J+1) total income terms. We can replace m/π_j in (24) with $\ln(m/\pi_j)$ to generate a rational polynomial in log-income, but this will again have a reduced rank when f has no curvature.

We conclude this section by showing that the translog (Christenson, Jorgenson, and Lau 1975), MAIDS and GEF (Cooper and McLaren 1992, 1996), fractional and rational rank four demand systems (Lewbel 1987b, 2003, 2004), reciprocal indirect generalized Leontief and minflex Laurent demand systems (Barnett and Lee 1985; Barnett, Lee and Wolfe 1985), and reciprocal indirect normalized quadratic (Diewert and Wales 1988) are all members this class of demand systems. As a consequence, nearly all commonly used demand models are encompassed by the class of rational demand systems presented above.[5]

Translog

$$v(\boldsymbol{p},m) = \boldsymbol{\alpha}^{\mathsf{T}}\ln(\boldsymbol{p}/m) + ½\ln(\boldsymbol{p}/m)^{\mathsf{T}}\boldsymbol{B}\ln(\boldsymbol{p}/m)\,. \tag{28}$$

The restrictions $\boldsymbol{i}^{\mathsf{T}}\boldsymbol{\alpha} = -1$ and $\boldsymbol{i}^{\mathsf{T}}\boldsymbol{B}\boldsymbol{i} = 0$ are typically imposed for identification and exact aggregation. Set $v(\boldsymbol{p},m) = u$, $m = e(\boldsymbol{p},u)$, impose these restrictions and solve for the log-expenditure function as

$$\ln e(\boldsymbol{p},u) = \frac{u - \boldsymbol{\alpha}^{\mathsf{T}}\ln\boldsymbol{p} - ½\ln\boldsymbol{p}^{\mathsf{T}}\boldsymbol{B}\ln\boldsymbol{p}}{1 - \boldsymbol{i}^{\mathsf{T}}\boldsymbol{B}\ln\boldsymbol{p}}\,. \tag{29}$$

---

[5] For brevity, we omit many Gorman polar forms – which are clearly rank two members of this class – such as the normalized quadratic expenditure function (Diewert and Wales 1988), and the translog, generalized Leontief, and generalized McFadden cost functions (Diewert and Wales 1987). Of course, extensions of these models that include both a linear and a quadratic term in output or log-output are not members of the class of rational demand systems developed in this paper.

Since $\prod_{i=1}^{n} p_i^{-\alpha_i} \equiv \pi(\boldsymbol{p})$ is 1° homogeneous, add $\left(\boldsymbol{i}^\top \boldsymbol{B} \ln \boldsymbol{p} - \boldsymbol{i}^\top \boldsymbol{B} \ln \boldsymbol{p}\right) \boldsymbol{\alpha}^\top \ln \boldsymbol{p}$ to the numerator, and move the term $\boldsymbol{\alpha}^\top \ln \boldsymbol{p} = -\ln \pi(\boldsymbol{p})$ to the left-hand side,

$$\ln\left(\frac{e(\boldsymbol{p},u)}{\pi(\boldsymbol{p})}\right) = \frac{u - ½ \ln \boldsymbol{p}^\top \boldsymbol{B} \ln \boldsymbol{p} - \boldsymbol{i}^\top \boldsymbol{B} \ln \boldsymbol{p}\left(\boldsymbol{\alpha}^\top \ln \boldsymbol{p}\right)}{1 - \boldsymbol{i}^\top \boldsymbol{B}(\ln \boldsymbol{p})}. \tag{30}$$

Define $\alpha(\boldsymbol{p}) = 1\Big/\sqrt{1 - \boldsymbol{i}^\top \boldsymbol{B}(\ln \boldsymbol{p})}$,

$\beta(\boldsymbol{p}) = -\left[½ \ln \boldsymbol{p}^\top \boldsymbol{B} \ln \boldsymbol{p} - \boldsymbol{i}^\top \boldsymbol{B} \ln \boldsymbol{p}\left(\boldsymbol{\alpha}^\top \ln \boldsymbol{p}\right)\right]\Big/\sqrt{1 - \boldsymbol{i}^\top \boldsymbol{B}(\ln \boldsymbol{p})}$,

$\gamma(\boldsymbol{p}) = 0$, and

$\delta(\boldsymbol{p}) = \sqrt{1 - \boldsymbol{i}^\top \boldsymbol{B}(\ln \boldsymbol{p})}$, to obtain the desired projective group format.[6]

If the translog indirect utility function is not exactly aggregable, it is convenient to apply a different normalization for identification, $\boldsymbol{i}^\top \boldsymbol{B} \boldsymbol{i} = 1$, to rewrite (28) as

$$(\ln e(\boldsymbol{p},u))^2 + 2(\boldsymbol{i}^\top \boldsymbol{\alpha} - \boldsymbol{i}^\top \boldsymbol{B} \ln \boldsymbol{p}) \ln e(\boldsymbol{p},u) + 2\boldsymbol{\alpha}^\top \ln \boldsymbol{p} + \ln \boldsymbol{p}^\top \boldsymbol{B} \ln \boldsymbol{p} - 2u = 0. \tag{31}$$

This can be solved for the log-expenditure function as

$$\ln e(\boldsymbol{p},u) = \boldsymbol{i}^\top \boldsymbol{B} \ln \boldsymbol{p} - \boldsymbol{i}^\top \boldsymbol{\alpha} \pm \sqrt{(\boldsymbol{i}^\top \boldsymbol{B} \ln \boldsymbol{p} - \boldsymbol{i}^\top \boldsymbol{\alpha})^2 + 2(u - \boldsymbol{\alpha}^\top \ln \boldsymbol{p}) - \ln \boldsymbol{p}^\top \boldsymbol{B} \ln \boldsymbol{p}}. \tag{32}$$

Since $\pi(\boldsymbol{p}) \equiv \mathrm{e}^{-\Sigma_{i=1}^{n}\alpha_i} \prod_{i=1}^{n} p_i^{\Sigma_{j=1}^{n} b_{ij}}$ is 1° homogeneous, subtract $\ln \pi(\boldsymbol{p})$ from both sides of (32) and square the result to obtain,

$$½\left[\ln\left(\frac{e(\boldsymbol{p},u)}{\pi(\boldsymbol{p})}\right)\right]^2 = u - \boldsymbol{\alpha}^\top \ln \boldsymbol{p} - ½ \ln \boldsymbol{p}^\top \boldsymbol{B} \ln \boldsymbol{p} + ½(\boldsymbol{i}^\top \boldsymbol{B} \ln \boldsymbol{p} - \boldsymbol{i}^\top \boldsymbol{\alpha})^2. \tag{33}$$

MAIDS

The indirect utility function for the MAIDS (Cooper and McLaren 1992) is

$$v(\boldsymbol{p},m) = \ln\left(\frac{m}{\pi_1(\boldsymbol{p})}\right)\Bigg/\left(\frac{m}{\pi_2(\boldsymbol{p})}\right)^\eta, \tag{34}$$

where $\ln \pi_1(\boldsymbol{p}) = \kappa + \boldsymbol{\alpha}^\top \ln \boldsymbol{p} + ½(\ln \boldsymbol{p})^\top \boldsymbol{C} \ln \boldsymbol{p}$, $\ln \pi_2(\boldsymbol{p}) = (\boldsymbol{\beta}^\top \ln \boldsymbol{p})/\eta$, $\boldsymbol{i}^\top \boldsymbol{\alpha} = 1$, $\boldsymbol{C} = \boldsymbol{C}^\top$, $\Sigma_{j=1}^{n} c_{ij} = 0,\ i = 1,\cdots,n$, and $\boldsymbol{i}^\top \boldsymbol{\beta} = \eta \in [0,1]$. Taking the log of both sides yields

$$\ln\left[\ln\left(\frac{e(\boldsymbol{p},u)}{\pi_1(\boldsymbol{p})}\right)\right] - \eta \ln\left(\frac{e(\boldsymbol{p},u)}{\pi_2(\boldsymbol{p})}\right) = u, \tag{35}$$

---

[6] Nearly identical steps put the nested AIDS and exactly aggregable translog model of Lewbel (1989b) in the same format, so we also omit this case.

with the normalization $\ln v(\boldsymbol{p},m)=u$.

GEF

The indirect utility function for the GEF (Cooper and McLaren 1996) is

$$v(\boldsymbol{p},m)=\left[\frac{\left(m/\pi_1(\boldsymbol{p})\right)^{\mu}-1}{\mu}\right]\left(\frac{m}{\pi_2(\boldsymbol{p})}\right)^{\eta}, \tag{36}$$

where $\pi_1,\pi_2:\mathscr{P}\rightarrow\mathbb{R}_{++}$ are 1° homogeneous, $\mu\geq-1$, and $\eta\in[0,1]$.[7] Taking the log of both sides then gives

$$\ln\left[\left(\frac{e(\boldsymbol{p},u)}{\pi_1(\boldsymbol{p})}\right)^{\mu}-1\right]+\eta\ln\left(\frac{e(\boldsymbol{p},u)}{\pi_2(\boldsymbol{p})}\right)=u\,, \tag{37}$$

with the normalization $\ln\left(\mu v(\boldsymbol{p},m)\right)=u$.

Fractional Demand Systems

The indirect utility functions for fractional demand systems that are presented in Lewbel (1987b) include homothetic, PIGL, PIGLOG, and translog demand systems – which have already been analyzed – plus three new ones:

1. LOG2 $v(\boldsymbol{p},m)=[A(\boldsymbol{p})+\ln m-1]m\mathrm{e}^{-B(\boldsymbol{p})}$;
2. EXP $v(\boldsymbol{p},m)=\left[m+A(\boldsymbol{p})m^{1+\kappa}\right]e^{-B(\boldsymbol{p})}$; and
3. TAN $v(\boldsymbol{p},m)=A(\boldsymbol{p})\sin(\tau\ln m)+B(\boldsymbol{p})\cos(\tau\ln m)$.

To put each of the new cases in the form of the above class of rational demand systems, we must impose the associated homogeneity conditions on the price functions.

For the LOG2 indirect utility function, 0° homogeneity requires $A(\boldsymbol{p})=-\ln\pi_1(\boldsymbol{p})$ and $\mathrm{e}^{B(\boldsymbol{p})}=\pi_2(\boldsymbol{p})$, where $\pi_1(\boldsymbol{p})$ and $\pi_2(\boldsymbol{p})$ are 1° homogeneous. This model can then be written in terms of the expenditure function as

$$\ln\left[\left(\frac{e(\boldsymbol{p},u)}{\pi_1(\boldsymbol{p})}\right)-1\right]+\ln\left(\frac{e(\boldsymbol{p},u)}{\pi_2(\boldsymbol{p})}\right)=u\,, \tag{38}$$

with the normalization $\ln v(\boldsymbol{p},m)=u$.

---

[7] Cooper and McLaren (1996) include a parameter, $\kappa\in[0,1]$ as a multiplicative scalar on $\pi_1(\boldsymbol{p})$ in the definition of the GEF. This parameter can not be identifed in empirical applications and has no effect on the structure of the indirect utility function. Hence, we omit it for compactness.

For the EXP indirect utility function, 0° homogeneity requires that $e^{B(\boldsymbol{p})} = \pi_1(\boldsymbol{p})$ and $A(\boldsymbol{p})e^{-B(\boldsymbol{p})} = \pi_2(\boldsymbol{p})^{-(1+\kappa)}$, where again $\pi_1(\boldsymbol{p})$ and $\pi_2(\boldsymbol{p})$ are 1° homogeneous. This model can then be written in terms of the expenditure function as

$$\left(\frac{e(\boldsymbol{p},u)}{\pi_1(\boldsymbol{p})}\right) + \left(\frac{e(\boldsymbol{p},u)}{\pi_1(\boldsymbol{p})}\right)^{\kappa+1} = u\,, \tag{39}$$

with the normalization $v(\boldsymbol{p},m) = u$.

For the TAN indirect utility function, Euler's equation for 0° homogeneity implies that two functions, $\pi_1 : \mathscr{P} \to \mathbb{R}_+$, $\pi_2 : \mathscr{P} \to \mathbb{R}$, exist such that $\pi_1(\boldsymbol{p})$ is 1° homogeneous, $\pi_2(\boldsymbol{p})$ is 0° homogeneous, and

$$v(\boldsymbol{p},m) = \pi_2(\boldsymbol{p})\left[\sin\left(\tau \ln\left(m/\pi_1(\boldsymbol{p})\right)\right) + \cos\left(\tau \ln\left(m/\pi_1(\boldsymbol{p})\right)\right)\right]. \tag{40}$$

Set $\alpha(\boldsymbol{p}) = \pi_2(\boldsymbol{p})^{-1}$ to write the TAN model in terms of the expenditure function as

$$\sin\left(\tau \ln\left(\frac{e(\boldsymbol{p},u)}{\pi_1(\boldsymbol{p})}\right)\right) + \cos\left(\tau \ln\left(\frac{e(\boldsymbol{p},u)}{\pi_1(\boldsymbol{p})}\right)\right) = \alpha(\boldsymbol{p})u\,. \tag{41}$$

Reciprocal Generalized Leontief and Minflex Laurent

The generalized Leontief reciprocal indirect utility function (Barnett and Lee 1985; and Barnett, Lee, and Wolfe 1985) is

$$\frac{1}{v(\boldsymbol{p},m)} = a_0 + \sum_{i=1}^{n} a_i \sqrt{p_i/m} + \sum_{i=1}^{m}\sum_{j=1}^{n} b_{ij} \sqrt{p_i p_j}\Big/m\,. \tag{42}$$

Define $\pi_1(\boldsymbol{p}) = \left[\sum_{i=1}^{n} a_i \sqrt{p_i}\right]^2$ and $\pi_2(\boldsymbol{p}) = \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}\sqrt{p_i p_j}$, so that we have

$$-\left(\frac{e(\boldsymbol{p},u)}{\pi_1(\boldsymbol{p})}\right)^{-½} - \left(\frac{e(\boldsymbol{p},u)}{\pi_2(\boldsymbol{p})}\right)^{-1} = \frac{a_0 u - 1}{u}\,. \tag{43}$$

Similarly, dropping the exponents on all coefficients in equation (3.1) of Barnett and Lee (1985), the minflex Laurent reciprocal indirect utility function is

$$\begin{aligned}\frac{1}{v(\boldsymbol{p},m)} &= a_0 + \sum_{i=1}^{n} a_i \sqrt{p_i/m} + \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij} \sqrt{p_i p_j}\Big/m \\ &+ \sum_{i=1}^{m} c_i\; p_i/m - \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}\; m\Big/\sqrt{p_i p_j}\,.\end{aligned} \tag{44}$$

The same definitions for $\pi_1(\boldsymbol{p})$ and $\pi_2(\boldsymbol{p})$, plus the new definitions $\pi_3(\boldsymbol{p}) = \sum_{i=1}^{n} c_i p_i$ and $\pi_4(\boldsymbol{p})^{-1} = \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij} \Big/ \sqrt{p_i p_j}$, imply that

$$-\left(\frac{e(\boldsymbol{p},u)}{\pi_1(\boldsymbol{p})}\right)^{-½} - \left(\frac{e(\boldsymbol{p},u)}{\pi_2(\boldsymbol{p})}\right)^{-1} - \left(\frac{e(\boldsymbol{p},u)}{\pi_3(\boldsymbol{p})}\right)^{-1} + \left(\frac{e(\boldsymbol{p},u)}{\pi_4(\boldsymbol{p})}\right) = \frac{a_0 u - 1}{u}. \qquad (45)$$

The linear dependence of the middle two terms on the left-hand side implies that this is a rank three system. This is verified by inspection of the budget shares in Barnett, Lee and Wolfe (1985, p. 8, equation (3.4)) and Barnett and Lee (1985, p. 1423). A class of up to rank six systems extending (43) and (45) can be obtained by including four unique powers of $\sqrt{m}$, with $\pi_1, \pi_2, \pi_3, \pi_4 : \mathscr{P} \to \mathbb{R}$ arbitrary, independent, 1° homogeneous price indices, and a projective group transformation on the right,

$$-\left(\frac{e(\boldsymbol{p},u)}{\pi_1(\boldsymbol{p})}\right)^{-1} - \left(\frac{e(\boldsymbol{p},u)}{\pi_2(\boldsymbol{p})}\right)^{-½} + \left(\frac{e(\boldsymbol{p},u)}{\pi_3(\boldsymbol{p})}\right)^{½} + \left(\frac{e(\boldsymbol{p},u)}{\pi_4(\boldsymbol{p})}\right) = \frac{\alpha(\boldsymbol{p})u + \beta(\boldsymbol{p})}{\gamma(\boldsymbol{p})u + \delta(\boldsymbol{p})}. \qquad (46)$$

This system has a maximum rank of six, rather than full rank seven, because the vector of price functions introduced by the term $m/\pi_4(\boldsymbol{p})$ on the left is linearly dependent with the vector of price functions associated with the constant (in *y*) function introduced by the projective group transformation on the right.

Reciprocal Indirect Normalized Quadratic

The reciprocal normalized quadratic indirect utility function is (Diewert and Wales 1988)

$$\frac{1}{v(\boldsymbol{p},m)} = b_o + \left(\frac{\boldsymbol{b}^{\mathsf{T}}\boldsymbol{p}}{m}\right) + ½\left(\frac{\boldsymbol{p}^{\mathsf{T}}\boldsymbol{B}\boldsymbol{p}/m^2}{\boldsymbol{\alpha}^{\mathsf{T}}\boldsymbol{p}/m}\right) + \boldsymbol{a}^{\mathsf{T}} \ln\left(\frac{\boldsymbol{p}}{m}\right). \qquad (47)$$

Since $\pi_1(\boldsymbol{p}) = \boldsymbol{b}^{\mathsf{T}}\boldsymbol{p}$, $\pi_2(\boldsymbol{p}) = ½\, \boldsymbol{p}^{\mathsf{T}}\boldsymbol{B}\boldsymbol{p}/\boldsymbol{\alpha}^{\mathsf{T}}\boldsymbol{p}$, and $\pi_3(\boldsymbol{p}) \equiv \prod_{i=1}^{n} p_i^{(a_i/\boldsymbol{a}^{\mathsf{T}}\boldsymbol{i})}$ are all 1° homogeneous, we can rewrite this in terms of the expenditure function as

$$-\left(\frac{e(\boldsymbol{p},u)}{\pi_1(\boldsymbol{p})}\right)^{-1} - \left(\frac{e(\boldsymbol{p},u)}{\pi_2(\boldsymbol{p})}\right)^{-1} + \boldsymbol{a}^{\mathsf{T}}\boldsymbol{i}\ln\left(\frac{e(\boldsymbol{p},u)}{\pi_3(\boldsymbol{p})}\right) = \frac{b_0 u - 1}{u}. \qquad (48)$$

Again, as can be verified by inspecting equation (7) in Diewert and Wales (1988), this is a rank two rational demand system due to the linear dependence of the first and second income functions on the left-hand side. A class of up to rank four systems of this general form is obtained with $\pi_1, \pi_2 : \mathscr{P} \to \mathbb{R}$ arbitrary, independent, 1° homogeneous price indices and a projective group transformation on the right,

$$-\left(\frac{e(\boldsymbol{p},u)}{\pi_1(\boldsymbol{p})}\right)^{-1}+\ln\left(\frac{e(\boldsymbol{p},u)}{\pi_2(\boldsymbol{p})}\right)=\frac{\alpha(\boldsymbol{p})u+\beta(\boldsymbol{p})}{\gamma(\boldsymbol{p})u+\delta(\boldsymbol{p})}. \tag{49}$$

This system has at most rank four, rather than full rank five, because the vector of price functions introduced by the term $\ln(m/\pi_2(\boldsymbol{p}))$ from the left-hand side is linearly dependent with the vector of price functions associated with the constant (in y) function introduced by the projective group transformation on the right-hand side.

The Rational Rank Four Demand System

The reciprocal indirect utility function for the rational rank four demand system (Lewbel 2003, 2004) is

$$\frac{1}{v(\boldsymbol{p},m)}=\left(\frac{\ln\left[m-a(\boldsymbol{p})\right]-b(\boldsymbol{p})}{c(\boldsymbol{p})}\right)^{-1}+d(\boldsymbol{p}). \tag{50}$$

Homogeneity requires that $a(\boldsymbol{p})$ is 1° homogeneous, $b(\boldsymbol{p})=\ln\pi_1(\boldsymbol{p})$, where $\pi_1(\boldsymbol{p})$ is 1° homogeneous, and $c(\boldsymbol{p}),d(\boldsymbol{p})$ are 0° homogeneous. Therefore, rewrite (50) as

$$\ln\left(\frac{e(\boldsymbol{p},u)-a(\boldsymbol{p})}{\pi_1(\boldsymbol{p})}\right)=\frac{c(\boldsymbol{p})u}{-d(\boldsymbol{p})u+1}=\frac{\sqrt{c(\boldsymbol{p})}u}{\left(-d(\boldsymbol{p})\big/\sqrt{c(\boldsymbol{p})}\right)u+\left(1\big/\sqrt{c(\boldsymbol{p})}\right)}, \tag{51}$$

so that $\alpha(\boldsymbol{p})=\sqrt{c(\boldsymbol{p})}$, $\beta(\boldsymbol{p})=0$, $\gamma(\boldsymbol{p})=-d(\boldsymbol{p})\big/\sqrt{c(\boldsymbol{p})}$, and $\delta(\boldsymbol{p})=1\big/\sqrt{c(\boldsymbol{p})}$ gives the projective group transformation on the right-hand side, while $f(x)=\ln x$, $\varphi_1(x)=x$, and $\varphi_2(\boldsymbol{p})=-a(\boldsymbol{p})/\pi_1(\boldsymbol{p})$ is independent of income and 0° homogeneous in $\boldsymbol{p}$ on the left-hand side.

This generates a rank four demand system because $\{1,x\}$ are linearly independent and $f(x)=\ln x$ has nonzero curvature, which combine to produce two linearly independent terms from the left-hand side, while the right-hand side has two independent price indices that define the projective group representation. A general class of rank four rational demand systems of this form is

$$\ln\left(\frac{e(\boldsymbol{p},u)-\pi_1(\boldsymbol{p})}{\pi_2(\boldsymbol{p})}\right)=\frac{\alpha(\boldsymbol{p})u+\beta(\boldsymbol{p})}{\gamma(\boldsymbol{p})u+\delta(\boldsymbol{p})}, \tag{52}$$

with demand equations given by

$$
\begin{aligned}
\boldsymbol{q} = \left(\pi_{1\boldsymbol{p}} - \pi_1 \boldsymbol{\alpha}_1\right) + \left(\frac{\pi_{2\boldsymbol{p}}}{\pi_2} + \boldsymbol{\alpha}_1\right) m \\
+ \boldsymbol{\alpha}_2 \left(m - \pi_1\right) \ln\left(\frac{m - \pi_1}{\pi_2}\right) + \boldsymbol{\alpha}_3 \left(m - \pi_1\right)\left(\ln\left(\frac{m - \pi_1}{\pi_2}\right)\right)^2 .
\end{aligned}
\tag{53}
$$

This is a rational demand system with budget shares on the left, and a simple quadratic polynomial in $\ln(m - \pi_1)$ with quantities or expenditures as dependent variables. The rank is at most four because the two terms introduced from the left in (52) span $\boldsymbol{\alpha}_1(m - \pi_1)$ from the projective group transformation on the right. Also, there are a total of six additively separable, independent income functions in the demand system,

$$
\left\{1, m, \ln\left((m - \pi_1)/\pi_2\right), m \ln\left((m - \pi_1)/\pi_2\right), \left[\ln\left((m - \pi_1)/\pi_2\right)\right]^2, m\left[\ln\left((m - \pi_1)/\pi_2\right)\right]^2\right\}.
$$

The choices $f(x) = \ln(x)$ and $\varphi_2(\pi_1(\boldsymbol{p})/\pi_2(\boldsymbol{p}))$, independent of *m*, therefore combine to reduce the rank of this demand system from six to a maximum of four.

Thus, in every case, the choice of the functional form for *f* and φ_j, $j = 1, \cdots, J$, determines the maximum rank of the demand system.

Conclusions

For several decades the pioneering conceptual work of Gorman (1953, 1961), Muellbauer (1975, 1976) and the later salient developments in Gorman (1981), have guided empirical analysis of demand systems. Gorman, with subsequent results by Lewbel (1987, 1989a, 1990) defined rank such that it is now commonly used to describe the flexibility of demand systems. Among Gorman's remarkable insights was that integrability substantially constrains the rank of a demand system. This paper synthesizes and extends work initiated by Russell (1983) using projective group transformations. After characterizing all full rank Gorman systems, a key insight is developed: all full rank nominal income Gorman systems as well as extended deflated income Gorman systems (Lewbel 1989a) can be represented as a projective group transformation of a specific function of deflated income. We develop rational demand systems with arbitrary rank by generalizing the dependent variable in such a projective group transformation, maintaining a one-to-one relationship between the number of independent price indices and rank of the system, yet encompassing virtually all extant applied demand models. This increases the flexibility to model demand behavior and facilitates testing for aggregability and the rank of a demand system (Lewbel 1991; Cragg and Donald 1997).

The primary focus of this paper is on full rank demand systems. Full rank models are parsimonious in the number of unknown parameters for a given level of flexibility, and as a result are of primary interest in econometric applications. It is easy to construct high rank demand systems that are not aggregable, but more difficult to obtain high full rank systems. One reasonable modeling strategy, therefore, would be to test Engel curve data for its rank, and if it is above rank four, implement a parsimonious rational full rank demand system in the class developed here.[8]

APPENDIX

Representation Algebra

To motivate the statement and proof of Lemma 1, we first show how to express any full rank three QES, generalized PIGL, or generalized PIGLOG nominal income Gorman system in the form

$$\frac{\partial}{\partial \boldsymbol{p}}\left(\frac{f(e(\boldsymbol{p},u))-\beta_1(\boldsymbol{p})}{\beta_3(\boldsymbol{p})}\right)=\left[\left(\frac{f(e(\boldsymbol{p},u))-\beta_1(\boldsymbol{p})}{\beta_3(\boldsymbol{p})}\right)^2+\theta(\beta_2(\boldsymbol{p}))\right]\frac{\partial \beta_2(\boldsymbol{p})}{\partial \boldsymbol{p}}. \qquad \text{(A.1)}$$

Throughout this section, subscripts denote partial derivatives, we use notation consistent with (A.1) to replace the corresponding notation in the original articles, and often omit the arguments of functions for compactness.

In van Daal and Merkies (1989), equation (2), group terms in $\beta_3(\boldsymbol{p})^{-1}$:

$$\boldsymbol{q}=\beta_3^{-1}\left(m^2\beta_{2\boldsymbol{p}}+m\beta_{3\boldsymbol{p}}-2m\beta_1\beta_{2\boldsymbol{p}}+\beta_1^2\beta_{2\boldsymbol{p}}-\beta_1\beta_{3\boldsymbol{p}}\right)+\beta_{1\boldsymbol{p}}+\theta(\beta_2)\beta_3\beta_{2\boldsymbol{p}}. \qquad \text{(A.2)}$$

where $\boldsymbol{p}$ subscripts denote differentiation. Regroup terms in the parentheses:

$$\boldsymbol{q}=\beta_3^{-1}\left[(m-\beta_1)^2\beta_{2\boldsymbol{p}}+(m-\beta_1)\beta_{3\boldsymbol{p}}\right]+\beta_{1\boldsymbol{p}}+\theta(\beta_2)\beta_3\beta_{2\boldsymbol{p}}. \qquad \text{(A.3)}$$

Gather terms in $\beta_{2\boldsymbol{p}}$, divide both sides by β_3, and isolate $\beta_{2\boldsymbol{p}}$ on the right:

$$\frac{\boldsymbol{q}-\beta_{1\boldsymbol{p}}}{\beta_3}-\frac{(m-\beta_1)\beta_{3\boldsymbol{p}}}{\beta_3^2}=\left[\left(\frac{m-\beta_1}{\beta_3}\right)^2+\theta(\beta_2)\right]\beta_{2\boldsymbol{p}}. \qquad \text{(A.4)}$$

To obtain (A.1), note that

[8] We are gratefully indebted to Arthur Lewbel for emphasizing each of the points in this final paragraph.

$$\frac{\partial}{\partial \boldsymbol{p}}\left(\frac{e(\boldsymbol{p},u)-\beta_1}{\beta_3}\right)=\frac{(\boldsymbol{q}-\beta_{1\boldsymbol{p}})}{\beta_3}-\frac{(m-\beta_1)\beta_{3\boldsymbol{p}}}{\beta_3^2}. \tag{A.5}$$

In Lewbel (1990), case iv, move $\tau m^{\tau-1}$ to the left-hand side, define $\tilde{\beta}_3(\boldsymbol{p})\equiv\beta_3(\boldsymbol{p})^{1/\tau}$ and $\tilde{\beta}_1(\boldsymbol{p})\equiv\beta_1(\boldsymbol{p})/\beta_3(\boldsymbol{p})$:

$$\begin{aligned}\tau m^{\tau-1}\boldsymbol{q}&=\tilde{\beta}_3^{\tau}\tilde{\beta}_{1\boldsymbol{p}}+\tilde{\beta}_1^2\tilde{\beta}_3^{\tau}\beta_{2\boldsymbol{p}}+\theta(\beta_2)\tilde{\beta}_3^{\tau}\beta_{2\boldsymbol{p}}\\&\quad+\left(\frac{\tau\tilde{\beta}_{3\boldsymbol{p}}}{\tilde{\beta}_3}-2\tilde{\beta}_1\beta_{2\boldsymbol{p}}\right)m^{\tau}+\frac{\beta_{2\boldsymbol{p}}}{\tilde{\beta}_3^{\tau}}m^{2\tau}.\end{aligned} \tag{A.6}$$

Group terms in $\beta_{2\boldsymbol{p}}$ and $\tilde{\beta}_3^{\tau}$:

$$\begin{aligned}\tau m^{\tau-1}\boldsymbol{q}&=\tilde{\beta}_3^{\tau}\left\{\left[\left(\frac{m^{2\tau}}{\tilde{\beta}_3^{2\tau}}\right)-2\tilde{\beta}_1\left(\frac{m^{\tau}}{\tilde{\beta}_3^{\tau}}\right)+\tilde{\beta}_1^2+\theta(\beta_2))\right]\beta_{2\boldsymbol{p}}+\tilde{\beta}_{1\boldsymbol{p}}\right\}+\tau\frac{\tilde{\beta}_{3\boldsymbol{p}}}{\tilde{\beta}_3}m^{\tau}\\&=\tilde{\beta}_3^{\tau}\left\{\left[\left(\frac{m^{\tau}-\tilde{\beta}_1\tilde{\beta}_3^{\tau}}{\tilde{\beta}_3^{\tau}}\right)^2+\theta(\beta_2)\right]\beta_{2\boldsymbol{p}}+\tilde{\beta}_{1\boldsymbol{p}}\right\}+\tau\frac{\tilde{\beta}_{3\boldsymbol{p}}}{\tilde{\beta}_3}m^{\tau}.\end{aligned} \tag{A.7}$$

Isolate the terms in $\beta_{2\boldsymbol{p}}$ on the right:

$$\frac{\tau m^{\tau-1}\boldsymbol{q}}{\tilde{\beta}_3^{\tau}}-\frac{\tau m^{\tau}\tilde{\beta}_{3\boldsymbol{p}}}{\tilde{\beta}_3^{\tau+1}}-\tilde{\beta}_{1\boldsymbol{p}}=\left[\left(\frac{m^{\tau}-\tilde{\beta}_1\tilde{\beta}_3^{\tau}}{\tilde{\beta}_3^{\tau}}\right)^2+\theta(\beta_2)\right]\beta_{2\boldsymbol{p}}. \tag{A.8}$$

Note that

$$\frac{\partial}{\partial \boldsymbol{p}}\left(\frac{e(\boldsymbol{p},u)^{\tau}}{\tilde{\beta}_3^{\tau}}-\tilde{\beta}_1\right)=\frac{\tau e(\boldsymbol{p},u)^{\tau-1}\boldsymbol{q}}{\tilde{\beta}_3^{\tau}}-\frac{\tau e(\boldsymbol{p},u)^{\tau}\tilde{\beta}_{3\boldsymbol{p}}}{\tilde{\beta}_3^{\tau+1}}-\tilde{\beta}_{1\boldsymbol{p}}. \tag{A.9}$$

Substitute $\beta_1(\boldsymbol{p})\equiv\tilde{\beta}_1(\boldsymbol{p})\tilde{\beta}_3(\boldsymbol{p})^{\tau}$ and $\beta_3(\boldsymbol{p})\equiv\tilde{\beta}_3(\boldsymbol{p})^{\tau}$ into (A.8) and (A.9) to get (A.1).

In Lewbel (1990), case v, fix a sign error and a typographical error on the right (see Lewbel 1990, p.297) and move $1/m$ to the left-hand side:

$$\begin{aligned}\frac{\boldsymbol{q}}{m}&=\frac{\beta_{2\boldsymbol{p}}}{\beta_3}(\ln\beta_1)^2-\frac{\beta_{3\boldsymbol{p}}}{\beta_3}\ln\beta_1+\frac{\beta_{1\boldsymbol{p}}}{\beta_1}+\theta(\beta_2)\beta_3\beta_{2\boldsymbol{p}}\\&\quad+\left(\frac{\beta_{3\boldsymbol{p}}-2\beta_{2\boldsymbol{p}}\ln\beta_1}{\beta_3}\right)\ln m+\frac{\beta_{2\boldsymbol{p}}}{\beta_3}(\ln m)^2.\end{aligned} \tag{A.10}$$

Group terms in $\beta_{2\boldsymbol{p}}$:

$$\frac{\boldsymbol{q}}{m} = \beta_3\left[\left(\frac{\ln(m/\beta_1)}{\beta_3}\right)^2 + \theta(\beta_2)\right]\beta_{2\boldsymbol{p}} + \frac{\beta_{1\boldsymbol{p}}}{\beta_1} + \frac{\ln(m/\beta_1)\beta_{3\boldsymbol{p}}}{\beta_3}. \tag{A.11}$$

Isolate the terms involving $\beta_{2\boldsymbol{p}}$ on the right-hand side:

$$\frac{\boldsymbol{q}}{\beta_3 m} - \frac{\beta_{1\boldsymbol{p}}}{\beta_1\beta_3} - \frac{\ln(m/\beta_1)\beta_{3\boldsymbol{p}}}{\beta_3^2} = \left[\left(\frac{\ln(m/\beta_1)}{\beta_3}\right)^2 + \theta(\beta_2)\right]\beta_{2\boldsymbol{p}}. \tag{A.12}$$

To obtain (A.1), note that

$$\frac{\partial}{\partial \boldsymbol{p}}\left(\frac{\ln[e(\boldsymbol{p},u)/\beta_1]}{\beta_3}\right) = \frac{\boldsymbol{q}}{\beta_3 e(\boldsymbol{p},u)} - \frac{\beta_{1\boldsymbol{p}}}{\beta_1\beta_3} - \frac{\ln[e(\boldsymbol{p},u)/\beta_1]\beta_{3\boldsymbol{p}}}{\beta_3^2}. \tag{A.13}$$

Proofs

Lemma 1: *If* $w:\mathcal{P}\times\mathcal{U}\to\mathbb{R}$, $\theta:\mathbb{R}\to\mathbb{R}$, $\eta:\mathcal{P}\to\mathbb{R}$, $w,\theta,\eta\in\mathcal{C}^\infty$, *satisfy*

$$\partial w(\boldsymbol{p},u)/\partial\boldsymbol{p} = \left[\theta(\eta(\boldsymbol{p})) + w(\boldsymbol{p},u)^2\right]\partial\eta(\boldsymbol{p})/\partial\boldsymbol{p},$$

and $\partial\eta(\boldsymbol{p})/\partial\boldsymbol{p}\neq\mathbf{0}$, *then* $w(\boldsymbol{p},u)\equiv g(\eta(\boldsymbol{p}),u)$ *with* $\partial g(x,u)/\partial x = \theta(x) + g(x,u)^2$.

Proof: Differentiating both sides of the system of partial differential equations,

$$\begin{aligned}\frac{\partial^2 w(\boldsymbol{p},u)}{\partial\boldsymbol{p}\partial\boldsymbol{p}^\top} &= \theta'(\eta(\boldsymbol{p}))\frac{\partial\eta(\boldsymbol{p})}{\partial\boldsymbol{p}}\frac{\partial\eta(\boldsymbol{p})}{\partial\boldsymbol{p}^\top} \\ &+\left[\theta(\eta(\boldsymbol{p})) + w(\boldsymbol{p},u)^2\right]\frac{\partial^2\eta(\boldsymbol{p})}{\partial\boldsymbol{p}\partial\boldsymbol{p}^\top} + 2w(\boldsymbol{p},u)\frac{\partial w(\boldsymbol{p},u)}{\partial\boldsymbol{p}}\frac{\partial\eta(\boldsymbol{p})}{\partial\boldsymbol{p}^\top}.\end{aligned} \tag{A.14}$$

Hence, $(\partial w/\partial\boldsymbol{p})\times(\partial\eta/\partial\boldsymbol{p})^\top$ is symmetric, so that $w(\boldsymbol{p},u) = g(\eta(\boldsymbol{p}),u)$ (Goldman and Uzawa 1964, Lemma 1). Differentiating with respect to prices then yields,

$$\frac{\partial w(\boldsymbol{p},u)}{\partial\boldsymbol{p}} = \frac{\partial g(\eta(\boldsymbol{p}),u)}{\partial\eta}\cdot\frac{\partial\eta(\boldsymbol{p})}{\partial\boldsymbol{p}} = \left[\theta(\eta(\boldsymbol{p})) + g(\eta(\boldsymbol{p}),u)^2\right]\frac{\partial\eta(\boldsymbol{p})}{\partial\boldsymbol{p}}.$$ ■

***Proposition* 1:** *Let* $\{w,\eta,\theta\}$ *satisfy lemma* 1*; let* $\pi:\mathcal{P}\to\mathbb{R}$, $\pi\in\mathcal{C}^\infty$, *be* 1° *homogeneous; let* $\alpha,\beta,\gamma,\delta:\mathcal{P}\to\mathbb{C}$, $\alpha,\beta,\gamma,\delta\in\mathcal{C}^\infty$, *be* 0° *homogeneous and satisfy* $\alpha(\boldsymbol{p})\delta(\boldsymbol{p}) - \beta(\boldsymbol{p})\gamma(\boldsymbol{p})\equiv 1$*; let* $\lambda\in\mathbb{R}$*; and let* $f:\mathbb{R}\to\mathbb{C}$, $f\in\mathcal{C}^\infty$, $f'\neq 0$. *Then the expenditure function of any full rank Gorman system can be written as:*

Rank 1 $e(\boldsymbol{p},u)/\pi(\boldsymbol{p}) = u$ *;*

Rank 2 $f\left(\frac{e(\boldsymbol{p},u)}{\pi(\boldsymbol{p})}\right) = u + \beta(\boldsymbol{p})$ *if* $f(x) \neq \ln x$ *, or*

$f\left(\frac{e(\boldsymbol{p},u)}{\pi(\boldsymbol{p})}\right) = \alpha(\boldsymbol{p})u$ *if* $f(x) = \ln x$ *;*

Rank 3 $f\left(\frac{e(\boldsymbol{p},u)}{\pi(\boldsymbol{p})}\right) = \alpha(\boldsymbol{p})u + \beta(\boldsymbol{p}),$ *if* $f(x) \notin \{\ln x, x^{\kappa}, x^{\iota\tau}\}$ *,*

$f\left(\frac{e(\boldsymbol{p},u)}{\pi(\boldsymbol{p})}\right) = \frac{\alpha(\boldsymbol{p})u + \beta(\boldsymbol{p})}{\gamma(\boldsymbol{p})u + \delta(\boldsymbol{p})}$ *if* $f(x) \in \{\ln x, x^{\kappa}\}$ *and* $\theta(\eta) \equiv \lambda$ *, or*

$f(x) = x^{\iota\tau}$ *, or*

$f\left(\frac{e(\boldsymbol{p},u)}{\pi(\boldsymbol{p})}\right) = \frac{\alpha(\boldsymbol{p})w(\eta(\boldsymbol{p}),u) + \beta(\boldsymbol{p})}{\gamma(\boldsymbol{p})w(\eta(\boldsymbol{p}),u) + \delta(\boldsymbol{p})}$ *, if* $f(x) \in \{\ln x, x^{\kappa}\}$ *and* $\theta'(\eta) \neq 0$ *;*

Rank 4 $f\left(\frac{e(\boldsymbol{p},u)}{\pi(\boldsymbol{p})}\right) = \frac{\alpha(\boldsymbol{p})u + \beta(\boldsymbol{p})}{\gamma(\boldsymbol{p})u + \delta(\boldsymbol{p})}$ *,* $\theta(\eta) \equiv \lambda$ *, or*

$f\left(\frac{e(\boldsymbol{p},u)}{\pi(\boldsymbol{p})}\right) = \frac{\alpha(\boldsymbol{p})w(\eta(\boldsymbol{p}),u) + \beta(\boldsymbol{p})}{\gamma(\boldsymbol{p})w(\eta(\boldsymbol{p}),u) + \delta(\boldsymbol{p})}$ *,* $\theta'(\eta) \neq 0$ *, and*

$f(x) \notin \{\ln x, x^{\kappa}, x^{\iota\tau}\}$ *in both of these cases.*

Proof: We first prove the representations for all nominal income systems and then for all deflated income systems.

Nominal Income Systems

Full rank one can always be written as $e(\boldsymbol{p},u)/\pi(\boldsymbol{p}) = u$, $\pi(\boldsymbol{p})$ 1° homogeneous. Adding up and ordinal utility imply that $f(x) = x$, without loss of generality (WLOG). Full rank two is only slightly more involved. For the PIGL model, we have

$$v(\boldsymbol{p},m) = [m^{\kappa} - \beta_1(\boldsymbol{p})]/\beta_2(\boldsymbol{p}), \tag{A.15}$$

with $\beta_1(\boldsymbol{p})$ and $\beta_2(\boldsymbol{p})$ $\kappa°$ homogeneous. Rewrite this in terms of deflated expenditure,

$$\left(\frac{e(\boldsymbol{p},u)}{\pi(\boldsymbol{p})}\right)^{\kappa} = u + \beta(\boldsymbol{p}), \tag{A.16}$$

with $\pi(\boldsymbol{p}) \equiv \beta_2(\boldsymbol{p})^{1/\kappa}$ 1° homogeneous and $\beta(\boldsymbol{p}) \equiv \beta_1(\boldsymbol{p})/\beta_2(\boldsymbol{p})$ 0° homogeneous. For the PIGLOG model, we have

$$v(\boldsymbol{p},m)=[\ln m-\beta_1(\boldsymbol{p})]/\beta_2(\boldsymbol{p}), \tag{A.17}$$

where $\beta_1(\boldsymbol{p})=\ln\tilde{\beta}_1(\boldsymbol{p})$, with $\tilde{\beta}_1(\boldsymbol{p})$ 1° homogeneous, $\beta_2(\boldsymbol{p})$ 0° homogeneous. Rewrite this in terms of deflated expenditure,

$$\ln\left(\frac{e(\boldsymbol{p},u)}{\pi(\boldsymbol{p})}\right)=\alpha(\boldsymbol{p})u, \tag{A.18}$$

where $\pi(\boldsymbol{p})\equiv\tilde{\beta}_1(\boldsymbol{p})$ is 1° homogeneous and $\alpha(\boldsymbol{p})\equiv\beta_2(\boldsymbol{p})$ is 0° homogeneous.

The van Daal and Merkies (1989) and Lewbel (1987, 1990) solution for full rank three nominal income systems with $f(m)\in\{m^{\kappa},\ln m\}$, $\kappa\in\mathbb{R}$, and $\theta(\beta_2(\boldsymbol{p}))\equiv\lambda$, is:

$$\int^{-\beta_3(\boldsymbol{p})/[f(e(\boldsymbol{p},u))-\beta_1(\boldsymbol{p})]}\frac{dw}{(1+\lambda w^2)}=\beta_2(\boldsymbol{p})+u. \tag{A.19}$$

We must put each of six cases in the form of a projective transformation group.

Generalized PIGL

For the generalized PIGL and $\lambda>0$, we will use:

$$\int_0^x\frac{ds}{(1+s^2)}=\tan^{-1}(x). \tag{A.20}$$

Let $\lambda=\mu^2>0$ and $s=\mu w$, so that (A.19) becomes

$$\int^{-\beta_3(\boldsymbol{p})/[e(\boldsymbol{p},u)^{\kappa}-\beta_1(\boldsymbol{p})]}\frac{dw}{(1+\lambda w^2)}=\frac{1}{\mu}\tan^{-1}\left\{\frac{-\mu\beta_3(\boldsymbol{p})}{e(\boldsymbol{p},u)^{\kappa}-\beta_1(\boldsymbol{p})}\right\}=\beta_2(\boldsymbol{p})+c(u). \tag{A.21}$$

The functions $\beta_1(\boldsymbol{p})$ and $\beta_3(\boldsymbol{p})$ are κ° homogeneous and $\beta_2(\boldsymbol{p})$ is 0° homogeneous. Define $\tilde{\beta}_1(\boldsymbol{p})\equiv\beta_1(\boldsymbol{p})^{1/\kappa}$ and $\tilde{\beta}_3(\boldsymbol{p})\equiv\beta_3(\boldsymbol{p})/\beta_1(\boldsymbol{p})$, so that $\tilde{\beta}_1(\boldsymbol{p})$ is 1° homogeneous, while $\tilde{\beta}_3(\boldsymbol{p})$ is 0° homogeneous. Rewrite (A.21) as

$$\frac{-\mu\tilde{\beta}_3(\boldsymbol{p})}{[e(\boldsymbol{p},u)/\tilde{\beta}_1(\boldsymbol{p})]^{\kappa}-1}=\frac{\tan[\mu c(u)]+\tan[\mu\beta_2(\boldsymbol{p})]}{1-\tan[\mu\beta_2(\boldsymbol{p})]\tan[\mu c(u)]}, \tag{A.22}$$

using the trigonometric rule for finding the tangent of the sum of two angles. Apply the normalization $c(u)=\mu^{-1}\tan^{-1}(u)$ with $\tan^{-1}(0)=0$, and rearrange terms to yield:

$$\left(\frac{e(\boldsymbol{p},u)}{\tilde{\beta}_1(\boldsymbol{p})}\right)^{\kappa}=\frac{\left\{\tan[\mu\beta_2(\boldsymbol{p})]+\mu\tilde{\beta}_3(\boldsymbol{p})\right\}u+\mu\tilde{\beta}_3(\boldsymbol{p})\tan[\mu\beta_2(\boldsymbol{p})]-1}{\tan[\mu\beta_2(\boldsymbol{p})]u-1}. \tag{A.23}$$

We have:

$$-\tan[\mu\beta_2(\boldsymbol{p})]-\mu\tilde{\beta}_3(\boldsymbol{p})-\left[\mu\tilde{\beta}_3(\boldsymbol{p})\tan(\mu\beta_2(\boldsymbol{p}))-1\right]\tan(\mu\beta_2(\boldsymbol{p})) = -\mu\tilde{\beta}_3(\boldsymbol{p})\big/\cos^2(\mu\beta_2(\boldsymbol{p})). \tag{A.24}$$

Define:

$$\begin{aligned}
&\pi(\boldsymbol{p})=\tilde{\beta}_1(\boldsymbol{p})\cdot\left[-\mu\tilde{\beta}_3(\boldsymbol{p})\big/\cos^2(\mu\beta_2(\boldsymbol{p}))\right]^{1/2\kappa};\\
&\alpha(\boldsymbol{p})=\left[\sin(\mu\beta_2(\boldsymbol{p}))+\mu\tilde{\beta}_3(\boldsymbol{p})\cos(\mu\beta_2(\boldsymbol{p}))\right]\Big/\sqrt{-\mu\tilde{\beta}_3(\boldsymbol{p})};\\
&\beta(\boldsymbol{p})=\left[\mu\tilde{\beta}_3(\boldsymbol{p})\sin(\mu\beta_2(\boldsymbol{p}))-\cos(\mu\beta_2(\boldsymbol{p}))\right]\Big/\sqrt{-\mu\tilde{\beta}_3(\boldsymbol{p})};\\
&\gamma(\boldsymbol{p})=\sin(\mu\beta_2(\boldsymbol{p}))\Big/\sqrt{-\mu\tilde{\beta}_3(\boldsymbol{p})};\text{ and}\\
&\delta(\boldsymbol{p})=-\cos(\mu\beta_2(\boldsymbol{p}))\Big/\sqrt{-\mu\tilde{\beta}_3(\boldsymbol{p})}.
\end{aligned} \tag{A.25}$$

Because $\tilde{\beta}_1(\boldsymbol{p})$ is 1° homogeneous, while $\beta_2(\boldsymbol{p})$ and $\tilde{\beta}_3(\boldsymbol{p})$ are 0° homogeneous, $\pi(\boldsymbol{p})$ is 1° homogeneous, while $\alpha(\boldsymbol{p})$, $\beta(\boldsymbol{p})$, $\gamma(\boldsymbol{p})$, and $\delta(\boldsymbol{p})$ are 0° homogeneous. A direct calculation yields $\alpha(\boldsymbol{p})\delta(\boldsymbol{p})-\beta(\boldsymbol{p})\gamma(\boldsymbol{p})\equiv 1$, as required. Hence, rewrite (A.23) as

$$\left(\frac{e(\boldsymbol{p},u)}{\pi(\boldsymbol{p})}\right)^{\kappa}=\frac{\alpha(\boldsymbol{p})u+\beta(\boldsymbol{p})}{\gamma(\boldsymbol{p})u+\delta(\boldsymbol{p})}. \tag{A.26}$$

The case where $\lambda=0$ is more straightforward, since

$$\frac{-\beta_3(\boldsymbol{p})}{e(\boldsymbol{p},u)^{\kappa}-\beta_1(\boldsymbol{p})}=\int^{-\beta_3(\boldsymbol{p})/[e(\boldsymbol{p},u)\kappa-\beta_1(\boldsymbol{p})]}dw=\beta_2(\boldsymbol{p})+c(u). \tag{A.27}$$

Define $\tilde{\beta}_1(\boldsymbol{p})$ and $\tilde{\beta}_3(\boldsymbol{p})$ as before and rearrange terms to obtain:

$$\left(\frac{e(\boldsymbol{p},u)}{\tilde{\beta}_1(\boldsymbol{p})}\right)^{\kappa}=1-\tilde{\beta}_3(\boldsymbol{p})\beta_2(\boldsymbol{p})-\tilde{\beta}_3(\boldsymbol{p})c(u). \tag{A.28}$$

The obvious normalization here is $c(u)=-u$, so that

$$\left(\frac{e(\boldsymbol{p},u)}{\tilde{\beta}_1(\boldsymbol{p})}\right)^{\kappa}=\tilde{\beta}_3(\boldsymbol{p})u+1-\tilde{\beta}_3(\boldsymbol{p})\beta_2(\boldsymbol{p}). \tag{A.29}$$

Define:

$$\begin{aligned}
&\pi(\boldsymbol{p})=\tilde{\beta}_1(\boldsymbol{p})\cdot\tilde{\beta}_3(\boldsymbol{p})^{1/2\kappa};\ \alpha(\boldsymbol{P})=\sqrt{\tilde{\beta}_3(\boldsymbol{p})};\\
&\beta(\boldsymbol{p})=\left[1-\tilde{\beta}_3(\boldsymbol{p})\beta_2(\boldsymbol{p})\right]\Big/\sqrt{\tilde{\beta}_3(\boldsymbol{p})};\\
&\gamma(\boldsymbol{p})=0;\ \delta(\boldsymbol{p})=1\Big/\sqrt{\tilde{\beta}_3(\boldsymbol{p})}.
\end{aligned} \tag{A.30}$$

Then we again obtain the group representation (A.26).

Next, let $\lambda = -\mu^2 < 0$ in (A.19), so that

$$\int^{-\beta_3(\boldsymbol{p})/[e(\boldsymbol{p},u)^\kappa - \beta_1(\boldsymbol{p})]} \frac{dw}{(1+\lambda w^2)} = \int^{-\beta_3(\boldsymbol{p})/[e(\boldsymbol{p},u)^\kappa - \beta_1(\boldsymbol{p})]} \frac{dw}{(1+\mu w)(1-\mu w)}$$
$$= \frac{1}{2\mu}\ln\left\{\frac{e(\boldsymbol{p},u)^\kappa - \beta_1(\boldsymbol{p}) - \mu\beta_3(\boldsymbol{p})}{e(\boldsymbol{p},u)^\kappa - \beta_1(\boldsymbol{p}) + \mu\beta_3(\boldsymbol{p})}\right\} = \beta_2(\boldsymbol{p}) + c(u). \tag{A.31}$$

Defining $\tilde{\beta}_1(\boldsymbol{p})$ and $\tilde{\beta}_3(\boldsymbol{p})$ as before, this can be rewritten as

$$\frac{\left[e(\boldsymbol{p},u)/\tilde{\beta}_1(\boldsymbol{p})\right]^\kappa - 1 - \mu\tilde{\beta}_3(\boldsymbol{p})}{\left[e(\boldsymbol{p},u)/\tilde{\beta}_1(\boldsymbol{p})\right]^\kappa - 1 + \mu\beta_3(\boldsymbol{p})} = \exp\left\{2\mu\left[\beta_2(\boldsymbol{p}) + c(u)\right]\right\}. \tag{A.32}$$

In this case, the obvious normalization is $c(u) = (\ln u)/2\mu$. Rearranging gives

$$\left(\frac{e(\boldsymbol{p},u)}{\tilde{\beta}_1(\boldsymbol{p})}\right)^\kappa = \frac{[1-\mu\beta_3(\boldsymbol{p})]\mathrm{e}^{2\mu\beta_2(\boldsymbol{p})}\cdot u - \left[1+\mu\tilde{\beta}_3(\boldsymbol{p})\right]}{\mathrm{e}^{2\mu\beta_2(\boldsymbol{p})}\cdot u - 1}. \tag{A.33}$$

We have:

$$-[1-\mu\beta_3(\boldsymbol{p})]\mathrm{e}^{2\mu\beta_2(\boldsymbol{p})} + \left[1+\mu\tilde{\beta}_3(\boldsymbol{p})\right]\mathrm{e}^{2\mu\beta_2(\boldsymbol{p})} = 2\mu\tilde{\beta}_3(\boldsymbol{p})\mathrm{e}^{2\mu\beta_2(\boldsymbol{p})}. \tag{A.34}$$

Define:

$$\begin{aligned}
\pi(\boldsymbol{p}) &= \tilde{\beta}_1(\boldsymbol{p})\cdot\left[2\mu\tilde{\beta}_3(\boldsymbol{p})\mathrm{e}^{2\mu\tilde{\beta}_3(\boldsymbol{p})}\right]^{1/2\kappa};\\
\alpha(\boldsymbol{P}) &= [1-\mu\beta_3(\boldsymbol{p})]\mathrm{e}^{\mu\beta_2(\boldsymbol{p})}\Big/\sqrt{2\mu\tilde{\beta}_3(\boldsymbol{p})};\\
\beta(\boldsymbol{p}) &= -\left[1+\mu\tilde{\beta}_3(\boldsymbol{p})\right]\Big/\left[\sqrt{2\mu\tilde{\beta}_3(\boldsymbol{p})}\mathrm{e}^{\mu\tilde{\beta}_3(\boldsymbol{p})}\right];\\
\gamma(\boldsymbol{p}) &= \mathrm{e}^{\mu\beta_2(\boldsymbol{p})}\Big/\sqrt{2\mu\tilde{\beta}_3(\boldsymbol{p})};\\
\delta(\boldsymbol{p}) &= -1\Big/\left[\sqrt{2\mu\tilde{\beta}_3(\boldsymbol{p})}\mathrm{e}^{\mu\tilde{\beta}_3(\boldsymbol{p})}\right].
\end{aligned} \tag{A.35}$$

Once again, we obtain the group representation (A.26).

Generalized PIGLOG

The same three cases apply here as for the generalized PIGL case, except that $\ln e(\boldsymbol{p},u)$ replaces m^κ everywhere, $\beta_1(\boldsymbol{p}) = \ln\tilde{\beta}_1(\boldsymbol{P})$ for some 1° homogeneous function, $\tilde{\beta}_1(\boldsymbol{p})$, while both $\beta_2(\boldsymbol{p})$ and $\beta_3(\boldsymbol{p})$ are 0° homogeneous. That is, when $\lambda > 0$, equation (A.21) becomes

$$\int^{-\beta_3(\boldsymbol{p})/\ln[e(\boldsymbol{p},u)/\tilde{\beta}_1(\boldsymbol{p})]} \frac{dw}{(1+\lambda w^2)} = \frac{1}{\mu}\tan^{-1}\left\{\frac{-\mu\beta_3(\boldsymbol{p})}{\ln[e(\boldsymbol{p},u)/\tilde{\beta}_1(\boldsymbol{p})]}\right\} = \beta_2(\boldsymbol{p}) + c(u). \quad \text{(A.36)}$$

Precisely the same steps as in the generalized PIGL case lead to

$$\ln\left(\frac{e(\boldsymbol{p},u)}{\pi(\boldsymbol{p})}\right) = \frac{\alpha(\boldsymbol{p})u + \beta(\boldsymbol{p})}{\gamma(\boldsymbol{p})u + \delta(\boldsymbol{p})}, \quad \text{(A.37)}$$

with the definitions for the price functions given in (A.25), except that $\pi(\boldsymbol{p}) \equiv \tilde{\beta}_1(\boldsymbol{p})$ and $\beta_3(\boldsymbol{p})$ replaces $\tilde{\beta}_3(\boldsymbol{p})$ throughout. Similarly, if $\lambda = 0$, we obtain (A.37) for the price functions defined in (A.30) with $\pi(\boldsymbol{p}) \equiv \tilde{\beta}_1(\boldsymbol{p})$ and $\beta_3(\boldsymbol{p})$ replacing $\tilde{\beta}_3(\boldsymbol{p})$ everywhere, while if $\lambda < 0$, we obtain (A.37) with the definitions (A.35), again with $\pi(\boldsymbol{p}) \equiv \tilde{\beta}_1(\boldsymbol{p})$ and $\beta_3(\boldsymbol{p})$ replacing $\tilde{\beta}_3(\boldsymbol{p})$ everywhere.

For the generalized PIGL (including the QES) and PIGLOG cases with $\theta'(\eta(\boldsymbol{p})) \neq 0$, combine (A.1) and Lemma 1 to write

$$w(\eta(\boldsymbol{p}),u) = \frac{f(e(\boldsymbol{p},u)) - \beta_1(\boldsymbol{p})}{\beta_3(\boldsymbol{p})}. \quad \text{(38)}$$

The same steps as above for the two functional forms for *f* give the group representations.

Trigonometric

From Lewbel (1988, 1990) the indirect utility function is

$$v(\boldsymbol{p},m) = \beta_2(\boldsymbol{p}) + \frac{\beta_3(\boldsymbol{p})\cos\left[\tau\ln\left(m/\beta_1(\boldsymbol{p})\right)\right]}{\left[1-\sin\left[\tau\ln\left(m/\beta_1(\boldsymbol{p})\right)\right]\right]}. \quad \text{(A.39)}$$

Apply the definitions of and rules for calculating sums and differences of sine and cosine functions (e.g., Abramowitz and Stegun 1972, pp.71-74) to rewrite (A.39) as

$$v(\boldsymbol{p},m) = \frac{\left[\beta_3(\boldsymbol{p}) - \iota\beta_2(\boldsymbol{p})\right]\times\left[m/\beta_1(\boldsymbol{p})\right]^{\iota\tau} + \beta_2(\boldsymbol{p}) - \iota\beta_3(\boldsymbol{p})}{1-\iota\left[m/\beta_1(\boldsymbol{p})\right]^{\iota\tau}}. \quad \text{(A.40)}$$

As before, to find the group representation, we need the appropriate transformation of deflated income. Setting $v(\boldsymbol{p},m) = u$ and $m = e(\boldsymbol{p},u)$ and inverting (A.40) yields:

$$\left(\frac{e(\boldsymbol{p},u)}{\beta_1(\boldsymbol{p})}\right)^{\iota\tau} = \frac{u - \left[\beta_2(\boldsymbol{p}) - \iota\cdot\beta_3(\boldsymbol{p})\right]}{\iota\cdot u - \iota\cdot\left[\beta_2(\boldsymbol{p}) + \iota\cdot\beta_3(\boldsymbol{p})\right]}. \quad \text{(A.41)}$$

We have

$$-\iota\cdot\left[\beta_2(\boldsymbol{p})+\iota\cdot\beta_3(\boldsymbol{p})\right]+\iota\cdot\left[\beta_2(\boldsymbol{p})-\iota\cdot\beta_3(\boldsymbol{p})\right]=2\beta_3(\boldsymbol{p}). \tag{A.42}$$

Define:

$$\begin{aligned}&\pi(\boldsymbol{p})=\beta_1(\boldsymbol{p})\cdot\left[2\beta_3(\boldsymbol{p})\right]^{1/(2\iota\tau)};\ \alpha(\boldsymbol{p})=1\big/\sqrt{2\beta_3(\boldsymbol{p})};\\&\beta(\boldsymbol{p})=-\left[\beta_2(\boldsymbol{p})-\iota\cdot\beta_3(\boldsymbol{p})\right]\big/\sqrt{2\beta_3(\boldsymbol{p})};\\&\gamma(\boldsymbol{p})=\iota\big/\sqrt{2\beta_3(\boldsymbol{p})};\ \text{and}\\&\delta(\boldsymbol{p})=-\iota\cdot\left[\beta_2(\boldsymbol{p})+\iota\cdot\beta_3(\boldsymbol{p})\right]\big/\sqrt{2\beta_3(\boldsymbol{p})}.\end{aligned} \tag{A.43}$$

This yields,

$$\left(\frac{e(\boldsymbol{p},u)}{\pi(\boldsymbol{p})}\right)^{\iota\tau}=\frac{\alpha(\boldsymbol{p})u+\beta(\boldsymbol{p})}{\gamma(\boldsymbol{p})u+\delta(\boldsymbol{p})}. \tag{A.44}$$

Deflated Income Systems

Deflated income Gorman systems can be written in the form

$$\boldsymbol{q}=\sum_{k=1}^{K}\boldsymbol{\alpha}_k(\boldsymbol{p})h_k\left(m/\pi(\boldsymbol{p})\right), \tag{A.45}$$

where $\pi:\mathcal{P}\to\mathbb{R}_+$, $\pi\in\mathcal{C}^\infty$, is 1º homogeneous. Adding up implies

$$m\equiv\sum_{k=1}^{K}\boldsymbol{p}^\top\boldsymbol{\alpha}_k(\boldsymbol{p})h_k\left(m/\pi(\boldsymbol{p})\right), \tag{A.46}$$

so that, by linear independence of $\{h_1(x),\dots,h_K(x)\}$, one (and only one) income function must be $m/\pi(\boldsymbol{p})$ and the associated vector of price functions must be $\partial\pi(\boldsymbol{p})/\partial\boldsymbol{p}$. WLOG, let this be the first one, and bring that to the left-hand side of (A.45), so that

$$\boldsymbol{q}-\frac{m}{\pi(\boldsymbol{p})}\frac{\partial\pi(\boldsymbol{p})}{\partial\boldsymbol{p}}=\sum_{k=2}^{K}\boldsymbol{\alpha}_k(\boldsymbol{p})h_k\left(m/\pi(\boldsymbol{p})\right). \tag{A.47}$$

Note that

$$\frac{\partial\left(e(\boldsymbol{p},u)/\pi(\boldsymbol{p})\right)}{\partial\boldsymbol{p}} = \frac{\boldsymbol{q}}{\pi(\boldsymbol{p})} - \frac{e(\boldsymbol{p},u)}{\pi(\boldsymbol{p})^2}\frac{\partial\pi(\boldsymbol{p})}{\partial\boldsymbol{p}}$$
$$= \sum_{k=2}^{K}\frac{\boldsymbol{\alpha}_k(\boldsymbol{p})}{\pi(\boldsymbol{p})}h_k\left(m/\pi(\boldsymbol{p})\right) \tag{A.48}$$
$$\equiv \sum_{k=2}^{K}\tilde{\boldsymbol{\alpha}}_k(\boldsymbol{p})h_k\left(m/\pi(\boldsymbol{p})\right).$$

These steps reduce the number of income terms on the right-hand side of the system by one, maintaining the Gorman structure in deflated income. This also results in a system in which symmetry is the only issue, as homogeneity and adding up have been addressed. This completely identifies the mechanism in which rank can increase by one additional linearly independent vector of price functions and one linearly independent income function in a deflated income Gorman system.

If $K\geq 2$, linear independence of $\{h_2(x),\ldots,h_K(x)\}$ implies that at least one cannot vanish. WLOG, let it be $h_2(x)$ and define

$$y(\boldsymbol{p},u) = f\left(e(\boldsymbol{p},u)/\pi(\boldsymbol{p})\right) = \int^{e(\boldsymbol{p},u)/\pi(\boldsymbol{p})}\frac{dx}{h_2(x)}. \tag{A.49}$$

By Leibnitz' rule, we have

$$\frac{\partial y(\boldsymbol{p},u)}{\partial\boldsymbol{p}} = \frac{1}{h_2(e(\boldsymbol{p},u)/\pi(\boldsymbol{p}))}\left[\frac{\boldsymbol{q}}{\pi(\boldsymbol{p})} - \frac{e(\boldsymbol{p},u)}{\pi(\boldsymbol{p})^2}\frac{\partial\pi(\boldsymbol{p})}{\partial\boldsymbol{p}}\right]$$
$$= \tilde{\boldsymbol{\alpha}}_2(\boldsymbol{p}) + \sum_{k=3}^{K}\tilde{\boldsymbol{\alpha}}_k(\boldsymbol{p})\frac{h_k\left(e(\boldsymbol{p},u/\pi(\boldsymbol{p})\right)}{h_2(e(\boldsymbol{p},u)/\pi(\boldsymbol{p}))} \tag{A.50}$$
$$\equiv \tilde{\boldsymbol{\alpha}}_2(\boldsymbol{p}) + \sum_{k=3}^{K}\tilde{\boldsymbol{\alpha}}_k(\boldsymbol{p})\tilde{h}_k\left(e(\boldsymbol{p},u/\pi(\boldsymbol{p})\right).$$

These steps reduce the problem to one in which the first income term on the right-hand side is identically one, while maintaining the Gorman structure. Since $h_2(x)\neq 0$, $f^{-1}(y)$ exists, so that

$$\frac{\partial y(\boldsymbol{p},u)}{\partial\boldsymbol{p}} = \tilde{\boldsymbol{\alpha}}_2(\boldsymbol{p}) + \sum_{k=3}^{K}\tilde{\boldsymbol{\alpha}}_k(\boldsymbol{p})\hat{h}_k\left(y(\boldsymbol{p},u)\right), \tag{A.51}$$

where $\hat{h}_k\left(y(\boldsymbol{p},u)\right) = \tilde{h}_k\left(f^{-1}\left(y(\boldsymbol{p},u)\right)\right)$, $\hat{h}_k:\mathbb{R}\to\mathbb{R}$, $\hat{h}_k\in\mathcal{C}^\infty$, $k=3,\cdots,K$.

We know from Gorman (1981) and Lewbel (1989) that $K\leq 4$. We solved K=1 to obtain (A.47). Hence, we must find all of the solutions to (A.51) for K=2,3,4. To simplify notation, rewrite (A.51) as

$$\frac{\partial y(\boldsymbol{p},u)}{\partial \boldsymbol{p}} = \boldsymbol{\alpha}_2(\boldsymbol{p}) + \sum_{k=3}^{K} \boldsymbol{\alpha}_k(\boldsymbol{p}) h_k\left(y(\boldsymbol{p},u)\right). \tag{A.52}$$

***K*=2:**

$$\frac{\partial y(\boldsymbol{p},u)}{\partial \boldsymbol{p}} = \boldsymbol{\alpha}_2(\boldsymbol{p}). \tag{A.53}$$

This implies

$$\frac{\partial^2 y}{\partial \boldsymbol{p} \partial \boldsymbol{p}^\top} = \frac{\partial \boldsymbol{\alpha}_2}{\partial \boldsymbol{p}^\top}, \tag{A.54}$$

so that $\partial \boldsymbol{\alpha}_2 / \partial \boldsymbol{p}^\top$ must be symmetric. This is necessary and sufficient for the existence of a function, $\beta : \mathscr{P} \to \mathbb{R}$, $\beta \in \mathcal{C}^\infty$, such that $\partial \beta(\boldsymbol{p}) / \partial \boldsymbol{p} = \boldsymbol{\alpha}_2(\boldsymbol{p})$. Integrating (A.53) yields

$$y(\boldsymbol{p},u) = \beta(\boldsymbol{p}) + u, \tag{A.55}$$

with an obvious normalization.

***K*=3:**

$$\frac{\partial y(\boldsymbol{p},u)}{\partial \boldsymbol{p}} = \boldsymbol{\alpha}_2(\boldsymbol{p}) + \boldsymbol{\alpha}_3(\boldsymbol{p}) h_3\left(y(\boldsymbol{p},u)\right). \tag{A.56}$$

This implies

$$\begin{aligned} \frac{\partial^2 y}{\partial \boldsymbol{p} \partial \boldsymbol{p}^\top} &= \frac{\partial \boldsymbol{\alpha}_2}{\partial \boldsymbol{p}^\top} + \frac{\partial \boldsymbol{\alpha}_3}{\partial \boldsymbol{p}^\top} h_3 + \boldsymbol{\alpha}_3 \boldsymbol{\alpha}_2^\top h_3' + \boldsymbol{\alpha}_3 \boldsymbol{\alpha}_3^\top h_3 h_3' \\ &= \frac{\partial \boldsymbol{\alpha}_2^\top}{\partial \boldsymbol{p}} + \frac{\partial \boldsymbol{\alpha}_3^\top}{\partial \boldsymbol{p}} h_3 + \boldsymbol{\alpha}_2 \boldsymbol{\alpha}_3^\top h_3' + \boldsymbol{\alpha}_3 \boldsymbol{\alpha}_3^\top h_3 h_3'. \end{aligned} \tag{A.57}$$

Subtracting the second line from the first implies,

$$\left(\boldsymbol{\alpha}_3 \boldsymbol{\alpha}_2^\top - \boldsymbol{\alpha}_2 \boldsymbol{\alpha}_3^\top\right) \tilde{h}_3' = \left(\frac{\partial \boldsymbol{\alpha}_2^\top}{\partial \boldsymbol{p}} - \frac{\partial \boldsymbol{\alpha}_2}{\partial \boldsymbol{p}^\top}\right) + \left(\frac{\partial \boldsymbol{\alpha}_3^\top}{\partial \boldsymbol{p}} - \frac{\partial \boldsymbol{\alpha}_3}{\partial \boldsymbol{p}^\top}\right) h_3. \tag{A.58}$$

Since $\{\boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3\}$ are linearly independent, $\boldsymbol{\alpha}_3 \neq c\boldsymbol{\alpha}_2$ for any $c \in \mathbb{R}$. Hence, $\boldsymbol{\alpha}_3 \boldsymbol{\alpha}_2^\top$ is not symmetric. Since $\{1, h_3(y)\}$ are linearly independent, $h_3' \neq 0$. Premultiply (A.58) by $\boldsymbol{\alpha}_3^\top$, postmultiply by $\boldsymbol{\alpha}_2$, and divide by $\boldsymbol{\alpha}_3^\top \boldsymbol{\alpha}_3 \boldsymbol{\alpha}_2^\top \boldsymbol{\alpha}_2 - (\boldsymbol{\alpha}_3^\top \boldsymbol{\alpha}_2)^2 \neq 0$ to obtain

$$h_3'(y) = c_1 + c_2 h_3(y), \tag{A.59}$$

where c_1 and c_2 must be absolute constants since $h_3(y)$ and $h_3'(y)$ are independent of $\boldsymbol{p}$.

If $c_2 \neq 0$, then applying the integrating factor $\mathrm{e}^{-c_2 y}$ implies that the general solution to this linear, first-order, ordinary differential equation has the form

$$h_3(y) = -(c_1/c_2) + c_3 e^{c_2 y}, \tag{A.60}$$

where c_3 is a constant of integration. Plugging this into (A.58)

$$\left(\boldsymbol{\alpha}_3\boldsymbol{\alpha}_2^\top - \boldsymbol{\alpha}_2\boldsymbol{\alpha}_3^\top\right) c_2 c_3 e^{c_2 y} = \left(\frac{\partial \boldsymbol{\alpha}_2^\top}{\partial \boldsymbol{p}} - \frac{\partial \boldsymbol{\alpha}_2}{\partial \boldsymbol{p}^\top}\right) + \left(\frac{\partial \boldsymbol{\alpha}_3^\top}{\partial \boldsymbol{p}} - \frac{\partial \boldsymbol{\alpha}_3}{\partial \boldsymbol{p}^\top}\right)\left[-(c_1/c_2) + c_3 e^{c_2 y}\right]. \tag{A.61}$$

This implies c_3=0, contradicting the linear independence of $\{1, h_3(y)\}$.

Therefore, it must be that c_2=0, and the solution to (A.59) is $h_3(y) = c_1 y$. This reduces the demand system to

$$\frac{\partial y(\boldsymbol{p},u)}{\partial \boldsymbol{p}} = \boldsymbol{\alpha}_2(\boldsymbol{p}) + \boldsymbol{\alpha}_3(\boldsymbol{p}) y(\boldsymbol{p},u). \tag{A.62}$$

Symmetry now reduces to

$$\frac{\partial^2 y}{\partial \boldsymbol{p}\partial \boldsymbol{p}^\top} = \frac{\partial \boldsymbol{\alpha}_2}{\partial \boldsymbol{p}^\top} + \boldsymbol{\alpha}_3\boldsymbol{\alpha}_2^\top + \left(\frac{\partial \boldsymbol{\alpha}_3}{\partial \boldsymbol{p}^\top} + \boldsymbol{\alpha}_3\boldsymbol{\alpha}_3^\top\right) y = \frac{\partial \boldsymbol{\alpha}_2^\top}{\partial \boldsymbol{p}} + \boldsymbol{\alpha}_2\boldsymbol{\alpha}_3^\top + \left(\frac{\partial \boldsymbol{\alpha}_3^\top}{\partial \boldsymbol{p}} + \boldsymbol{\alpha}_3\boldsymbol{\alpha}_3^\top\right) y. \tag{A.63}$$

Equating like powers in *y*, $\partial \boldsymbol{\alpha}_3 / \partial \boldsymbol{p}^\top$ is symmetric. Hence, a 0º homogeneous function, $\delta : \mathscr{P} \to \mathbb{R}$, $\delta \in \mathcal{C}^\infty$, exists such that $\partial\delta(\boldsymbol{p})/\partial\boldsymbol{p} = \boldsymbol{\alpha}_3(\boldsymbol{p})$ and $\partial \boldsymbol{\alpha}_2/\partial \boldsymbol{p}^\top - \boldsymbol{\alpha}_2\, \partial\delta/\partial\boldsymbol{p}^\top$ is symmetric. Applying the integrating factor $e^{-\delta}$ to

$$\frac{\partial}{\partial \boldsymbol{p}} y(\boldsymbol{p},u) e^{-\delta(\boldsymbol{p})} = \left[\frac{\partial y(\boldsymbol{p},u)}{\partial \boldsymbol{p}} - y(\boldsymbol{p},u)\frac{\partial \delta(\boldsymbol{p})}{\partial \boldsymbol{p}}\right] e^{-\delta(\boldsymbol{p})}, \tag{A.64}$$

and

$$\frac{\partial}{\partial \boldsymbol{p}^\top}\left[\boldsymbol{\alpha}_2(\boldsymbol{p}) e^{-\delta(\boldsymbol{p})}\right] = \left[\frac{\partial \boldsymbol{\alpha}_2(\boldsymbol{p})}{\partial \boldsymbol{p}^\top} - \boldsymbol{\alpha}_2(\boldsymbol{p})\frac{\partial \delta(\boldsymbol{p})}{\partial \boldsymbol{p}^\top}\right] e^{-\delta(\boldsymbol{p})}, \tag{A.65}$$

implies that there is a 0º homogeneous function, $\gamma : \mathscr{P} \to \mathbb{R}$, $\gamma \in \mathcal{C}^\infty$, such that

$$y(\boldsymbol{p},u) = u e^{\delta(\boldsymbol{p})} + \gamma(\boldsymbol{p}) e^{\delta(\boldsymbol{p})} \equiv \alpha(\boldsymbol{p}) u + \beta(\boldsymbol{p}), \tag{A.66}$$

with *u* as the constant of integration.

***K*=4:**

$$\frac{\partial y(\boldsymbol{p},u)}{\partial \boldsymbol{p}} = \boldsymbol{\alpha}_2(\boldsymbol{p}) + \boldsymbol{\alpha}_3(\boldsymbol{p}) h_3\left(y(\boldsymbol{p},u)\right) + \boldsymbol{\alpha}_4(\boldsymbol{p}) h_4\left(y(\boldsymbol{p},u)\right). \tag{A.67}$$

We have

$$\frac{\partial^2 y}{\partial p_i \partial p_j} = \frac{\partial \alpha_{i2}}{\partial p_j} + \sum_{k=3}^{4} \frac{\partial \alpha_{ik}}{\partial p_j} h_k + \sum_{k=3}^{4} \alpha_{ik} h_k' \left(\alpha_{j2} + \sum_{\ell=3}^{4} \alpha_{j\ell} h_\ell \right)$$
$$= \frac{\partial \alpha_{j2}}{\partial p_i} + \sum_{k=3}^{4} \frac{\partial \alpha_{jk}}{\partial p_i} h_k + \sum_{k=3}^{4} \alpha_{jk} h_k' \left(\alpha_{i2} + \sum_{\ell=3}^{4} \alpha_{i\ell} h_\ell \right) = \frac{\partial^2 y}{\partial p_j \partial p_i}, \ \forall \ i \neq j. \tag{A.68}$$

Rewrite this in terms of ½n(n–1) vanishing differences,

$$0 = \frac{\partial \alpha_{i2}}{\partial p_j} - \frac{\partial \alpha_{j2}}{\partial p_i} + \left(\frac{\partial \alpha_{i3}}{\partial p_j} - \frac{\partial \alpha_{j3}}{\partial p_i} \right) h_3 + \left(\frac{\partial \alpha_{i4}}{\partial p_j} - \frac{\partial \alpha_{j4}}{\partial p_i} \right) h_4$$
$$+ \left(\alpha_{i3}\alpha_{j2} - \alpha_{i2}\alpha_{j3} \right) h_3' + \left(\alpha_{i4}\alpha_{j2} - \alpha_{i2}\alpha_{j4} \right) h_4' \tag{A.69}$$
$$+ \sum_{k=3}^{4} \sum_{\ell=3}^{4} \alpha_{ik}\alpha_{j\ell} \left(h_k' h_\ell - h_k h_\ell' \right), \ \forall \ j < i = 2, \cdots, n.$$

If $k = \ell$ in the double sum, then $\alpha_{ik}\alpha_{jk}$ is multiplied $h_k' h_k - h_k h_k' = 0$, while if $k \neq \ell$, then $h_k' h_\ell - h_k h_\ell'$ is multiplied once by $\alpha_{ik}\alpha_{j\ell}$ and once by $-\alpha_{i\ell}\alpha_{jk}$. Rewrite (A.69) as

$$0 = \frac{\partial \alpha_{i2}}{\partial p_j} - \frac{\partial \alpha_{j2}}{\partial p_i} + \left(\frac{\partial \alpha_{i3}}{\partial p_j} - \frac{\partial \alpha_{j3}}{\partial p_i} \right) h_3 + \left(\frac{\partial \alpha_{i4}}{\partial p_j} - \frac{\partial \alpha_{j4}}{\partial p_i} \right) h_4$$
$$+ \left(\alpha_{i3}\alpha_{j2} - \alpha_{i2}\alpha_{j3} \right) h_3' + \left(\alpha_{i4}\alpha_{j2} - \alpha_{i2}\alpha_{j4} \right) h_4' \tag{A.70}$$
$$+ \left(\alpha_{i4}\alpha_{j3} - \alpha_{i3}\alpha_{j4} \right) \left(h_3' h_4 - h_3 h_4' \right), \ \forall \ j < i = 2, \cdots, n.$$

Define

$$\boldsymbol{B} = \begin{bmatrix} \alpha_{23}\alpha_{12} - \alpha_{22}\alpha_{13} & \alpha_{24}\alpha_{12} - \alpha_{22}\alpha_{14} & \alpha_{24}\alpha_{13} - \alpha_{23}\alpha_{14} \\ \alpha_{33}\alpha_{12} - \alpha_{32}\alpha_{13} & \alpha_{34}\alpha_{12} - \alpha_{32}\alpha_{14} & \alpha_{34}\alpha_{13} - \alpha_{13}\alpha_{34} \\ \vdots & \vdots & \vdots \\ \alpha_{n,3}\alpha_{n-1,2} - \alpha_{n,2}\alpha_{n-1,3} & \alpha_{n,4}\alpha_{n-1,2} - \alpha_{n,2}\alpha_{n-13} & \alpha_{n,4}\alpha_{n-1,3} - \alpha_{n-1,3}\alpha_{n,4} \end{bmatrix}, \tag{A.71}$$

$$\boldsymbol{C} = \begin{bmatrix} \frac{\partial \alpha_{22}}{\partial p_2} - \frac{\partial \alpha_{12}}{\partial p_1} & \frac{\partial \alpha_{23}}{\partial p_2} - \frac{\partial \alpha_{13}}{\partial p_1} & \frac{\partial \alpha_{24}}{\partial p_2} - \frac{\partial \alpha_{14}}{\partial p_1} \\ \vdots & \vdots & \vdots \\ \frac{\partial \alpha_{n,2}}{\partial p_{n-1}} - \frac{\partial \alpha_{n-1,2}}{\partial p_n} & \frac{\partial \alpha_{n,3}}{\partial p_{n-1}} - \frac{\partial \alpha_{n-1,3}}{\partial p_n} & \frac{\partial \alpha_{n,4}}{\partial p_{n-1}} - \frac{\partial \alpha_{n-1,4}}{\partial p_n} \end{bmatrix}, \tag{A.72}$$

$\boldsymbol{h} = [1 \quad h_3 \quad h_4]^\top$, and $\tilde{\boldsymbol{h}} = [h_3' \quad h_4' \quad h_3' h_4 - h_3 h_4']^\top$. $\boldsymbol{B}$ is ½$n(n-1) \times 3$, $\boldsymbol{C}$ is ½$n(n-1) \times 3$, $\boldsymbol{h}$ is 3×1, and $\tilde{\boldsymbol{h}}$ is 3×1. Full rank requires n≥3.

This gives the symmetry equations as $\boldsymbol{B}\tilde{\boldsymbol{h}} = \boldsymbol{C}\boldsymbol{h}$. Premultiply both sides by $\boldsymbol{B}^\top$ to obtain $\boldsymbol{B}^{\mathsf{T}}\boldsymbol{B}\tilde{\boldsymbol{h}} = \boldsymbol{B}^{\mathsf{T}}\boldsymbol{C}\boldsymbol{h}$. The matrix $\boldsymbol{B}^{\mathsf{T}}\boldsymbol{B}$ is 3×3 and has rank 3, so that the solution for $\tilde{\boldsymbol{h}}$ in terms of $\boldsymbol{h}$ is

$$\tilde{\boldsymbol{h}} = (\boldsymbol{B}^{\mathsf{T}}\boldsymbol{B})^{-1}\boldsymbol{B}^{\mathsf{T}}\boldsymbol{C}\boldsymbol{h} \equiv \boldsymbol{D}\boldsymbol{h}. \tag{A.73}$$

The vectors $\tilde{\boldsymbol{h}}$ and $\boldsymbol{h}$ depend only on y and not on $\boldsymbol{p}$. The matrix $\boldsymbol{D}$ depends only on $\boldsymbol{p}$ and not on y. It follows that the elements of $\boldsymbol{D}$ are absolute constants.

The implications of symmetry on the income functions therefore can be written as

$$\begin{aligned} &h_3'(y) = d_{11} + d_{12}h_3(y) + d_{13}h_4(y), \\ &h_4'(y) = d_{21} + d_{22}h_3(y) + d_{23}h_4(y), \\ &h_3(y)h_4'(y) - h_3'(y)h_4(y) = d_{31} + d_{32}h_3(y) + d_{34}h_4(y), \end{aligned} \tag{A.74}$$

where the $\{d_{ij}\}$ are constants and cannot all be zero in any given equation. The first two equations form a complete system of linear, ordinary differential equations with constant coefficients. This system is constrained by the third equation, which restricts the values that the $\{d_{ij}\}$ can assume in an integrable system.

Differentiate the first differential equation and substitute out $h_3'(y)$ and $h_3(y)$,

$$\begin{aligned} h_3''(y) &= d_{12}h_3'(y) + d_{13}h_4'(y) \\ &= d_{12}h_3'(y) + d_{13}\left[d_{21} + d_{22}h_3(y) + d_{23}h_4(y)\right] \\ &= d_{13}d_{21} + d_{12}h_3'(y) + d_{13}d_{22}h_3(y) + d_{23}\left[h_3'(y) - d_{11} - d_{12}h_3(y)\right] \\ &= d_{13}d_{21} - d_{22}d_{11} + (d_{11} + d_{22})h_3'(y) + (d_{13}d_{22} - d_{23}d_{12})h_3(y). \end{aligned} \tag{A.75}$$

The homogeneous differential equation is

$$h_3''(y) - (d_{11} + d_{22})h_3'(y) - (d_{13}d_{22} - d_{23}d_{12})h_3(y) = 0, \tag{A.76}$$

with characteristic equation

$$\lambda^2 - (d_{11} + d_{22})\lambda - (d_{13}d_{22} - d_{23}d_{12}) = 0, \tag{A.77}$$

and characteristic roots

$$\lambda = ½\left[d_{11} + d_{12} \pm \sqrt{\left(d_{11} + d_{12}\right)^2 + 4(d_{13}d_{22} - d_{23}d_{12})}\right]. \tag{A.78}$$

If $\lambda = 0$ is the only root, the complete solution has the form

$$\begin{aligned} h_3(y) &= a_1 + b_1 y + c_1 y^2, \\ h_4(y) &= a_2 + b_2 y + c_2 y^2. \end{aligned} \tag{A.79}$$

We prove that this is the only possibility. With distinct roots, the complete solution to the ordinary differential equations is

$$h_3(y) = a_1 + b_1 e^{\lambda_1 y} + c_1 e^{\lambda_2 y},$$
$$h_4(y) = a_2 + b_2 e^{\lambda_1 y} + c_2 e^{\lambda_2 y}. \quad \text{(A.80)}$$

The first income function is unity, hence, set $h_3(y) = e^{\lambda_1 y}$ and $h_4(y) = e^{\lambda_2 y}$, WLOG. The equation for $h_3 h_4' - h_3' h_4$ is

$$(\lambda_2 - \lambda_1) e^{(\lambda_1 + \lambda_2) y} = d_{31} + d_{32} e^{\lambda_1 y} + d_{33} e^{\lambda_2 y}, \quad \text{(A.81)}$$

with $\lambda_2 - \lambda_1 = \sqrt{(d_{11} + d_{12})^2 + 4(d_{13} d_{22} - d_{23} d_{12})} \neq 0$ and $\lambda_1 + \lambda_2 = d_{11} + d_{12} \neq \lambda_1 \neq \lambda_2$, a contradiction for all $(\lambda_1, \lambda_2) \neq (0,0)$.

Hence, the roots must be equal, $\lambda = ½(d_{11} + d_{12})$. The complete solution then is

$$h_3(y) = a_1 + b_1 e^{\lambda y} + c_1 y e^{\lambda y},$$
$$h_4(y) = a_2 + b_2 e^{\lambda y} + c_2 y e^{\lambda y}. \quad \text{(A.82)}$$

Again WLOG, set $h_3(y) = e^{\lambda y}$ and $h_4(y) = y e^{\lambda y}$. The equation for $h_3 h_4' - h_3' h_4$ is

$$e^{2\lambda y} = d_{31} + d_{32} e^{\lambda y} + d_{33} y e^{\lambda y}, \quad \text{(A.83)}$$

a contradiction for all $\lambda \neq 0$. Hence, only a repeated vanishing root is possible and

$$\frac{\partial y}{\partial \boldsymbol{p}} = \boldsymbol{\alpha}_2 + \boldsymbol{\alpha}_3 y + \boldsymbol{\alpha}_4 y^2. \quad \text{(A.84)}$$

This system has the same form and solutions as a nominal income full rank QES. ■

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