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Immigration and demographics: can high immigrant fertility explain voter support for immigration?*

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Abstract

First generation immigrants to the U.S. have higher fertility rates than natives. This paper analyzes to what extent this factor provides political support for immigration, using an overlapping generation model with production and capital accumulation. In this setting, immigration represents a dynamic trade-off for native workers as more immigrants decrease current wages but increase the future return on their savings. We find that immigrant fertility has surprisingly strong effects on voter incentives, especially when there is persistence in the political process. If fertility rates are sufficiently high, native workers support immigration. Persistence, either due to inertia induced by frictions in the legal system or through expectational linkages, significantly magnifies the effects. Entry of immigrants with high fertility has redistributive impacts across generations similar to pay-as-you-go social security: initial generations are net winners while latter generations are net losers.

Keywords: immigration, political economy model, overlapping generations, immigrant fertility rates, intergenerational redistribution.

JEL codes: E24, F22.

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1 Introduction

This paper examines the political economy of immigration policy. In virtually all democratic societies, working-age cohorts account for a large majority of voters. Because immigration reduces the capital-labor ratio and therefore tends to reduce wages, one might expect the working-age majority to oppose immigration. Yet many countries are remarkably open to immigration, or at least tolerate immigration by not enforcing immigration restrictions.

When immigrants have children, voting on immigration is complicated by intertemporal considerations. Because a lower capital-labor ratio also increases the return on capital, voters are supportive of policies that increase the working-age population in the future, at a time when they are retired and expect to earn capital income. A decision for or against immigrants with non-zero fertility is in effect a decision not only about current wages but also about the future return on capital. This motivates the paper's question: Can high immigrant fertility explain voter support for immigration?

We examine the dynamic trade-offs induced by immigration in a two-period overlapping generation model with neoclassical production and capital accumulation. This is arguably the most simple environment for voting with endogenous factor prices. In the model, immigrants decrease current wages, which reduces native workers' utility. However, immigration has two effects that decrease the future capital-labor ratio. First, reduced wages translate into lower savings per worker, and thus lower aggregate savings relative to a case in which wages don't change. Second, if first-generation immigrants have more children than natives, the future labor force is higher relative to a case where first generation immigrants have the same fertility as natives. These two effects decrease the future capital-labor ratio and raise current native workers' utility by increasing their future return on capital.

Additional effects arise when voting decisions are linked over time, either through game theoretic arguments or because of frictions in political decision making, and when the saving rate responds to changes in the future interest rate. Intertemporal linkages between voting decisions matter because future immigration chosen by the next generation benefits the current one. We first examine voting under log-utility, which simplifies the analysis due to a constant savings rate, and then consider preferences with constant relative risk aversion (CRRA). If relative risk aversion exceeds one, as the empirical literature suggests, a falling saving rate magnifies the positive effect of immigration on the next period's capital-labor ratio and therefore increases voting support for high-fertility immigration.

The paper makes several contributions. First, we show that if fertility rates of immigrants are high enough, the young are in favor of some immigration irrespective of what future generations do. Second, voter support for immigration is enhanced (a) if there is a positive probability that the next generation will be unable to change the immigration quota; or (b) if voters expect the next generation to condition their vote on the current voting outcomes. Third, we use numerical calculations to show that the model version with political frictions can explain immigration at rates observed in the U.S. We also show that immigration policy raises game theoretic issues similar to voting over pay-as-you-go social security. Just like social security, immigration benefits the retiree generation but imposes cost on workers. For this reason, current votes on immigration depend importantly on expectations about future immigration.[1]

Related literature on political-economic equilibrium models of immigration includes Benhabib (1996) which studies endogenous political-economic equilibrium immigration policy when agents are heterogeneous; Ortega (2005, 2010), who studies the Markov political-economic equilibrium of endogenous immigration policies with skill-upgrading in the presence of differentiated labor. Russo (2008) studies endogenous immigration policy with skilled and unskilled labor in a model where agents display immigration aversion of unskilled labor. Dolmas and Huffman (2004) study the joint political decision of immigration and redistribution policy, while Sand and Razin (2007) study the Markov equilibria of immigration and a pay-as-you-go social security system. Papers on the dynamic economic effects of specific immigration policies include Storesletten (2000), Lee and Miller (2000), and Ben-Gad's (2004, 2008).

A crucial assumption driving the results is that first-generation immigrants have higher fertility rates than natives. This assumption has been used in the theoretical model of Sand and Razin (2007), as well as in a calibrated model in the paper of Lee and Miller (2000), who back up this assumption from cross-section data.[2] Evidence documenting higher fertility of immigrants in recent years is provided by Livingston and Cohn (2012) who use data from the National Center for Health statistics, the US census (1990) and the American Community Survey (2010). Swicegood et al. (2006) use the American Community Survey for the period 2000-2004, and Sevak and Schmidt (2008) use many data sources for their estimates. For demographic-projection purposes, the US bureau of the census allows fertility rates to vary over time for several racial/ethnic groups and to converge with national levels in the long run (see Hollman et al. (2000)). Other evidence includes Hill and Johnson's (2002) time series analysis of total fertility rate in California that documents that first generation immigrant women have significantly higher total fertility rates than native women for their 1982-1998 analysis period, with no significant difference of second and successive generations; and Bean et al. (2000) cross-sectional study of fertility of Mexican origin-women in the US which shows similar findings.

The model abstracts from fiscal policy. This is in part to show that a standard dynamic macro model augmented by political decisions and differential

[1] See Boldrin and Rustichini (2000) and Cooley and Soares (1999) for analogous reasoning about social security. We abstract from social security for clarity, to avoid mixing the factor price effects of immigration with the effects of social security.

[2] Lee and Miller's assumption come from the calculations by the panel on the demographic and economic impacts of immigration made for the book "The New Americans" (1997). The data source used to compute fertility of different groups is the June 1994 Current Population Survey and tabulations from the National Center for Health and Statistics. The cross section analysis implies higher fertility for the first-generation immigrants, with roughly the same fertility for the third-generation immigrants and natives, while the 2nd generation immigrants has a number between the two.

fertility has the potential to explain immigration without having to invoke redistributional concerns. In part we exclude fiscal issues because they could enter in so many ways that the paper would lose focus.[3]

The paper is organized as follows. Section 2 describes the economic environment for the dynamic voting games. Section 3 introduces the baseline model that identifies the main trade-offs faced by the median voter, using logarithmic utility to simplify the exposition. Section 4 presents two variations of the baseline model that both create persistence in immigration policy. One version assumes inertia in the political process, motivated by the idea that there are often "checks and balances" embedded in the political system that make it difficult to change laws. The second version assumes that each generation votes on whether to continue or not an immigration policy set in the past, and where a generation has the option to restart the system (choose the quota) if the previous generation did not allow immigration. Section 5 examines the model with more general CRRA preferences. The analysis is more complicated and the conceptual insights are similar to log-utility, but the generalization is quantitatively important: a low elasticity of intertemporal substitution turns out to strengthen the impact of high immigrant fertility. Section 6 concludes. Proofs are presented in the appendix.

2 The Economic Framework

Economic agents live for two periods. In the first period (young age) they work, earn a wage and choose how much to consume and save. In the second period (old age) they retire and consume out of their first-period savings. We refer to the period-t young as generation t.

The labor force at time t, denoted L_t, consists of N_t young natives plus M_t immigrants, which are assumed to be young[4]:

$$L_t = N_t + M_t = N_t (1 + \theta_t) \tag{1}$$

where

$$\theta_t = \frac{M_t}{N_t} \tag{2}$$

[3] Considerations of fiscal policy might increase or decrease voter support for immigration. Potentially relevant issues include the existence and design of public pensions, the design of (static) redistributional taxes and transfers, and the nature of public goods. As noted in the literature discussed above, public pensions have different ramifications depending on pay-as-you-go versus funded designs and defined contributions versus defined benefits. For redistributional taxes or transfers, it matters to what extent immigrants qualify for benefits and/or they have a different income distribution than natives. Public goods interact with immigration if there are economies or diseconomies of scale in their production. It is beyond the scope of this paper to address all these issues adequately; social security and redistributional taxes are arguably studied well enough that their inclusion would complicate the model without providing much new insight.

[4] This assumption is consistent with the age distribution of US immigrants, which is heavily skewed toward working years. See for example Smith and Edmonston (1997), editors. Pp 55.

is the ratio of immigrant workers to native workers. We assume that the pool of potential immigrants is large enough that θ_t is generally determined by a limit (the immigration quota) imposed by the host country. In cases when immigration is unlimited, we assume that M_t/N_t takes an exogenous "large" but finite value denoted $\bar{\theta}$.[5]

A key assumption is that natives have η children per agent whereas immigrants have $\varepsilon\eta$ children, where $\varepsilon > 1$ parameterizes the fertility of immigrants relative to natives. Second generation immigrants (children of immigrants) are considered naturalized and identical to natives.[6] Given these assumptions, the evolution of the young native cohort is given by

$$N_{t+1} = \eta\left(1 + \varepsilon\theta_t\right) N_t \tag{3}$$

Production is Cobb-Douglas with capital share α:

$$F(K_t, L_t) = K_t^{\alpha} L_t^{1-\alpha}.$$

Factors are paid their marginal products: $R_t = \alpha\left(L_t/K_t\right)^{1-\alpha}$ and $w_t = (1-\alpha)\left(K_t/L_t\right)^{\alpha}$. We assume capital depreciates fully after one generation, so R_t is the gross return to capital. The young supply one unit of labor and earn a wage w_t.

The consumption/saving problem of individuals is

$$\begin{aligned} U_t &= \underset{c_t^1, c_{t+1}^2, s_t}{Max}\left\{u(c_t^1) + \beta u(c_{t+1}^2)\right\} \qquad (4) \\ s.t. \quad c_t^1 + s_t &= w_t \text{ and } c_{t+1}^2 = R_{t+1} s_t, \end{aligned}$$

where c_t^1 and c_{t+1}^2 are consumption when young and old during periods t and $t+1$, respectively. The period utility u is assumed to have constant-relative-risk-aversion (CRRA):

$$\begin{aligned} u(c_t; \gamma) &= \frac{(c_t)^{1-\gamma}}{1-\gamma} \qquad \text{if } \ 0 < \gamma \ \ \& \ \gamma \neq 1 \\ &= \ln c_t \qquad \text{if } \quad \gamma = 1 \end{aligned}$$

where γ can be interpreted as relative risk-aversion and $\frac{1}{\gamma}$ as elasticity of intertemporal substitution. Maximizing U_t yields savings $s_t = \sigma_t w_t$ with optimal savings rate

$$\sigma_t = \frac{\beta^{\frac{1}{\gamma}}\left(R_{t+1}\right)^{\frac{1}{\gamma}-1}}{1 + \beta^{\frac{1}{\gamma}}\left(R_{t+1}\right)^{\frac{1}{\gamma}-1}}. \tag{5}$$

[5] The country is assumed to be "rich" enough that individuals from other countries try to immigrate. One may think of the upper bound $\bar{\theta}$ as determined by the supply of migrants or by physical constraints on the country's ability to absorb immigrants.

[6] We abstract from the possibility of differential labor supply in order to keep the model simple. Empirically, there is evidence (see for example, Ribar (2012)) that shows that first generation male immigrants to the US tend to devote more time to market labor activities than natives. However, women tend to devote less time than natives. Since these forces go in opposite directions, the net effect is ambiguous.

The maximum utility for given (w_t, R_{t+1}) defines an indirect utility function that can be written as

$$
\begin{aligned}
U(w_t, R_{t+1}) &= \left(1+\beta^{\frac{1}{\gamma}} R_{t+1}^{\frac{1}{\gamma}-1}\right)^{\gamma} \frac{w_t^{1-\gamma}}{1-\gamma} \qquad \text{if } 0<\gamma \ \& \ \gamma \neq 1 \\
&= (1+\beta)\ln w_t + \beta \ln R_{t+1} \qquad \text{if} \quad \gamma = 1
\end{aligned}
\tag{6}
$$

Aggregate capital next period equals savings per worker times the number of workers:

$$K_{t+1} = N_t (1+\theta_t) s_t. \tag{7}$$

Let $\kappa_t = \frac{K_t}{N_t}$ denote the capital stock per native worker and let $k_t = \frac{K_t}{L_t} = \frac{K_t}{N_t(1+\theta_t)}$ denote capital divided by the entire labor force. Then the evolution of capital per worker is given by

$$k_{t+1} = \frac{\kappa_{t+1}}{1+\theta_{t+1}} = \frac{(1+\theta_t)}{\eta(1+\varepsilon\theta_t)(1+\theta_{t+1})}\sigma_t w_t. \tag{8}$$

where the component $\frac{(1+\theta_t)}{\eta(1+\varepsilon\theta_t)} w_t$ is pre-determined from period-t savings and immigration.

The wage

$$w_t = (1-\alpha)\left(\frac{\kappa_t}{1+\theta_t}\right)^{\alpha} \tag{9}$$

depends positively on κ_t and negatively on current immigration θ_t.[7] The return

$$R_{t+1} = \alpha\left(\frac{1+\theta_{t+1}}{\kappa_{t+1}}\right)^{1-\alpha} \tag{10}$$

depends on κ_{t+1} and on the future immigration quota θ_{t+1}. Hence the indirect utility of the young depends on their own voting decision when young, θ_t, and on the voting decision of the next generation, θ_{t+1}; it can be written as a function

$$V_t = V(\theta_t, \theta_{t+1}, \kappa_t). \tag{11}$$

Whenever V is invoked below, the first argument is θ_t and the second argument is θ_{t+1}; when dependence on κ_t is inessential, we omit the third argument for simplicity. Because θ_{t+1} enters (6) positively via R_{t+1}, $\frac{\partial V}{\partial \theta_{t+1}} > 0$ always holds (given θ_t). The sign of $\frac{dV}{d\theta_t}$ is ambiguous, however, and to be determined in the analysis.

[7] The empirical literature is not unanimous in whether an immigration flow decreases the average wages of natives. Borjas (2003) and Aydemir and Borjas (2007) find that immigration decrease the wages of natives. Card (2001) and Friedberg (2001) do not find any effects. Ottaviano and Peri (2012) find a positive effect on native's wages and negative on previous immigrants' wages. To keep the model tractable, we follow the macroeconomic literature in assuming that the average wage depends negatively on labor supply (see for example Storesletten (2000), Dolmas and Huffman (2004), Sand and Razin (2007), Boldrin and Montes (2015)).

3 A Baseline Model of Immigration

For this and the next section, we assume logarithmic utility (case $\gamma = 1$). This simplifies the analysis considerably and provides the main conceptual insights, as we will verify later. Notably, it turns out that the optimal immigration quota under log utility does not depend on capital per worker or on expectations about future policy.

3.1 Voting in the Log-Utility Case

With log-utility, (5) implies a constant savings rate $\beta/(1+\beta)$. The evolution of capital-per-worker and capital per native worker can be written in closed form as

$$k_{t+1} = \frac{\beta(1-\alpha)}{(1+\beta)} \frac{(1+\theta_t)}{\eta(1+\varepsilon\theta_t)(1+\theta_{t+1})} k_t^{\alpha} \tag{12}$$

and

$$\kappa_{t+1} = \frac{\beta(1-\alpha)}{(1+\beta)} \frac{(1+\theta_t)^{1-\alpha}}{\eta(1+\varepsilon\theta_t)} \kappa_t^{\alpha}. \tag{13}$$

After substituting $w_t(\kappa_t, \theta_t)$ and $R_{t+1}(k_{t+1}(\kappa_t, \theta_t, \theta_{t+1}))$ into (6), indirect utility can be written as

$$V(\theta_t, \theta_{t+1}) = A + \chi \ln \kappa_t - (1+\chi) \ln(1+\theta_t) + \ln(1+\varepsilon\theta_t) + \ln(1+\theta_{t+1}) \tag{14}$$

where κ_t is omitted as argument in V because it enters separably and therefore does not influence voting incentives, and where

$$\chi = \frac{\alpha(1+\alpha\beta)}{\beta(1-\alpha)} \tag{15}$$

and A are constants (A is unimportant; see appendix for derivation.) Inspection of the indirect utility shows that V is unambiguously increasing in the future quota of immigration (θ_{t+1}), reflecting a reduced future capital-per-worker ratio, whereas the current immigration quota (θ_t) has terms going in opposite directions. The intuition for the former is the positive effect of a lower future capital-per-worker ratio on the return on capital; the intuition for the latter is explained below.

Allowing immigrants decrease current wages (w_t), which impacts negatively the native workers. There are, however, two forces that will decrease the future capital-labor ratio, and this ultimately impacts positively the current generation. First, the lower wages translate into lower savings per worker, which decreases aggregate savings relative to the case in which wages don't change. Second, as immigrants are assumed to have more children than natives by a factor $\varepsilon > 1$, the future labor force is higher (relative to a case of same number of children as natives), which also decreases the future capital-labor ratio. We can therefore summarize the trade-offs of immigration: lower current wages in exchange of a higher future return on capital at the time of retirement. The net effect in lifetime utility depends on the relative magnitudes of χ and ε.

We maintain throughout the paper that voting outcomes are determined by the young generation. This captures the empirical fact that even in countries with very low birth rates, a large majority of voters are working-age.[8] Thus immigration θ_t is chosen to maximize the utility of a representative generation-t voter. Voters rank allocations by immigration quotas, and they favor the one that maximizes their indirect utility.

Maximizing the indirect utility (14) yields the optimal policy

$$\theta_t = \theta^0 \equiv \max\left[0\ ,\ \frac{\varepsilon - \chi - 1}{\varepsilon \cdot \chi}\right] \tag{16}$$

taking into account the constraint $\theta_t \geqslant 0$.[9] Therefore the most preferred immigration level for the young is (i) positive if $\varepsilon > 1 + \chi$, (ii) increasing in the fertility rate of immigrants, and (iii) bounded above by $\frac{1}{\chi}$. Because each generation has the *dominant* strategy to choose this specific (constant) level of immigration, the system is also politically sustainable.

Over time, an economy with immigration rate $\theta^0 > 0$ converges to a lower steady state capital-labor ratio. Provided the economy is dynamically efficient (which requires $\chi > 1 - \alpha$; see appendix), steady state utility is lower than without immigration. Thus starting from a steady state without immigration, some "transitional" generations enjoy increased utility, but later generations would be better off without immigration. This pattern of transitory welfare gains followed by longer term welfare losses is similar to the welfare effects of a pay-as-you-go social security system; both immigration and social security redistribute welfare from future generations to the earlier ones.

3.2 Simple Calculations for the U.S. (and Europe)

In the basic model, immigration occurs if the fertility rate of immigrants exceeds $1 + \chi$. This turns out to be a high hurdle for empirically plausible parameters.

Swicegood et al. (2006) estimate that the total fertility rate of U.S. immigrant women is 27% higher than fertility of native women using 2000-2004 data from the American Community Survey (ACS). Sevak and Schmidt (2008) estimate a total fertility rate that is 56% higher for immigrants in 1990, and 43%

[8] For simplicity, the model assumes working age and retirement periods of equal length and no early mortality. Taking these assumptions literally, $\eta > 1$ is required for a working-age majority. Empirically, a working-age majority is a robust finding when mortality is modeled more realistically, e.g., by assuming stochastic mortality at the start of retirement (which stays within a two-period setting). We show in the appendix that such an extended model reduces to the same economic framework as our model provided β is suitably reinterpreted, and that a working-age majority is obtained for the US for $\eta > 1/2$.

Note that immigration cycles (as in Ortega 2005) are ruled out when the majority is working-age.

[9] Note that the optimization treats θ_{t+1} as given and as being unaffected by θ_t. This is rational because the resulting optimal quota θ^0 applies for all initial conditions (all κ_t). Put differently, the constant function $\theta_t = \theta^0$ can be interpreted as equilibrium in Markov strategies, because $\theta_t = \theta^0$ is optimal if the next generation follows the same strategy $\theta_{t+1} = \theta^0$. Hence the seemingly myopic solution (taking θ_{t+1} as given) is Markovian. Taking θ_{t+1} as given also rules out trigger strategies (see Section 4.2).

higher in 2000 than for natives using census data; and Lee and Miller (2000) have used a fertility factor of 1.35 for first generation immigrants with 1994 cross-section data from the ACS for the demographic projections of their model asessing the fiscal impacts of immigration.[10]

More recently, Livingston and Cohn (2012) estimate with census data that the general fertility rate is 70% higher for immigrants in 1990 and 49% higher in 2010.[11] Since the general fertility rate doesn't take into account the age-composition of women, we construct total fertility rates with the data they present. This yields total fertility rates of 1.96 children for native-born women and 3.37 children for foreign-born women in 1990, while numbers for 2010 are 1.755 for native-born women and 2.696 for foreign women. These numbers imply fertility factors of 1.72 in 1990 and 1.53 in 2010.

In summary, the empirical evidence for the U.S. suggests values of ε between 1.3 and 1.7. We take $\varepsilon = 1.5$ as baseline for calibration.

Sobotka (2008) summarizes the fertility rates of native citizens and immigrants in several European countries, using data from several sources. Taking ratios from his values, we obtain implied values for ε that range from 1.17 for Sweden to 2.07 for Italy. (See appendix for details.) Taking a simple average across the European data, the average ratio of immigrant to native fertility is 1.53. While we focus on the U.S. (to maintain a coherent calibration), its worth noting that ratios of immigrant-to-native fertility of 1.5 (or higher) are not special to the U.S.

Regarding immigration, Ben-Gad (2008) reports that the net rate of U.S. immigration between 1991 and 2000 was 3.2 per thousand annually.[12] Lee and Miller (2000) assume 900,000 immigrants annually in net immigration, in line with census projections. That rate also represents 3.2 immigrants per thousand natives in year 2000. This represents a flow of 8.3% for a generational period of 25 years and includes illegal immigration. Thus a successful model should explain values for θ of around 8%.

Optimal immigration in the model (θ^0) generally depends on α and β, which determine χ. We follow the literature in setting the capital share at $\alpha = \frac{1}{3}$. For the calibration of β, we exploit that under log utility and Cobb-Douglas production, the gross return to capital in steady state is given by $R = \frac{\alpha(1+\beta)}{(1-\alpha)\beta} \cdot \eta(1+\varepsilon\theta)$. This can be solved for β to obtain:

$$\beta = \frac{\alpha}{(1-\alpha) r - \alpha}, \text{ where } r \equiv \frac{R}{\eta(1+\varepsilon\theta)}$$

[10] Using the general fertility rate, Swicegood et al. estimate that the difference is 40%, as opposed to 27% when the total fertility rate is used. Sevak and Schmidt in turn estimate general fertility rates that are 56% higher in 1990 and 57% higher in 2010. The fertility estimates for Lee and Miller's are more thoroughly discussed in Smith and Edmonston (1997).

[11] Livingston and Cohn report birth rates in 1990 of 66.5 children for each 1000 native born women, and 112.8 children for each 1000 foreign-born women. The implied fertility factor is $112.8/66.5 = 1.696$. Similarly for 2010 they report birth rates of 58.9 children for each 1000 native-born women, and 87.8 for each 1000 foreign-born women. We estimate total fertility rates with the information on children per women disaggregated by age.

[12] That number includes illegal immigration to the US.

The ratio r relates the return on capital to the rate of economic growth and turns out to be an insightful moment to match with data because it can be measured directly and because this approach will help maintain a consistent calibration for β when we consider general CRRA preferences. If the annual difference between the return on capital and the growth rate is around $.01-.03$ (as empirically reasonable for broad concepts of capital, accounting for productivity growth), compounding for 25 years implies an r-ratio in the range $1.01^{25} = 1.28$ to $1.03^{25} = 2.09$. We use $r = 1.5$ as baseline for calibration. Together with $\alpha = \frac{1}{3}$, this implies $\beta = \frac{1}{2}$ and $\chi = \frac{7}{6} = 1.17$.

Using the equation for θ^0, the baseline values $\varepsilon = 1.5$ and $\chi = 1.17$, the young optimally choose zero immigration. Non-zero immigration in this simple model would require $\varepsilon > 2.17$; an optimal level of 8% immigration would require $\varepsilon = 2.39$.

Alternatively, if one considers the entire range $[1.28, 2.09]$ for r, implied values for β are in $[0.24, 0.69]$ and values for χ are the range $[0.95, 1.76]$. For $\chi \geq 0.95$, the basic model requires $\varepsilon \geq 1.95$ to explain immigration. While $\chi < 0.95$ may be relevant for economies with strong desire to save (perhaps aging societies in the future), and $\varepsilon \geq 1.95$ may be relevant for countries with very low domestic fertility (e.g. in parts of Europe), neither is empirically plausible for the United States.

Thus, while the baseline model explains conceptually why high immigrant fertility favors immigration, an expanded model will be needed to explain U.S. immigration quantitatively.

4 Persistence in the Political Process

This section shows that intertemporal linkages in the voting decisions significantly strengthen the impact of immigrant fertility. We study two types of linkages. One version considers persistence of the law and the other considers trigger strategies. We will show that both versions can produce voting equilibria with positive immigration at empirically observed levels of immigrant fertility.

4.1 Persistence of the Law and Immigration

Some nations like the US have multiple "checks and balances" embedded in their political systems that make it difficult to enact new laws, as would be required to change immigration rules. In the U.S., for example, a new law must be approved by the House, the Senate, and the President, each of which are separately elected; in addition, there are procedural rules (e.g. the filibuster in the Senate) that allow minorities to block new laws. The main visa program in the US was enacted in 1965 and although there have been some changes, it still regulates the bulk of (legal) immigration. This section examines the ramification of political frictions that create persistence in immigration rules.

To model persistence in immigration law, assume that opportunities to change the law arise at uncertain times. Every period, there is an exogenous probability p that new legislation cannot be enacted, so the immigration quota remains at the previous value, $\theta_t = \theta_{t-1}$. This restriction influences the choice of a new quota because voters know that with positive probability, their choice will remain in effect during their own retirement.

The specific timing is as follows. At the beginning of a period, before individuals take economic and political decisions, nature reveals the state, which is either that immigration law can be changed (state I) or not (state II). If the law can be changed, then the young choose their most preferred immigration level, so $\theta_t = \theta_t^I$ is optimally chosen. Otherwise, $\theta_t = \theta_t^{II} = \theta_{t-1}$ is predetermined.

Consider the optimization problem of the period-t young in state I, when they can set θ_t^I. Current consumption depends on immigration as in the previous section, $c_t^1 = c_t^1(\theta_t^I)$. With probability $1-p$, state I is realized in period-$t+1$, so $\theta_{t+1} = \theta_{t+1}^I$ will be determined by the young in period-$(t+1)$. Then as in the previous section, $c_{t+1}^2 = c_{t+1}^2(\theta_t^I, \theta_{t+1}^I)$ depends on both generations' choices (via R_{t+1}). With probability p, state II is realized in period-$t+1$. Then $\theta_{t+1} = \theta_t^I$ is determined by generation t, so $c_{t+1}^2 = c_{t+1}^2(\theta_t^I, \theta_t^I)$.

Thus voting behavior is obtained by maximizing

$$\underset{\theta_t^I \geq 0}{Max} \ln\left[c_t^1(\theta_t^I)\right] + \beta(1-p)\left[\ln c_{t+1}^2(\theta_t^I, \theta_{t+1}^I)\right] + \beta p\left[\ln c_{t+1}^2(\theta_t^I, \theta_t^I)\right] \qquad (17)$$

taking θ_{t+1}^I as given. Using the closed-form solutions for factor prices, capital-per-worker, and savings, the objective function can be written as indirect utility over voting decisions:

$$V(\theta_t^I, \theta_{t+1}^I) = -(1+\chi-p)\ln(1+\theta_t^I) + \ln(1+\varepsilon\theta_t^I) + (1-p)\ln(1+\theta_{t+1}^I), \qquad (18)$$

where an intercept and a separable κ_t-term are omitted (see appendix).

Depending on parameters, three cases may apply: (i) If $p \leq \varepsilon - 1 - \chi$, then $\theta_t^I = 0$ is optimal, so there is no immigration. (ii) If $p > \varepsilon - \chi - 1$ and $p < \chi$, then there is an interior optimum: $\theta_t^I = \frac{\varepsilon-\chi-1+p}{\varepsilon(\chi-p)}$. (iii) If $p \geq \chi$ (which implies $1-p < \varepsilon - \chi$ because $\varepsilon > 1$), then unrestricted immigration is optimal, so $\theta_t^I = \bar{\theta}$ is bounded only by the immigrant pool. Since the numerical calculation suggest $\chi \geq 1$, we henceforth disregard case (iii). Then the model implies immigration quotas of

$$\theta_t = \theta^p \equiv \max\left[0\ ,\quad \frac{\varepsilon-\chi-1+p}{\varepsilon\,(\chi-p)}\right]. \qquad (19)$$

The higher the probability p that future generations cannot change the law, the higher is the immigration quota. A higher probability p also expands the set of parameters (ε, χ) for which immigration is positive. Thus persistence of the law unambiguously favors more immigration.

To illustrate this numerically, consider again $\alpha = \frac{1}{3}$, $\beta = \frac{1}{2}$ (so $\chi = 1.167$) and suppose $\varepsilon = 1.5$. Then positive immigration occurs for probabilities $p >$

$\varepsilon - 1 - \chi = \frac{2}{3}$. We obtain 8% immigration (the U.S. value) for a persistence of $p = 0.72$.[13]

The magnitude of θ is quite sensitive to the parameter p and to the difference $\varepsilon - \chi$. For example, $p = 0.695$ would imply 4% immigration and $p = 0.76$ would imply 16% immigration.[14] If ε is high and χ is small, immigration occurs even when p is relatively low; e.g., for all $p > 0.3$ when $\varepsilon = 1.7$ and $\chi = 1$. If ε is relatively low, there are empirically plausible values for χ that do not support immigration for any p; e.g., $\theta^p = 0$ for all p when $\varepsilon = 1.3$ and $\chi \geq 1.3$.

4.2 Expectational Linkages

Expectational linkages are an alternative mechanism to generate persistence. For this section, we consider a simple set of trigger strategies and implied expectations: each generation expects the next generation to leave the immigration quota unchanged, provided the immigration quota remains unchanged in the current period. If the quota is changed, however, the next generation is expected to disregard expectational linkages, i.e., to optimize as in the baseline model. The underlying immigration quota is set by some initial "starting" generation t_0.[15]

In game-theoretic terms, the task is to show under what conditions immigration can be sustained as a subgame-perfect equilibrium of this repeated voting game. Much of the reasoning is analogous to voting over social security (Cooley and Soares (1999), Boldrin and Rustichini (2000)), and here adapted to immigration.[16] Hence the exposition is brief.

The new element of this game is that decisions can be conditioned on history, and so we define $V(\theta_t, \theta_{t+1}|h_{t-1})$ as the utility that a member of generation-t obtains when the equilibrium immigration quotas are θ_t and θ_{t+1}, given the history h_{t-1} of previous generations' immigration choices. We continue to assume logarithmic utility, so κ_t is separable in the utility function and irrelevant for voting incentives.

In general, the optimal strategy of generation t_0 involves choosing a sustainable policy that is utility maximizing. By sustainable we mean one that future generations will not repudiate along an equilibrium path. As candidate for the starting generation's optimal choice, let θ^1 be the utility-maximizing immigration quota for generation t_0 if sustainability is taken for granted. That is, let θ^1

[13] We discuss the history of U.S. immigration law in section 5.4 below and explain why p is difficult to calibrate. Hence we calibrate other parameters and use implied values of p to illustrate the magnitude of the effects involved.

[14] This sensitivity applies mostly to log-utility. Sensitivity to p is substantially less under CRRA preferences with lower elasticities of substitution; see analysis in Section 5.3.

[15] One may assume that if policy is changed and a generation reoptimizes, the following generation is allowed to restart the dynamic game, but this is immaterial because it will not occur in equilibrium.

[16] The interaction of immigration and pay-as-you-go social security is much discussed in the literature (e.g., Sand and Razin (2007)). Without disputing that social security would affect voting over immigration, social security is excluded here because it would obscure the factor price effects caused by immigration per se.

maximize

$$V(\theta,\theta) = -\chi \ln(1+\theta) + \ln(1+\varepsilon\theta).$$

This implies

$$\theta^1 = \max\left[0\ ,\ \frac{\varepsilon-\chi}{\varepsilon(\chi-1)}\right]. \tag{20}$$

for $\chi > 1$, and $\theta^1 = \bar{\theta}$ for $\chi \leq 1$.

Notice that θ^1 equals the limiting value of (19) as $p \to 1$. The immigration quota is positive for lower values of ε than in the previous sections. For $\varepsilon > \chi$, θ^1 is strictly greater than the immigration quotas in (16) and in (19) for all $p < 1$.

It is straightforward to verify that θ^1 is sustainable. If a generation t deviated and set $\theta_t \neq \theta^1$, it must expect θ_{t+1} to be given by (16), which means $\theta_{t+1} = \theta^0$. Because conditional on $\theta_t \neq \theta^1$, θ_{t+1} does not depend on θ_t, generation t would also choose $\theta_t = \theta^0$ according to (16), so $\theta_t = \theta_{t+1} = \theta^0$. The best deviation from θ^1 thus yields utility $V(\theta^0,\theta^0)$. Because θ^0 is in the feasible set for maximizing $V(\theta,\theta)$ and $\theta^0 \neq \theta^1$ whenever $\theta^1 > 0$, $V(\theta^1,\theta^1) > V(\theta^0,\theta^0)$ holds for all $\theta^1 > 0$.

Thus generation t_0 sets $\theta_{t_0} = \theta^1$. All subsequent generations follow the strategy of setting $\theta_t = \theta^1$ if $\theta_{t-1} = \theta^1$ and setting $\theta_t = \theta^0$ if $\theta_{t-1} \neq \theta^1$. Hence the voting outcome is $\theta_t = \theta^1$ for all t.

Quantitatively, expectational linkages are capable of producing extremely high immigration quotas for empirically plausible fertility rates. In the setting with $\varepsilon = 1.5$, $\alpha = \frac{1}{3}$, and $\beta = \frac{1}{2}$, one finds $\theta^1 = 1.33$.

One objection to trigger strategies is that they may not be perfectly credible in reality. We have in effect analyzed this objection already, because imperfect credibility can be modeled as a positive probability $1-p$ that the game re-starts. Then the analysis of the previous section would apply, and immigration would be determined by (19).

In summary, the analysis suggests that a model with persistence in setting immigration policy – through legislative frictions, expectational linkages, or a combination thereof – can explain voter support for positive immigration quotas.

5 CRRA Preferences: More Support of Immigration if $\gamma > 1$

In this section we examine voting over immigration under more general CRRA preferences. The analysis is more complicated because savings generally depend on expectations about the returns to savings, which are endogenous. This extension is important, however, because elasticity of intertemporal substitution regulates to what extent invidual utility is affected by immigration-induced changes in the returns to savings.

The problem of young voters with general CRRA utility ($\gamma \neq 1$) is to maximize U_t in (6), where R_{t+1} is determined implicitly by (10), (8), and (5).

There are two critical differences to logarithmic utility. First, the maximization problem of voters is not separable in capital per native worker (κ_t) and the immigration quotas (θ_t, θ_{t+1}). Even if expectations about θ_{t+1} are taken as given, θ_t is generally a function of κ_t and not a constant. Second, since θ_{t+1} will generally depend on κ_{t+1}, and κ_{t+1} will depend on the time-t savings rate, period-t voters may recognize that their savings have an impact on θ_{t+1} even in absence of other expectational linkages.

To separate these two issues, we consider two scenarios for voting. First we consider optimal choices when voters take θ_{t+1} as given. Second, we consider voting when voters recognize the functional dependence of θ_{t+1} on the capital per native worker. The latter is formalized as voting over Markov strategies. We introduce both scenarios without persistence and then add persistence to the Markov setting.

5.1 Voting with Static Expectations

Voters have an incentive to approve immigration because immigration raises the return on capital R_{t+1}. Under CRRA preferences, the saving rate responds to changes in this return; and since current immigration affects R_{t+1}, the choice of θ_t will affect the saving rate. This effect is given by

$$\frac{d\sigma_t}{d\theta_t} = \frac{d\sigma_t}{dR_{t+1}} \frac{dR_{t+1}}{d\theta_t} = \left(\frac{1}{\gamma} - 1\right) \frac{\sigma_t(1-\sigma_t)}{R_{t+1}} \frac{dR_{t+1}}{d\theta_t} \tag{21}$$

where $\sigma_t(1-\sigma_t)/R_{t+1} > 0$ and $dR_{t+1}/d\theta_t > 0$.

If the intertemporal substitution $1/\gamma$ is less than one (i.e., if $\gamma > 1$), the saving rate responds negatively to higher R_{t+1} and hence to higher θ_t. In turn, reduced savings raise R_{t+1}, so the impact of immigration on R_{t+1} is greater than under log-utility. The incentive to accept high-fertility immigrants is increased. In contrast, if $1/\gamma > 1$, the savings rate would respond positively to R_{t+1} and to θ_t, so the the impact of immigration on R_{t+1} is reduced, and the incentive to accept high-fertility immigrants is also reduced. Empirical evidence favors an intertemporal substitution less than one.[17] Hence CRRA preferences tend to strengthen voter incentives to approve immigration.

In more detail, suppose for this section that generation t takes future immigration θ_{t+1} as given (static expectations). An interior solution to their voting problem implies the first order condition

$$\beta^{-\frac{1}{\gamma}} \alpha^{1-\frac{1}{\gamma}} \left(\frac{1+\theta_{t+1}}{\kappa_{t+1}}\right)^{\phi} = \frac{\frac{(1-\alpha)}{\alpha} \frac{(\varepsilon-1)}{(1+\varepsilon\theta_t)} - \alpha}{1-\phi} \tag{22}$$

where $\phi = (1-\alpha)(1-\frac{1}{\gamma}) > 0$,[18] and that capital per native worker can be written as

$$\kappa_{t+1} = \frac{\alpha\left[(1-\alpha)+\alpha\gamma\right](1+\theta_t)^{1-\alpha}}{\eta\left[\varepsilon\alpha(1+\theta_t)+(\gamma-\alpha)(\varepsilon-1)\right]} \kappa_t^{\alpha}. \tag{23}$$

[17] Classic references are Hall (1988) and Ogaki and Reinhart (1998).

[18] All derivations are in the appendix. Corner solutions ($\theta_t = 0$) are omitted here; they apply if the interior solution would imply $\theta_t < 0$.

These difference equations define a perfect foresight path for $\{\theta_t, \kappa_t\}_{t \geq t_0}$. The system is saddle-path stable for a wide range of parameters and hence converges to a steady state. Given convergence, most insights derive from comparing steady states.

The primitive parameters in the CRRA model are $\{\alpha, \beta, \gamma, \eta, \varepsilon\}$. We use $\gamma = 4$ as baseline value, and again assume $\alpha = \frac{1}{3}$ and set β to match $r = 1.5$. The latter is more subtle than under log-utility because r is influenced by all five parameters. It turns out, however, that all combinations of β and of population growth $\eta(1+ \varepsilon\theta)$ that imply a common steady state value for r also imply the same dynamics around the steady state. Hence population growth becomes irrelevant when β is calibrated to match r. For reporting β, we assume $\eta(1+ \varepsilon\theta) = 1.25$ (about 1% population growth per year); then using the equation for calibrating β (see the appendix for details), one obtains $\beta = 0.412$,. Using $\varepsilon = 1.5$ and $\theta = 0.08$ (as discussed in Section 3.2) η is calibrated as $\eta = \frac{1.25}{1+1.5 \cdot 0.08} = 1.116$.

For these baseline parameters, we find that the model requires $\varepsilon = 1.78$ to obtain 8% immigration in steady state. This is substantially less than the corresponding value under log-utility (recall $\varepsilon = 2.39$ in Section 3.2), though still greater than the empirical fertility ratios in most countries.[19]

To characterize immigration policy out of steady state, we compute log-linearized approximations to the policy function around the steady state. Importantly, immigration varies positively with deviations of κ_{t+1} from its steady state; e.g., for the baseline parameters and $\varepsilon = 1.78$, we find an elasticity $\frac{\partial \ln(1+\theta_{t+1})}{\partial \ln(\kappa_{t+1})} = 0.077$. Since κ_{t+1} depends on θ_t, current immigration affects θ_{t+1}, and this conflicts with the assumption of static expectations.

5.2 Voting with Markov Strategies

Voters who understand the model's dynamics should expect future generations to condition the immigration quota θ_{t+1} on κ_{t+1}. Assuming no other expectational linkages, optimal voting behavior is then defined by a voting equilibrium under Markov strategies.

Without persistence, the analysis of Markov strategies is straightforward. A voting equilibrium is a function $g : \theta_t = g(\kappa_t)$ such that θ_t is optimal for any κ_t under the expectation that $\theta_{t+1} = g(\kappa_{t+1})$ is determined by the same function g. In technical terms, g must solve a functional problem in the space of functions that map the state κ into an immigration quota θ:

$$g(\kappa) = \arg\max_{\theta \geq 0} U(\kappa, \theta) \tag{24}$$

[19] Comparisons between CRRA and log-utility are similar for other parameters (details omitted). Numerical analysis (also not reported in detail) shows that when immigration is increased, the utility of the first few "transition" generations is higher than the utility of later generations, and (as under log-utility) the steady state utility with optimal immigration (for current voters) is lower than in an economy without immigration.

where $U(\kappa,\theta) = \dfrac{1}{1-\gamma}\left\{1+\beta^{\frac{1}{\gamma}}\alpha^{\left(\frac{1}{\gamma}-1\right)}\left(\dfrac{1+g\left(\Psi\left(\kappa,\theta\right)\right)}{\Psi\left(\kappa,\theta\right)}\right)^{-\phi}\right\}^{\gamma}\left(\dfrac{\kappa}{1+\theta}\right)^{\alpha(1-\gamma)}$

is indirect utility and

$$\Psi\left(\kappa,\theta\right) = \frac{\left(1-\alpha\right)\left(1+\theta\right)^{1-\alpha}\kappa^{\alpha}}{\eta\left(1+\varepsilon\theta\right)\left(1+\beta^{-\frac{1}{\gamma}}\alpha^{1-\frac{1}{\gamma}}\left(\frac{1+g(\Psi(\kappa,\theta))}{\Psi(\kappa,\theta)}\right)^{\phi}\right)}$$

specifies the next period's capital per native worker as a function of the current state κ and an immigration quota θ, $\kappa_{t+1} = \Psi\left(\kappa_t,\theta_t\right)$.

Assuming g is continuous and differentiable, an interior solution to voters' utility maximization problem implies a first-order condition that involves the derivative of g:

$$\beta^{-\frac{1}{\gamma}}\alpha^{1-\frac{1}{\gamma}}\left(\frac{1+\theta_{t+1}}{\kappa_{t+1}}\right)^{\phi} = \frac{\left[1-\alpha+\left(\frac{1-\alpha}{\alpha}\right)\frac{(\varepsilon-1)}{(1+\varepsilon\theta_t)}\right](1-\lambda_{t+1})-1}{1-\phi(1-\lambda_{t+1})} \tag{25}$$

where $\lambda_{t+1} = g'(\kappa_{t+1})\frac{\kappa_{t+1}}{1+g(\kappa_{t+1})}$ captures voters' recognition that current immigration will impact next period's capital per worker and hence the next generation's vote on immigration. If $g'(\kappa) = 0$, (25) reduces to (22).

We obtain numerical solutions for g using projection methods similar to den Haan and Marcet's parameterized expectations approach. (See appendix for specifics). Once an approximate solution for g is obtained, the functions g and Ψ imply sequences $\{\kappa_t,\theta_t\}$ from any starting value κ_{t_0}. Provided the sequences converge, the objects of main interest are the steady state values, denoted (κ^*,θ^*).

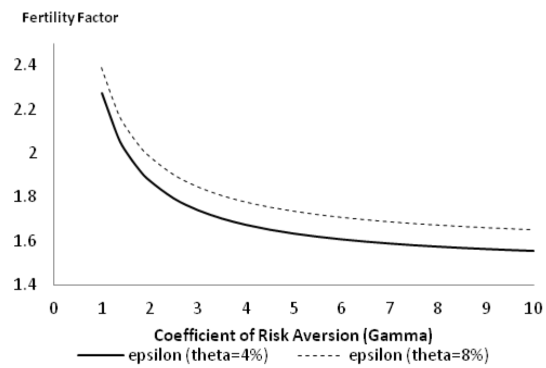
Our numerical findings are as follows. For the baseline parameters, we find that $\varepsilon = 1.80$ is required for $\theta^* = 4\%$ and that $\varepsilon = 1.93$ is required for $\theta^* = 8\%$. These fertility factors are higher than under static expectations but less than under log-utility.

The key insight for the intuition is that the policy function g has a positive derivative, so $g' > 0$ in (25).[20] Recall that (i) generation-t voter benefit from high immigration in the next period, which raises the return on capital; and (ii), period-t immigration reduces capital per native worker in the next period ($\partial\kappa_{t+1}/\partial\theta_t < 0$ in (23)). When $g' > 0$, higher θ_t, by reducing κ_{t+1}, has an undesirable negative impact on θ_{t+1}, and this discourages voting for immigration.

We have explored to what extent the baseline results are sensitive to parameters. To illustrate the role of CRRA preferences, Figure 1 shows the immigrant fertility ratios required to explain immigration rates of 4% and 8% for different levels of intertemporal substitution. Required fertility ratios drop sharply as γ rises above one and then flatten out. Hence the baseline case ($\gamma = 4$) is broadly representative for substitution elasticities in the empirically plausible range (say, $\gamma \geq 2$).

[20] This finding is robust across parameter settings provided $\gamma > 1$.

Figure 1. Parameter combinations of intertemporal substitution and fertility that provide voter support for immigration



5.3 Voting with Markov Strategies and Persistence

Persistence is modeled as an exogenous probability p that political frictions prevent a change in immigration policy in any given period, as in Section 4.1.

With persistence, rational voters will anticipate that if a vote about immigration takes place in the next period (in state I), future voters will set θ^I_{t+1} as function of the state variable κ_{t+1}. Hence optimal voting behavior is again defined by a Markovian function g_p such that $\theta_t = g_p(\kappa_t)$ is optimal under the expectation that with probability $1-p$, $\theta^I_{t+1} = g_p(\kappa_{t+1})$ is determined by the same function, and that with probability p, $\theta^{II}_{t+1} = \theta_t$ is predetermined. Interior solutions again imply a first-order condition that can be solved numerically (see appendix).

We present our analysis of this case in two steps. In the first step we study the persistence required by the model in order to be consistent with US immigration rates. In the second step, presented in the following section, we discuss the US historical record of immigration reforms and show that the spacing of reforms is broadly consistent with the persistence required by the model.

For the baseline calibration, we obtain $\theta^* = 8\%$ for persistence $p = 0.303$. That is, *the observed U.S. immigration is optimal in the model, if voters believe that there is about a* 30% *chance of no change in immigration policy within the next generation.*

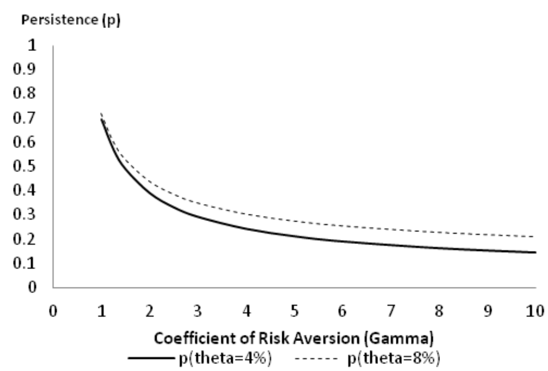
The exact persistence required to explain observed immigration depends of course on model parameters, but the p-values turn out to be quite robust. Figure 2 illustrates the dependence of persistence on preferences, showing values of p that explain $\theta^* = 8\%$ and (for comparison) $\theta^* = 4\%$ for a range of γ-values.

The figure suggests that for a broad range of preference parameters, U.S. immigration is consistent with optimal voting when voters expect political frictions and estimate odds of no policy change between 20% and 40%.

To explore sensitivity to other parameters, we consider several parametric variations from the baseline and compute persistence values required for 8%

immigration. For the capital share, we find $p = 0.28$ for $\alpha = 0.3$ and $p = 0.35$ for $\alpha = 0.4$. For the return-to-growth ratio r, we find $p = 0.20$ for $r = 1.3$ and $p = 0.40$ for $r = 1.7$. These p-values are again in the 20% to 40% range. For the fertility ratio, we find $p = 0.45$ for $\varepsilon = 1.3$ and $p = 0.16$ for $\varepsilon = 1.7$. The latter confirms that the fertility ratio–our key variable–has indeed a substantial impact. Since $[1.3, 1.7]$ is the empirically relevant range for the fertility ratio, we find that some persistence is needed for the fertility argument to provide a positive theory of immigration.

Figure 2. Parameter combinations of intertemporal substitution and persistence (p) that provide voter support for immigration



5.4 US Immigration Law and Persistence

To infer persistence from historical data, note that if there is a constant annual probability q_a that immigration law will change, the number of years that the law remains unchanged (denoted X) is distributed geometrically with mean $E[X] = 1/q_a$. One may estimate q_a by average number of years that the law remains unchanged, and one may infer $q = (1 - q_a)^{25}$ as the probability that the law remains in place for a generation (the base period in our model).

Immigration to the US at the end of the 19th century and in the early 20th century was largely unregulated, though Congress enacted several laws aiming to excluding certain groups of people, including Asians (1875 and 1917), poligamists and sick people (1891), the Chinese specifically (1882), and anarchists, beggars and importers of prostitutes (1903)).[21]

In 1921, Congress attempted to limit and regulate immigration by passing the *Emergency Quota Act*, which set numerical quotas for immigration based on nationality and the composition of US population. These quotas were set at 3% of the foreign-born population of the respective nationalities in the 1910 census and total annual immigration was capped at 350,000. This law was revisited

[21] See Cohn (2016) for a summary of these laws.

in 1924 when the *National Origins Quota Act* was passed, which revised the quotas at a level of 2% and total annual immigration was capped at 165,000.

The reforms in the 1920's determined immigration until 1965, when the *Immigration and Nationality Act* replaced the quota system and which created different immigration categories with an emphasis in family reunification and skilled immigration. Current immigration policy is in large part still determined by the architecture of this law. *The Immigration Act* of 1990 revised somewhat the preference categories, created temporary visas H1B for skilled workers and H2B for seasonal, non-agricultural workers, and increased the annual immigration cap in the total number of visas where the majority are designated for family sponsored immigration. In addition, there were smaller reforms aimed at solving particular temporary problems like the bracero program (1942 - 1964) and the *Immigration Reform and Control Act* of 1986 (IRCA) which provided a path for illegal immigrants to regularize their status.

The U.S. experience shows that immigration laws changed quite frequently in the 1875-1925 period, but only rarely since then, suggesting a break in persistence. Using data since World War II to estimate persistence in modern times, one finds three reforms (1965, 1986, 1990) in the 72-year period of 1945-2016, suggesting $\widehat{q}_a = 1/24 = 4.2\%$ and $\widehat{p} = 35\%$. Alternatively, if one uses the Great Depression as starting point, one finds four reforms (1942, 1965, 1986, 1990) over 87 years (1930-2016), suggesting $\widehat{q}_a = 4.6\%$ and $\widehat{q} = 31\%$. These values are well within the range of p-values required by the model.

One major caveat is that unchanged immigration law might not indicate frictions but deliberate political choices not to make changes. Hence we interpret $\widehat{q}$ cautiously as upper bound for p and do not attempt to *calibrate* p. However, there are indications of frictions, especially since World War II. For example, the bracero program started in 1942 as a solution to wartime labor shortages but was left unchanged until long after the war (1965). More recently, there were major attempts at overhauling immigration policy in 2006 (*Comprehensive Immigration Reform Act* which passed the US Senate but not the House), in 2007 (*Secure Borders, Economic Opportunity and Immigration Reform Act* which was introduced but never voted on) and in 2013 (*Border Security, Economic Opportunity, and Immigration Modernization Act* which passed the Senate but was not considered by the House).[22]

This history of failed reforms suggest that persistence is significantly due to exogenous frictions, so the model parameter p is close to $\widehat{q}$. In this sense, our quantitative model is consistent with the U.S. experience.

[22] In addition, the executive decisions by the Obama administration in 2012 (*Deferred Action for Childhood Arrivals*) and 2014 (*Deferred action for Parents of Americans and Lawful Permanent Residents*) that give temporary relief from deportation to persons brought to the US illegally as children and for parents that have small children born in the US, respectively, attest to the difficulty of passing comprehensive immigration reform and thus consistent with a view that the law is persistent.

6 Conclusions

The paper shows that a fertility differential between immigrants and natives rates can help explain voter support for immigration. In the model, the median voter is a young worker who trades off the negative wage effects of current immigration against the higher returns to savings implied by an increase in next period's labor force. High immigrant fertility favors immigration because more children imply a greater increase in next period's labor force per current (working) immigrant.

High immigrant fertility is empirically relevant. According to a variety of sources, U.S. immigrants have fertility rates about 50% higher than U.S. natives ($\pm$20%); similar differences have been found in Europe.

Persistence in immigration policy is a quantitatively important supporting factor. We model persistence in two ways, as political friction that may, with positive probability, prevent a policy change, and as result of expectational linkages. Regardless of the underlying causes, persistence favors immigration because it magnifies the impact of voting for immigration on future labor supply and hence on voters' returns to savings.

We first present the conceptual points in a simple setting with log-utility. Log-utility simplifies the analysis significantly because the savings rate is then constant, capital drops out as state variable, and so optimal immigration is a number. Log-utility is quantitative unappealing, however, because it understates the utility value of high returns to savings, as compared to preferences with lower elasticity of intertemporal substitution. Put differently, explaining observed U.S. immigration rates (about 8% per generation) in a log-utility model requires rather high persistence–probabilities of no policy change of 70% or more.

Hence we also examine immigration in a more general setting with CRRA preferences, focusing on empirically relevant cases of elasticities of intertemporal substitution less than one. Optimal immigration is then a function of the initial capital per native worker. If voters expect that future immigration is determined similarly, a voting equilibrium is defined by a Markovian policy function that is optimal under the expectation that future voters use the same function. Compared to log-utility, low elasticities of substitution favor immigration, but the endogeneity of future immigration turns out to be a deterrent. (We isolate the former effect in a version with static expectations.) For empirically plausible elasticities of intertemporal substitution (1/2 or less), we find that observed U.S. immigration rates are consistent with optimal voting if voters expect political frictions to generate a probability of no policy change (at the generational level) around 30%. This probability is not inconsistent with the persistence inferred from US history on immigration reform.

The model clearly abstracts from many other issues that may be relevant for immigration policy. This is for clarity, to show that differential fertility is an important factor. Our modeling of persistence may be interpreted broadly as illustrating how differential fertility may interact with other forces that might enter into a more elaborate model of immigration.

Our analysis also shows that immigration policy has redistributive effects across generations that have similarities to pay-as-you-go social security. That is, if the economy is dynamically efficient, voting to allow immigration increase the utility of current voters, and possibly the utility of a few succeeding generations; but utility in steady state is less than in an economy without immigration, so after a transition phase, all future generations are worse off. The analogy to social security facilitates our modeling because we can apply game-theoretic approaches to expectational linkages developed by the social security literature.

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7 Appendix (Not for Publication)

The appendix is included here to facilitate review. We intend to provide it as online resource.

7.1 Fertility Data for Europe

Sobotka (2008) compiles empirical evidence from several studies on total fertility rates for several European countries. The table below reproduces Sobotka's estimates from his tables 2.a and 2.b and the implied fertility factor (ε) is computed. In every case fertility is higher for immigrants. There are two types of estimates: countries labeled with **(a)** show the comparison between fertility of "native" versus fertility of "immigrant" women, while countries labeled with **(b)** compare "native" nationals with "foreign" nationals. For more details on the particular data sets used for the estimate for each country, see Sobotka (2008).

Table A.1. Total Fertility Rates by Native and Immigrant Status
Several European Countries and Years

Country	Native Fertility	Immigrant Fertility	Ratio (ε)	Year
Austria (b)	1.29	2.03	1.57	2001 – 2005
Belgium (b)	1.49	2.13	1.43	2001 – 2005
Flanders (Belgium)(b)	1.5	3	2	1995
Denmark (a)	1.69	2.43	1.44	1999 – 2003
England and Wales (a)	1.6	2.2	1.38	2001
France (a)	1.65	2.50	1.52	1991 – 1998
France (ii) (a)	1.70	2.16	1.27	1991 – 1998
France (b)	1.72	2.80	1.63	1999
France (b)	1.80	3.29	1.83	2004
Italy (b)	1.26	2.61	2.07	2004
Netherlands (a)	1.65	1.97	1.19	2005
Norway (a)	1.76	2.42	1.38	1997 – 1998
Spain (b)	1.19	2.12	1.78	2002
Sweden (a)	1.72	2.01	1.17	2005
Switzerland (b)	1.34	1.86	1.39	1997

Source: Sobotka (2008) tables 2.a and 2.b.

(a): Native vs. immigrant women. (b) : Native nationals vs. foreign nationals.

For (ii), data is adjusted for age of arrival and duration of stay in France.

7.2 Analysis in Sections 2-3

7.2.1 Utility as Function of Factor Prices

We claim in Section 2 that (4) implies the indirect utility (6).

Proof: For log-utility ($\gamma = 1$), maximizing

$$U\left(c_t^1, c_{t+1}^2\right) = \ln c_t^1 + \beta \ln c_{t+1}^2 \tag{A.1}$$

straightforwardly yields optimal consumption and savings

$$c_t^1 = \frac{1}{1+\beta} w_t,\ c_{t+1}^2 = \frac{\beta}{1+\beta} w_t R_{t+1}, \text{ and } s_t = \frac{\beta}{1+\beta} w_t.$$

Inserting these expressions into (A.1), using rules of logarithms and collecting similar terms one obtains

$$\begin{aligned} U\left(w_t, R_{t+1}\right) &= A_1 + (1+\beta)\ln w_t + \beta \ln R_{t+1} \\ A_1 &= \left[\beta \ln \beta - (1+\beta)\ln(1+\beta)\right] \end{aligned} \tag{A.2}$$

The constant $A_1 = [\beta \ln \beta - (1+\beta)\ln(1+\beta)]$ is inessential and omitted from (6) for simplicity.

For power utility ($\gamma \neq 1$), the first order conditions of (4) imply

$$s_t = \frac{(\beta R_{t+1})^{1/\gamma}}{R_{t+1} + (\beta R_{t+1})^{1/\gamma}} w_t,\ c_{1,t} = \frac{R_{t+1}}{R_{t+1} + (\beta R_{t+1})^{1/\gamma}} w_t, \text{ and } c_{2,t+1} = (\beta R_{t+1})^{\frac{1}{\gamma}} c_{1,t}.$$

Substituting consumption into the utility function, one obtains:

$$U\left(w_t, R_{r+1}\right) = \frac{1}{1-\gamma} \frac{\left\{1 + \beta^{\frac{1}{\gamma}} R_{t+1}^{\frac{1}{\gamma}-1}\right\}}{\left(1 + \beta^{\frac{1}{\gamma}} R_{t+1}{}^{\frac{1}{\gamma}-1}\right)^{1-\gamma}} w_t^{1-\gamma} = \frac{1}{1-\gamma} \left\{1 + \beta^{\frac{1}{\gamma}} R_{t+1}^{\frac{1}{\gamma}-1}\right\}^{\gamma} w_t^{1-\gamma},$$

as claimed. *QED*.

7.2.2 Extensions with stochastic mortality

We claim in footnote 8 that our maintained assumption of a working-age majority is a robust result in extended versions with stochastic mortality. This appendix provides an illustration.

To model stochastic mortality within a two-period setting, suppose there is idiosyncratic uncertainty about survival at the end of working age: individuals die with probability $(1-\pi)$ and survive with probability π. These assumption largely follow Bohn (2001). The consumption/saving problem of individuals is then

$$\begin{aligned} U_t &= \underset{c_t^1, \hat{c}_{t+1}^2, s_t}{Max} \left\{u(c_t^1) + E[\hat{\beta} u(\hat{c}_{t+1}^2)]\right\} \\ &= \underset{c_t^1, c_{t+1}^2, s_t}{Max} \left\{u(c_t^1) + \pi \hat{\beta} u(c_{t+1}^2)\right\} \\ s.t. \quad c_t^1 + s_t &= w_t \text{ and } c_{t+1}^2 = \hat{R}_{t+1} s_t + Q_{t+1}, \end{aligned} \tag{A.3}$$

where $E[\cdot]$ is the expectation over survival, $\hat{\beta}$ pure time preference, $\hat{c}^2_{t+1} = c^2_{t+1}$ in case of survival, $\hat{c}^2_{t+1} = 0$ otherwise, and Q_{t+1} denotes bequests (if any). The return $\hat{R}_{t+1}$ conditional on survival depends on the availability of annuity markets. With actuarially fair annuities, $\hat{R}_{t+1} = R_{t+1}/\pi$, as all savings are allocated to survivors. Without annuities, $\hat{R}_{t+1} = R_{t+1}$ is the return on savings, and one must make assumptions about the disposal of deceased agents' assets.

In general, denote survivors return to saving by $\hat{R}_{t+1} = R_{t+1}/\pi^a$ where $a \in [0, 1]$ admits intermediate degrees of annuitization. Then the first order condition for optimal saving is $u'(c^1_t) = \pi\hat{\beta}\hat{R}_{t+1}u'(c^2_{t+1}) = \pi^{1-a}\hat{\beta}R_{t+1}u'(c^2_{t+1})$. Accidental bequests are $(R_{t+1}/\pi - \hat{R}_{t+1})s_t$. Without much loss of generality, assume accidental bequests are shared by surviving members of the old generation. Then $c^2_{t+1} = \hat{R}_{t+1}s_t + Q_{t+1} = R_{t+1}s_t/\pi$ applies regardless of annuitization, and optimal saving imply $(w_t - s_t)^{-\gamma} = \pi^{1-a}\hat{\beta}R_{t+1}(R_{t+1}s_t/\pi)^{-\gamma}$, and hence $\sigma_t = \frac{s_t}{w_t} = \frac{\hat{B}_t}{1+\hat{B}_t}$, where $\hat{B}_t = \left(\pi^{1-a+\gamma}\hat{\beta}\right)^{\frac{1}{\gamma}}(R_{t+1})^{\frac{1-\gamma}{\gamma}}$. Thus, individual consumption and saving are the same as in the model with 100% survival and time preference $\beta = \pi^{1-a+\gamma}\hat{\beta}$.

Since the voting population consists of $N_t = N_{t-1}\eta\,(1 + \varepsilon\theta_{t-1})$ young and πN_{t-1} old agents, the voting share of the young is given by $\eta\,(1 + \varepsilon\theta_{t-1})\,/[\pi + \eta\,(1 + \varepsilon\theta_{t-1})]$. Since $\theta_{t-1} \geq 0$, the voting share of the young is bounded below by $\eta/(\pi + \eta)$. Empirically, U.S. life expectancy at age 65 is about 20 years (male and female averaged according to the Social Security Administration[23]), so π can be estimated as (remaining life expectancy)/(number of years in workforce), which yields $\pi \approx 1/2$. Hence $\eta > 1/2$ is a sufficient condition for the young to be the majority in absence of immigration. In section 5.1 we calibrate $\eta = 1.116$, which rules out immigration cycles by a wide margin.[24]

7.2.3 Utility as Function of Immigration

Section 3 claims that utility in terms of immigration quotas is given by (14) and that dynamically efficiency corresponds to $\chi \geq 1 - \alpha$.

[23] According to the Social Security Administration, conditional on reaching 65 years of age, a man in the US is expected to live until age 84.3, while a woman is expected to live until age 86.6. See https://www.ssa.gov/planners/lifeexpectancy.html .

[24] The particular assumptions required for immigration cycles are: (i) Fertility and mortality such that the old generation is the majority in the absence of immigration: $N_{t+1} = \eta N_t < \pi N_t$, which implies that $\eta < \pi$; and (ii) a policy space with maximum immigration quota $\theta^{\max}$ high enough that in the next period there are more young agents than (alive) old agents: $N_{t+1} = N_t\eta\,(1 + \varepsilon\theta^{\max}) > \pi N_t$ (implies $\eta\,(1 + \varepsilon\theta^{\max}) > \pi$). Under these assumptions, the initial old majority would choose the maximum quota available ($\theta^*_t = \theta^{\max}$). Then since the majority in next period is the young cohort because $\eta\,(1 + \varepsilon\theta^{\max}) > \pi$, this young majority restricts immigration in order to remain the majority in the following period (when they are old), which is a period in which they liberalize immigration. That particular cohort controls policy when young and when old. The immigration quota that the young majority selects is either $\theta^*_{t+1} = 0$ or a sligthly positive number (depending on how large ε is), taking into account that the they will be in power the next period and thus choosing the maximum quota in that period ($\theta^*_{t+2} = \theta^{\max}$). A cycle of restriction and then liberalization repeats, tracking the life cycle of cohorts that remain in power during all their lifetimes and similarly, each cycle sees a generation that is never in power, and so on.

Proof of (14): Inserting (9) into (8), one obtains

$$k_{t+1} = \frac{\beta(1-\alpha)}{\eta\,(1+\beta)} \frac{(1+\theta_t)}{(1+\varepsilon\theta_t)\,(1+\theta_{t+1})} k_t^{\alpha} \tag{A.4}$$

Substituting this into $R_{t+1} = \alpha\,(k_{t+1})^{-1+\alpha}$ and substitution R_{t+1} into the utility function (A.2), we obtain

$$\begin{aligned} \widehat{V}\,(\theta_t, \theta_{t+1}, k_t) &= A_1 + (1+\beta)\ln(1-\alpha)\,k_t^{\alpha} \\ &\quad -\beta\,(1-\alpha)\ln\left(\frac{\beta\,(1-\alpha)}{(1+\beta)} \frac{(1+\theta_t)}{\eta\,(1+\varepsilon\theta_t)\,(1+\theta_{t+1})} k_t^{\alpha}\right) \end{aligned}$$

Finally, substituting $k_t = \frac{\kappa_t}{1+\theta_t}$, collecting similar terms, and dividing the equation by the constant $\beta\,(1-\alpha)$, we obtain the indirect utility function that depends on θ_t, θ_{t+1} and κ_t :

$$V\,(\theta_t, \theta_{t+1}, \kappa_t) = A + \chi \ln \kappa_t - (1+\chi)\ln(1+\theta_t) + \ln(1+\varepsilon\theta_t) + \ln(1+\theta_{t+1})\,. \tag{A.5}$$

where χ is a constant given by $\chi = \frac{\alpha(1+\beta\alpha)}{\beta(1-\alpha)}$. We write $V\,(\theta_t, \theta_{t+1}, \kappa_t)$, where we condition on κ_t since agents take it as given. Q.E.D.

Regarding dynamic efficiency, note that the steady state ratio of return to capital is greater than (or equal to) population growth if and only if $\chi > 1-\alpha$ (or $\chi = 1-\alpha$). In detail, (A.4) implies that k_{t+1} converges to $k = \left(\frac{\beta}{1+\beta}\frac{1-\alpha}{\eta(1+\varepsilon\theta)}\right)^{1/(1-\alpha)}$ for any constant $\theta_t = \theta$, and hence R_{t+1} converges to $R = \alpha/\left(\frac{\beta}{1+\beta}\frac{1-\alpha}{\eta(1+\varepsilon\theta)}\right)$. Since $\eta\,(1+\varepsilon\theta)$ is population growth, the return-to-growth ratio is $\frac{R}{\eta(1+\varepsilon\theta)} = \frac{\alpha}{1-\alpha}\frac{1+\beta}{\beta}$. Moreover,

$$\chi + \alpha = \frac{\alpha\,(1+\alpha\beta)}{\beta\,(1-\alpha)} + \alpha = \frac{\alpha}{\beta\,(1-\alpha)}\left[1+\alpha\beta+\beta\,(1-\alpha)\right] = \frac{\alpha}{1-\alpha}\frac{1+\beta}{\beta}.$$

Hence $\frac{R}{\eta(1+\varepsilon\theta)} = 1$ iff $\chi = 1-\alpha$ and $\frac{R}{\eta(1+\varepsilon\theta)} > 1$ iff $\chi > 1-\alpha$.

7.2.4 Welfare Comparisons

In the baseline model, we claim that a transition from zero immigration to $\theta^0 = \frac{\varepsilon-(1+\chi)}{\chi\varepsilon} > 0$ will (a) increase welfare for one or more generations and (b) reduce welfare in the long run, provided the economy is dynamically efficient. The following provides a constructive proof:

First we write the evolution of capital per-native worker by substituting $k_t = \frac{\kappa_t}{(1+\theta_t)}$ into equation (A.4). We obtain

$$\kappa_{t+1} = \varpi \frac{(1+\theta_t)^{1-\alpha}}{(1+\varepsilon\theta_t)} \kappa_t^{\alpha}, \text{ where } \varpi = \frac{\beta(1-\alpha)}{\eta\,(1+\beta)}. \tag{A.6}$$

Since we will compare the lifetime utility of agents with and without immigration, it is convenient to define $V_t^0 = V\left(\theta^0, \theta^0, \kappa_t\right)$ as lifetime utility in the regime with immigration and to define $\tilde{V}_t = V\left(0, 0, \kappa_t\right)$ as lifetime utility in a regime without immigration. Using (A.5), lifetimes utility in the immigration regime is given by

$$V_t^0 = V\left(\theta^0, \theta^0, \kappa_t\right) = A + \chi \ln \kappa_t + \Omega \tag{A.7}$$

(since $\theta_t = \theta^0$ is constant), where $\Omega = \ln\left(1 + \varepsilon\theta^0\right) - \chi \ln\left(1 + \theta^0\right)$. Note that $\varepsilon > (1 + \chi)$ implies $\Omega > 0$.

Taking logs in equation (A.6), we can write the evolution of κ_t in the regime with immigration as

$$\begin{aligned} \ln \kappa_{t+1} &= \ln \varpi - \Delta + \alpha \ln \kappa_t \\ \text{where } \Delta &= \ln\left(1 + \varepsilon\theta^0\right) - (1 - \alpha) \ln\left(1 + \theta^0\right) \end{aligned}$$

Note that $\Delta > \Omega$ for $\chi > (1 - \alpha)$ and $\theta^0 > 0$, which applies under conditions of dynamic efficiency.

To compare utilities across regimes, we write lifetime utility in terms of the initial value κ_0 in some starting period labeled $t = 0$. For all $t > 0$, we have:

$$\ln \kappa_t = \frac{(1 - \alpha^t)}{1 - \alpha} \ln \varpi - \frac{(1 - \alpha^t)}{1 - \alpha} \Delta + \alpha^t \ln \kappa_0.$$

Using (A.7), the sequence of lifetime utilities is

$$V_t^0 = A + \chi \frac{(1 - \alpha^t)}{1 - \alpha} \ln \varpi + \alpha^t \ln \kappa_0 + \left[\Omega - \chi \frac{(1 - \alpha^t)}{1 - \alpha} \Delta\right].$$

In a regime without immigration, analogous dynamics apply with $\Omega = \Delta = 0$, so the sequence of lifetime utilities would be

$$\tilde{V}_t = A + \chi \frac{(1 - \alpha^t)}{1 - \alpha} \ln \varpi + \alpha^t \ln \kappa_0$$

Hence the difference between lifetime utilities with and without immigration is

$$V_t^0 - \tilde{V}_t = \left[\Omega - \chi \frac{(1 - \alpha^t)}{1 - \alpha} \Delta\right]$$

Notice that $V_0^0 - \tilde{V}_0 = \Omega > 0$ is positive for the generation $t = 0$ and that because $\Delta > 0$, $V_t^0 - \tilde{V}_t$ declines monotonely over time. As $t \to \infty$, $(V_t^0 - \tilde{V}_t) \to \Omega - \frac{\chi\Delta}{1-\alpha}$. Note that

$$
\begin{aligned}
\Omega-\frac{\chi\Delta}{1-\alpha} &= -(1+\chi)\ln\left(1+\theta^0\right)+\ln\left(1+\varepsilon\theta^0\right)+\ln\left(1+\theta^0\right) \\
&\quad -\frac{\chi}{1-\alpha}\left\{\ln\left(1+\varepsilon\theta^*\right)-(1-\alpha)\ln\left(1+\theta^0\right)\right\} \\
&= \left(1-\frac{\chi}{1-\alpha}\right)\ln\left(1+\varepsilon\theta^0\right)
\end{aligned}
$$

Hence $\lim_{t\to\infty}(V_t^0-V_t)<0$ and only if $\chi>(1-\alpha)$. Monotonicity then implies there is a date $\bar{t}$ such that $V_t^0-V_t>0$ for all $t<\bar{t}$, whereas $V_t^0-V_t<0$ for all $t>\bar{t}$. Q.E.D.

7.3 Analysis in Section 4

Section 4.1 asserts that indirect utility is (18). Proof: Let $j=I,II$ denote the states, $p_I=1-p$, $p_{II}=p$. Then the individual problem for general CRRA utility is

$$
\begin{aligned}
U_t &= \underset{c_t^1,c_{t+1,i}^2,s_t}{Max}\left\{u(c_t^1)+\beta\sum_j p_j\cdot u(c_{t+1,j}^2)\right\} \qquad \text{(A.8)} \\
s.t.\quad c_t^1+s_t &= w_t,\; c_{t+1,j}^2=R_{t+1,j}s_t
\end{aligned}
$$

The first order condition for optimal savings is $u'(w_t-s_t)=\beta\sum_j p_j\cdot R_{t+1,j}u'(R_{t+1,j}s_t)$, and it implicit defines the optimal savings rate

$$
\sigma_t=\frac{s_t}{w_t}=\frac{B_t}{1+B_t},\text{ where} \qquad \text{(A.9)}
$$

$$
B_t=\beta^{\frac{1}{\gamma}}\left[\sum_j p_j\cdot(R_{t+1,j})^{1-\gamma}\right]^{\frac{1}{\gamma}} \qquad \text{(A.10)}
$$

For log-utility ($\gamma=1$), this reduces to $B_t=\beta$. Substituting consumption and savings (A.9) into (A.8), one obtains

$$
U_t=(1+\beta)\ln w_t+\beta E\left[\ln R_{t+1}\right]+const
$$

Substituting wages (10) and returns (10), this implies

$$
\begin{aligned}
\widehat{V} &= -\left(\alpha\left(1+\beta\alpha\right)+\beta\left(1-\alpha\right)\right)\ln\left(1+\theta_t\right)+\beta\left(1-\alpha\right)\ln\left(1+\varepsilon\theta_t\right) \\
&\quad +\beta\left(1-\alpha\right)E\left[\ln\left(1+\widehat{\theta}_{t+1}\right)\right]+exog
\end{aligned}
$$

where immigration $\widehat{\theta}_{t+1}$ is treated as random variable and *exog* summarizes exogenous terms (inessential constants and initial conditions). The possible realizations for $\widehat{\theta}_{t+1}$ are $\widehat{\theta}_{t+1} = \theta^I_{t+1}$ with probability $1-p$ (chosen by gen. $t+1$) and $\widehat{\theta}_{t+1} = \theta^{II}_{t+1} = \theta_t$ with probability p. Therefore $\widehat{V}$ can be written as

$$\begin{aligned}\widehat{V} &= -\left(\alpha\left(1+\beta\alpha\right)+\beta\left(1-\alpha\right)\right)\ln\left(1+\theta_t\right)+\beta\left(1-\alpha\right)\ln\left(1+\varepsilon\theta_t\right)\\ &\quad +\beta\left(1-\alpha\right)\left(1-p\right)\ln\left(1+\theta_{t+1}\right)+\beta\left(1-\alpha\right)p\ln\left(1+\theta_t\right)+exog\end{aligned}$$

Dividing by $\beta\left(1-\alpha\right)$ and simplifying, one obtains

$$V\left(\theta_t,\theta^I_{t+1}\right) = -\left(1+\chi-p\right)\ln\left(1+\theta_t\right)+\ln\left(1+\varepsilon\theta_t\right)+(1-p)\ln\left(1+\theta^I_{t+1}\right)+exog$$

If state I applies in period t, $\theta_t = \theta^I_t$ is set by generation t and $V\left(\theta^I_t,\theta^I_{t+1}\right)$ is the relevant indirect utility. *QED.*

We also claim that $\theta^p = \frac{\varepsilon-1-\chi+p}{\varepsilon(\chi-p)}$ is increasing in p and in ε. As proof, note that

$$\begin{aligned}\frac{d\theta^p}{dp} &= \frac{\varepsilon\left(\chi-p\right)+\left(\varepsilon-1-\chi+p\right)\varepsilon}{\varepsilon^2\left(\chi-p\right)^2} = \frac{\left(\varepsilon-1-\chi+p\right)\varepsilon}{\varepsilon\left[\left(\chi+p\right)-1\right]^2} = \frac{\varepsilon-1}{\varepsilon\left(\chi-p\right)^2} > 0\\ \frac{d\theta^p}{d\varepsilon} &= \frac{\varepsilon\left(\chi-p\right)-\left(\varepsilon-1-\chi+p\right)\left(\chi-p\right)}{\varepsilon^2\left(\chi-p\right)^2} = \frac{\chi+1-p}{\varepsilon^2\left(\chi-p\right)} > 0.\end{aligned}$$

for all (ε,χ,p) such that $0<\theta^p<\bar{\theta}$ is not a corner solution. *QED.*

7.4 Details on Calibrating the CRRA model in Section 5

The parameters to be calibrated are $\{\alpha,\beta,\varepsilon,\gamma,\eta\}$. We explain each of them in this section.

Externally calibrated parameters. The externally calibrated parameters are α, ε and γ. We use baseline values $\alpha = \frac{1}{3}$ and $\varepsilon = 1.5$ as discussed in Section 3.2. Regarding γ, typical overlapping generation economies in the macroeconomics and the finance literature use risk aversion levels between 2 and 5 (see for example Auerbach and Kotlikoff (1987), Rios-Rull (1996), Constantinides et al. (2002), Conesa and Garriga (2008) and Evans et al. (2012)). We use a baseline of $\gamma = 4$, but also explore the sensitivity of the fertility required to different levels of risk aversion.

Internally calibrated parameters. The internally calibrated parameters are η and β. The total-growth factor of the population in the model is given by $\eta^+ \equiv \eta\left(1+\varepsilon\theta\right)$, which we set at 1.25 (about 1% growth per year), which is consistent with US population growth.[25] Given η^+, one can infer η from ε and θ by writing $\eta = \eta^+/\left(1+\varepsilon\theta\right)$. Using $\varepsilon = 1.5$ and $\theta = .08$, this yields $\eta = \frac{1.25}{1+1.5\cdot0.08} = 1.116$.

[25] Using US census data from 1970 to 2010, the annual population growth rate for this 40 year period is $\left(\frac{308.45}{205.52}\right)^{1/40} - 1 = 1.02\%$.

For the calibration of β, we show that one can derive a calibrating expression that is invariant to the particular equilibrium concept used. Start with equation (8), which shows the evolution of capital per native worker in the CRRA case, given by

$$\kappa_{t+1} = \frac{(1+\theta_t)^{1-\alpha}}{\eta(1+\varepsilon\theta_t)}\sigma_t w_t.$$

replacing $w_t = (1-\alpha)\kappa_t^{\alpha}$, the evolution of capital per native worker can be written as

$$\kappa_{t+1} = \frac{(1-\alpha)(1+\theta_t)^{1-\alpha}}{\eta(1+\varepsilon\theta_t)}\sigma_t \kappa_t^{\alpha}$$

where σ_t is the saving rate. At steady state, this equation can be solved for $\frac{1}{\sigma}$ as

$$\frac{1}{\sigma} = \frac{(1-\alpha)}{\eta(1+\varepsilon\theta)}\left(\frac{\kappa}{1+\theta}\right)^{\alpha-1}$$

Since the gross interest rate at steady state R is given by $R = \alpha\left(\frac{\kappa}{1+\theta}\right)^{\alpha-1}$. The above expression can be written as

$$\frac{1}{\sigma} = \frac{(1-\alpha)}{\alpha}\frac{R}{\eta(1+\varepsilon\theta)}. \tag{A.11}$$

From the definition of σ_t in equation (5), at steady state $\sigma = \frac{\beta^{\frac{1}{\gamma}}R^{\frac{1}{\gamma}-1}}{1+\beta^{\frac{1}{\gamma}}R^{\frac{1}{\gamma}-1}}$, which implies that the term $\frac{1}{\sigma}$ is also given by

$$\frac{1}{\sigma} = 1 + \frac{1}{\beta^{\frac{1}{\gamma}}R^{\frac{1}{\gamma}-1}}. \tag{A.12}$$

Therefore, at steady state equating (A.11) and (A.12) yields an equality $1 + \frac{1}{\beta^{\frac{1}{\gamma}}R^{\frac{1}{\gamma}-1}} = \frac{(1-\alpha)}{\alpha}\frac{R}{\eta(1+\varepsilon\theta)}$ that can be solved for β as

$$\beta = \frac{R^{\gamma-1}}{\left[\frac{(1-\alpha)}{\alpha}\frac{R}{\eta(1+\varepsilon\theta)} - 1\right]^{\gamma}}.$$

Thus the calibration of β depends on the ratio $r \equiv \frac{R}{\eta(1+\varepsilon\theta)}$ as explained in the text. Since $\eta^{+} = \eta(1+\varepsilon\theta)$ is also empirically observed, R can be replaced by $r\cdot\eta(1+\varepsilon\theta)$ and hence β can be calibrated as

$$\widehat{\beta} = \frac{\left[(r\cdot\eta(1+\varepsilon\theta))\right]^{\gamma-1}}{\left[\frac{(1-\alpha)}{\alpha}r - 1\right]^{\gamma}}. \tag{A.13}$$

In the case of log utility ($\gamma = 1$), this simplifies to

$$\widehat{\beta}_{(\gamma=1)} = \frac{\alpha}{(1-\alpha)\, r - \alpha}.$$

as claimed in Section 3.2. For general γ and in the context of sensitivity analyses, we vary β as implied by the parameters on the r.h.s. of $\widehat{\beta}$.

7.5 Analysis in Section 5.1

7.5.1 Derivation of $\frac{dR_{t+1}}{d\theta_t}$

Section 5.1 claims that $\frac{dR_{t+1}}{d\theta_t} > 0$. Proof: Combining (8), (A.9), (A.10), (9), and writing (10) as $k_{t+1} = \left(\frac{\alpha}{R_{t+1}}\right)^{\frac{1}{1-\alpha}}$, one obtains

$$\left(\frac{\alpha}{R_{t+1}}\right)^{\frac{1}{1-\alpha}} = k_{t+1} = \frac{(1-\alpha)\,(1+\theta_t)^{1-\alpha}\,(\kappa_t)^{1-\alpha}}{\eta\,(1+\varepsilon\theta_t)\,(1+\theta_{t+1})} \frac{\beta^{\frac{1}{\gamma}}\,(R_{t+1})^{\frac{1}{\gamma}-1}}{1+\beta^{\frac{1}{\gamma}}\,(R_{t+1})^{\frac{1}{\gamma}-1}}.$$

Taking logs and differentiating with respect to θ_t, one obtains

$$\begin{aligned}
-\frac{1}{1-\alpha}\frac{d\ln R_{t+1}}{d\theta_t} &= \frac{1-\alpha}{1+\theta_t} - \frac{\varepsilon}{1+\varepsilon\theta_t} + (\frac{1}{\gamma}-1)\frac{1}{1+B_t}\frac{d\ln R_{t+1}}{d\theta_t} \\
\frac{d\ln R_{t+1}}{d\theta_t} &= (1-\alpha)\left[\frac{\varepsilon}{(1+\varepsilon\theta_t)} - \frac{(1-\alpha)}{(1+\theta_t)}\right] - \phi\frac{1}{1+B_t}\frac{d\ln R_{t+1}}{d\theta_t} \\
\frac{d\ln R_{t+1}}{d\theta_t} &= \frac{(1-\alpha)}{1+\phi/(1+B_t)}\left[\frac{\varepsilon}{(1+\varepsilon\theta_t)} - \frac{(1-\alpha)}{(1+\theta_t)}\right],
\end{aligned}$$

$$\frac{dR_{t+1}}{d\theta_t} = \frac{(1-\alpha)\,R_{t+1}}{1+\phi/(1+B_t)}\left[\frac{\varepsilon}{(1+\varepsilon\theta_t)} - \frac{(1-\alpha)}{(1+\theta_t)}\right]. \tag{A.14}$$

The sign is determined by $\left[\frac{\varepsilon}{(1+\varepsilon\theta_t)} - \frac{(1-\alpha)}{(1+\theta_t)}\right]$, which is positive iff $\varepsilon > \frac{(1-\alpha)}{(1+\alpha\theta_t)}$. Since $\varepsilon > 1$ and $\frac{1-\alpha}{1+\alpha\theta_t} \leq 1-\alpha$, $\frac{dR_{t+1}}{d\theta_t} > 0$ follows. *QED.*

7.5.2 The First Order Condition

Section 5.1 asserts the optimality condition (22). Proof: Maximizing (6) with respect to θ_t implies (using (A.10))

$$\begin{aligned}
\frac{dU_t}{d\theta_t} &= \{1+B_t\}^{\gamma}\, w_t^{-\gamma}\frac{dw_t}{d\theta_t} + \{1+B_t\}^{\gamma-1}\, w_t^{1-\gamma}\beta^{\frac{1}{\gamma}} R_{t+1}^{\frac{1}{\gamma}-2}\frac{dR_{t+1}}{d\theta_t} \quad \text{(A.15)} \\
&= \{1+B_t\}^{\gamma}\, w_t^{1-\gamma}\left[\frac{1}{w_t}\frac{dw_t}{d\theta_t} + \frac{B_t}{1+B_t}\frac{1}{R_{t+1}}\frac{dR_{t+1}}{d\theta_t}\right]
\end{aligned}$$

Hence $\frac{dU_t}{d\theta_t} = 0 \Leftrightarrow \frac{B_t}{1+B_t}\frac{1}{R_{t+1}}\frac{dR_{t+1}}{d\theta_t} = -\frac{1}{w_t}\frac{dw_t}{d\theta_t}$. Note that $\frac{1}{w_t}\frac{dw_t}{dk_t} = \frac{\alpha}{k_t}$ and $\frac{dk_t}{d\theta_t} = -\frac{k_t}{(1+\theta_t)}$, so $\frac{1}{w_t}\frac{dw_t}{d\theta_t} = -\frac{\alpha}{(1+\theta_t)} < 0$. Replacing $\frac{dR_{t+1}}{d\theta_t}$ by (A.14), one finds that $\frac{dU_t}{d\theta_t}=0$ is equivalent to

$$\frac{\alpha}{(1+\theta_t)} = (1-\alpha)\left[\frac{\varepsilon}{(1+\varepsilon\theta_t)} - \frac{(1-\alpha)}{(1+\theta_t)}\right] / \left(1+\left(\frac{1}{\gamma}+\alpha-\frac{\alpha}{\gamma}\right)\frac{1}{B_t}\right),$$

$$1+\left(\frac{1}{\gamma}+\alpha-\frac{\alpha}{\gamma}\right)\frac{1}{B_t} = \left(\frac{1-\alpha}{\alpha}\right)\left[\frac{\varepsilon(1+\theta_t)}{(1+\varepsilon\theta_t)}-(1-\alpha)\right] = \left(\frac{1-\alpha}{\alpha}\right)\left[\alpha-\frac{1-\varepsilon}{(1+\varepsilon\theta_t)}\right],$$

$$\left(\frac{1}{\gamma}+\alpha-\frac{\alpha}{\gamma}\right)\frac{1}{B_t} = \frac{1-\phi}{B_t} = \left[\left(\frac{1-\alpha}{\alpha}\right)\frac{\varepsilon-1}{(1+\varepsilon\theta_t)}\right]-\alpha.$$

Dividing by $(1-\phi)$, using (A.10) and (10) to replace B_t, one obtains (22). *QED*.

Note that the corner solution $\theta_t=0$ applies if $\frac{dU_t}{d\theta_t}\leq 0$ at $\theta_t=0$, which is equivalent to

$$\frac{\alpha}{(1+\theta_t)} - (1-\alpha)\left[\frac{\varepsilon}{(1+\varepsilon\theta_t)} - \frac{(1-\alpha)}{(1+\theta_t)}\right] / \left(1+(1-\phi)\,\beta^{-\frac{1}{\gamma}}R_{t+1}^{1-\frac{1}{\gamma}}\right) \geq 0, \text{ or}$$

$$\frac{1}{B_t} \geq \frac{1}{1-\phi}\left[\left(\frac{1-\alpha}{\alpha}\right)(\varepsilon-1)-\alpha\right].$$

7.5.3 The Dynamics of Capital per Worker

Proof of (23): For CRRA utility, the evolution of capital is given by

$$k_{t+1} = \frac{(1+\theta_t)}{\eta\,(1+\varepsilon\theta_t)\,(1+\theta_{t+1})}s_t = \frac{(1+\theta_t)}{\eta\,(1+\varepsilon\theta_t)\,(1+\theta_{t+1})}\frac{B_t}{1+B_t}w_t \qquad \text{(A.16)}$$

If optimal immigration has an interior solution, (22) implies

$$1+\frac{1}{B_t} = 1+\frac{\left(\frac{1-\alpha}{\alpha}\right)\frac{\varepsilon-1}{(1+\varepsilon\theta_t)}-\alpha}{1-\phi} = \frac{\frac{1}{\gamma}(1-\alpha)+\left(\frac{1-\alpha}{\alpha}\right)\frac{\varepsilon-1}{(1+\varepsilon\theta_t)}}{1-\phi}$$

Substituting into (A.16) and simplifying implies

$$k_{t+1} = \frac{(1+\theta_t)}{\eta\,(1+\theta_{t+1})}\frac{\alpha(1-\alpha+\alpha\gamma)}{\varepsilon\alpha(1+\theta_t)+(\gamma-\alpha)(\varepsilon-1)}k_t^{\alpha}, \qquad \text{(A.17)}$$

$$\kappa_{t+1} = \frac{1}{\eta}\frac{\alpha(1-\alpha+\alpha\gamma)}{\varepsilon\alpha(1+\theta_t)+(\gamma-\alpha)(\varepsilon-1)}\,(1+\theta_t)^{1-\alpha}\,\kappa_t^{\alpha}, \qquad \text{(A.18)}$$

which is (23). *QED* .

Note that for corner solutions with $\theta_t=0$, (A.16) and $R_{t+1}=\alpha k_{t+1}^{\alpha-1}$ imply

$$\kappa_{t+1}+\beta^{-\frac{1}{\gamma}}\alpha^{1-\frac{1}{\gamma}}(\kappa_{t+1})^{\phi}(1+\theta_{t+1})^{(1-\alpha)(1-\frac{1}{\gamma})} = \frac{1}{\eta}(1-\alpha)\kappa_t^{\alpha}.$$

Since $\phi>0$, the l.h.s. is strictly increasing in κ_{t+1}, so κ_{t+1} is determined uniquely.

7.5.4 Steady State

Section 5.1 claims there is a unique steady state (θ^0, κ^0). Proof: The steady state conditions are obtained by setting (θ, κ) constant in (22) and (23), which yields:

$$\kappa^0 = \frac{1}{\eta}\frac{\alpha(1-\alpha+\alpha\gamma)}{\varepsilon\alpha\theta^0 + (\gamma-\alpha)(\varepsilon-1)}\left(1+\theta^0\right)^{1-\alpha}(\kappa^0)^{\alpha}, \quad \text{(A.19)}$$

$$\beta^{-\frac{1}{\gamma}}(R^0)^{1-\frac{1}{\gamma}} = \beta^{-\frac{1}{\gamma}}\alpha^{1-\frac{1}{\gamma}}\left(\frac{\kappa^0}{1+\theta^0}\right)^{-\phi} = \frac{\left(\frac{1-\alpha}{\alpha}\right)\frac{\varepsilon-1}{1+\varepsilon\theta^0} - \alpha}{1-\phi} \quad \text{(A.20)}$$

where (A.19) can be simplified to obtain

$$\frac{\kappa^0}{1+\theta^0} = \left\{\frac{\alpha\left[(1-\alpha)+\alpha\gamma\right]}{\eta\left(\varepsilon\alpha(1+\theta^0) + (\gamma-\alpha)(\varepsilon-1)\right)}\right\}^{\frac{1}{1-\alpha}} \quad \text{(A.21)}$$

Substituting (A.21) into (A.20), one obtains

$$\beta^{-\frac{1}{\gamma}}\alpha^{1-\frac{1}{\gamma}}\left(\frac{\eta\left(\varepsilon\alpha(1+\theta^0) + (\gamma-\alpha)(\varepsilon-1)\right)}{\alpha\left[(1-\alpha)+\alpha\gamma\right]}\right)^{(1-\frac{1}{\gamma})} = \frac{\left(\frac{1-\alpha}{\alpha}\right)\frac{\varepsilon-1}{(1+\varepsilon\theta^0)} - \alpha}{1-\phi},$$

$$\beta^{-\frac{1}{\gamma}}\left(\frac{\eta\left(\varepsilon\alpha(1+\theta^0) + (\gamma-\alpha)(\varepsilon-1)\right)}{\left[(1-\alpha)+\alpha\gamma\right]}\right)^{(1-\frac{1}{\gamma})} = \frac{\left(\frac{1-\alpha}{\alpha}\right)\frac{\varepsilon-1}{(1+\varepsilon\theta^0)} - \alpha}{1-\phi}$$

which is a univariate equation for θ^0; the solution is unique since the l.h.s. is strictly increasing in θ^0 for $\gamma > 1$ whereas the r.h.s. is strictly decreasing. Given θ^0, (A.21) provides solutions for κ^0 and $k^0 = \kappa^0/(1+\theta^1)$. *QED.*

7.5.5 Convergence and Stability

Section 5.1 claims that the perfect foresight path $\{\theta_t, \kappa_t\}_{t\geq t_0}$ converges to (θ^0, κ^0). To streamline the algebra, we sometimes work with $x_t = 1+\theta_t$ and $k_t = \frac{\kappa_t}{x_t}$ (since we can always recover $\{\theta_t, \kappa_t\}_{t\geq t_0}$ from $\{x_t, k_t\}_{t\geq t_0}$). To streamline, we use the constants

$$\begin{aligned}
\phi_1 &= \frac{\alpha\left[(1-\alpha)+\alpha\gamma\right]}{\eta} > 0 \\
\phi_2 &= (\gamma-\alpha)(\varepsilon-1) \\
\phi_3 &= \frac{\eta^{\phi}\left[\alpha^{1+\alpha-\frac{\alpha}{\gamma}}\right]\left[(1-\alpha)+\alpha\gamma\right]^{\alpha+\frac{1-\alpha}{\gamma}}}{\beta^{\frac{1}{\gamma}}\gamma(1-\alpha)(\varepsilon-1)} > 0 \\
\phi_4 &= \frac{\alpha^2}{(1-\alpha)(\varepsilon-1)} > 0
\end{aligned}$$

and we omit superscripts for variables in steady state.

The dynamic system (A.16) and (22) can be written in terms of $\{x_t, k_t\}$ as

$$\begin{aligned} k_{t+1} &= \phi_1 \frac{x_t}{x_{t+1}\left[\varepsilon\alpha x_t + \phi_2\right]} k_t^{\alpha} \\ \frac{1}{\varepsilon x_t - (\varepsilon - 1)} &= \phi_3 \left[\frac{x_{t+1}\left[\varepsilon\alpha x_t + \phi_2\right]}{x_t k_t^{\alpha}}\right]^{\phi} + \phi_4, \end{aligned}$$

and the steady-state values are

$$\begin{aligned} k^1 &= \left\{\frac{\phi_1}{\left[\varepsilon\alpha x + \phi_2\right]}\right\}^{\frac{1}{1-\alpha}} \\ \frac{1}{\varepsilon x^1 - (\varepsilon - 1)} &= \phi_3 \phi_1^{-\frac{\alpha\phi}{1-\alpha}} \left[\varepsilon\alpha x^1 + \phi_2\right]^{\frac{\phi}{1-\alpha}} + \phi_4. \end{aligned}$$

To determine the stability of the system, we take a log-linear approximation around the steady state. Denote the percentage deviations from steady state by "^", e.g. $\widehat{z}_t = \ln z_t - \ln z^1$ for generic variable z_t. We obtain

$$\begin{aligned} \widehat{\kappa}_{t+1} &= \alpha\widehat{k}_t + b_0\widehat{x}_t = \alpha\widehat{\kappa}_t + (b_0 - \alpha)\widehat{x}_t \\ \widehat{x}_{t+1} &= \alpha\widehat{k}_t - (1/b_1)\widehat{x}_t = \alpha\widehat{\kappa}_t - (\alpha + 1/b_1)\widehat{x}_t, \end{aligned}$$

where

$$b_0 = \frac{\phi_2}{\varepsilon\alpha x^1 + \phi_2} = \frac{1}{1 + \frac{\varepsilon\alpha x^1}{(\gamma-\alpha)(\varepsilon-1)}} \tag{A.22}$$

$$b_1 = \frac{\left[\varepsilon x^1 - (\varepsilon - 1)\right]}{\left(\frac{1}{\phi}\left\{\frac{\varepsilon x^1}{1-\varepsilon x^1\phi_4+(\varepsilon-1)\phi_4}\right\} - \left[\frac{[\varepsilon x^1-(\varepsilon-1)](\gamma-\alpha)(\varepsilon-1)}{(\varepsilon\alpha x^1+(\gamma-\alpha)(\varepsilon-1))}\right]\right)} \tag{A.23}$$

In matrix form, this is

$$\begin{bmatrix} \widehat{\kappa}_{t+1} \\ \widehat{x}_{t+1} \end{bmatrix} = \begin{bmatrix} \alpha & (b_0 - \alpha) \\ \alpha & -\left(\alpha + \frac{1}{b_1}\right) \end{bmatrix} \begin{bmatrix} \widehat{\kappa}_t \\ \widehat{x}_t \end{bmatrix}$$

Stability requires that the system has one characteristic inside the unit circle and the other outside. Here the characteristic equation is $\mu^2 + \frac{\mu}{b_1} - \frac{\alpha(1+b_0b_1)}{b_1} = 0$, which has roots

$$\mu_{1,2} = -\frac{1}{2b_1} \pm \sqrt{\frac{1}{4}\left(\frac{1}{b_1}\right)^2 + \frac{\alpha\left(1 + b_0 b_1\right)}{b_1}}.$$

The properties of $\mu_{1,2}$ require tedious derivations, which we report in a series of Lemmas below; in combination, the Lemmas provide conditions for $\mu_1 < -1$ and $0 < \mu_2 < 1$, which are sufficient conditions for saddle-path stability and convergence.

Lemma A1: $\gamma > \alpha$ implies $0 < b_0 < 1$. Proof: Follows from $\phi_2 = (\gamma - \alpha)(\varepsilon - 1) > 0$ and $x^1 \geq 1$. *QED.*

Lemma A2: $1 - \phi_4 \left[\varepsilon x^1 - (\varepsilon - 1)\right] > 0$, provided $x^1 > 1$. Proof: The steady state satisfies

$$\frac{1}{\varepsilon x - (\varepsilon - 1)} = \phi_3 \phi_1^{-\frac{\alpha\phi}{1-\alpha}} \left[\varepsilon \alpha x + \phi_2\right]^{\frac{\phi}{1-\alpha}} + \phi_4$$

$$\Leftrightarrow 1 - \phi_4 \left[\varepsilon x - (\varepsilon - 1)\right] = \phi_3 \phi_1^{-\frac{\alpha\phi}{1-\alpha}} \left[\varepsilon x - (\varepsilon - 1)\right] \left[\varepsilon \alpha x + \phi_2\right]^{\frac{\phi}{1-\alpha}}$$

The r.h.s. is positive because $\varepsilon\alpha x + \phi_2 > 0$, $\varepsilon x - (\varepsilon - 1) > 0$ and $\phi_3 \phi_1^{-\frac{\alpha\phi}{1-\alpha}} > 0$. Hence $1 - \phi_4 \left[\varepsilon x - (\varepsilon - 1)\right] > 0$. *QED.*

Lemma A3: $b_1 > 0$, provided $x^1 > 1$. Proof: b_1 can be written as

$$b_1 = 1/\left(\left\{\frac{\varepsilon x}{\left[1 - \phi_4 \left[\varepsilon x - (\varepsilon - 1)\right]\right] \phi \left[\varepsilon x - (\varepsilon - 1)\right]}\right\} - \left[\frac{\phi_2}{(\varepsilon \alpha x + \phi_2)}\right]\right)$$

Therefore $b_1 > 0$ if $\left\{\frac{\varepsilon x}{\left[1 - \phi_4 \left[\varepsilon x - (\varepsilon - 1)\right]\right] \phi \left[\varepsilon x - (\varepsilon - 1)\right]}\right\} > \left[\frac{\phi_2}{(\varepsilon \alpha x + \phi_2)}\right] = b_0$. Using Lemma A2, this is equivalent to

$$\varepsilon x \left(\varepsilon \alpha x + \phi_2\right) > \phi_2 \phi \left[1 - \phi_4 \left[\varepsilon x - (\varepsilon - 1)\right]\right] \left[\varepsilon x - (\varepsilon - 1)\right], \text{ or}$$

$$\alpha \left(\varepsilon x\right)^2 + \varepsilon x \phi_2 + \phi_2 \phi \phi_4 \left[\varepsilon x - (\varepsilon - 1)\right]^2 > \phi_2 \phi \left[\varepsilon x - (\varepsilon - 1)\right].$$

Since $\phi_2 = (\gamma - \alpha)(\varepsilon - 1) > 0$ and $0 < \phi = (1 - \alpha)\left(1 - \frac{1}{\gamma}\right) < 1$ if $\gamma > 1$, one can divide both sides by $\phi_2 \phi$ and simplify to obtain

$$\frac{\alpha \left(\varepsilon x\right)^2}{\phi_2 \phi} + \varepsilon x \left(\frac{1 - \phi}{\phi}\right) + \phi_4 \left[\varepsilon x - (\varepsilon - 1)\right]^2 + (\varepsilon - 1) > 0.$$

The r.h.s. is positive, because $0 < \phi < 1$ and because ϕ_2, ϕ_4 and ε are all greater than one. By equivalence, $b_1 > 0$. *QED.*

Lemma A4: $0 < \mu_2 < 1$, provided $0 < b_0 < 1$ and $b_1 > 0$. Proof: Since $b_0 > 0$ and $b_1 > 0$, we have $\frac{\alpha(1+b_0 b_1)}{b_1} > 0$, which implies $\mu_2 = -\frac{1}{2b_1} + \sqrt{\frac{1}{4}\left(\frac{1}{b_1}\right)^2 + \frac{\alpha(1+b_0 b_1)}{b_1}} > 0$. Given $\mu_2 > 0$ and $b_1 > 0$, $\mu_2 + \frac{1}{2b_1} > 0$, so $\mu_2 = -\frac{1}{2b_1} + \sqrt{\frac{1}{4}\left(\frac{1}{b_1}\right)^2 + \frac{\alpha(1+b_0 b_1)}{b_1}} < 1 \Leftrightarrow \frac{1}{4}\left(\frac{1}{b_1}\right)^2 + \frac{\alpha(1+b_0 b_1)}{b_1} < \left(1 + \frac{1}{2b_1}\right)^2$, $\Leftrightarrow 1 - \alpha b_0 + \frac{1-\alpha}{b_1} > 0$, which is implied by $b_0 < 1$ and $b_1 > 0$. *QED.*

Lemma A5: $\mu_1 < -1$, provided $0 < b_0 < 1$ and $0 < b_1 < 1$. Proof: Since $b_0 > 0$ and $b_1 > 0$ imply $\frac{\alpha(1+b_0 b_1)}{b_1} > 0$, $\mu_2 < -\frac{1}{2b_1} - \sqrt{\frac{1}{4}\left(\frac{1}{b_1}\right)^2} < -\frac{1}{b_1}$, so $\mu_2 < -\frac{1}{b_1} < -1$ for $b_1 < 1$. *QED.*

Lemma A6: Define $\rho = \frac{\alpha}{\gamma-\alpha}$ and $z(\rho) = (2+3\rho) - 2\sqrt{2\rho(1+\rho)} = \left(\sqrt{2(1+\rho)} - \sqrt{\rho}\right)^2$. Then sufficient conditions for $b_1 < 1$ are that

$$(1-\frac{1}{\gamma})z(\rho)\left(1-\alpha-\frac{\alpha^2}{\varepsilon-1}\right) \quad < \quad 1, \text{ or} \tag{A.24}$$

$$(1-\frac{1}{\gamma})z(\rho) \quad < \quad \frac{1}{1-\alpha} \tag{A.25}$$

Proof: Since (A.25) implies (A.24) holds for $\varepsilon > 1$, it suffices to prove (A.24). Since $b_1 > 0$, the restriction $b_1 < 1 \Leftrightarrow \left\{\frac{\varepsilon x}{[1-\phi_4[\varepsilon x-(\varepsilon-1)]]\phi[\varepsilon x-(\varepsilon-1)]}\right\} - \frac{\phi_2}{\varepsilon\alpha x+\phi_2} > 1$ $\Leftrightarrow \left\{\frac{\varepsilon x}{[1-\phi_4[\varepsilon x-(\varepsilon-1)]]\phi[\varepsilon x-(\varepsilon-1)]}\right\} > \left(\frac{\varepsilon\alpha x+2\phi_2}{\varepsilon\alpha x+\phi_2}\right) \Leftrightarrow \left\{\frac{\left(\frac{\varepsilon\alpha x}{(\gamma-\alpha)(\varepsilon-1)}+1\right)}{\left(\frac{\varepsilon\alpha x}{(\gamma-\alpha)(\varepsilon-1)}+2\right)}\frac{\varepsilon x}{[\varepsilon x-(\varepsilon-1)]}\frac{1}{\phi}\right\} +$ $\phi_4\left[\varepsilon x-(\varepsilon-1)\right] > 1$

Define $\Psi = \frac{(\varepsilon-1)}{\varepsilon x}$, $\rho = \frac{\alpha}{(\gamma-\alpha)}$, and

$$H(\Psi) = \frac{\left(\frac{\rho}{\Psi}+1\right)}{\left(\frac{\rho}{\Psi}+2\right)}\frac{1}{1-\Psi} = \frac{(\rho+\Psi)}{(\rho+2\Psi)}\frac{1}{1-\Psi}.$$

Then

$$b_1 < 1 \Leftrightarrow \frac{1}{\phi}H(\Psi) + \phi_4\left[\varepsilon x^1 - (\varepsilon-1)\right] > 1.$$

Given $\varepsilon > 1$ and given an interior solution $x^1 > 1$, we have $0 < \Psi < 1$. The terms $\frac{1}{[1-\Psi]}$ and $\frac{1}{\phi}$ are both greater than 1, while the term $\frac{\left[\frac{\rho}{\Psi}+1\right]}{\left[\frac{\rho}{\Psi}+2\right]}$ is necessarily less than 1. Note that $H(\Psi)$ has a minimum in the interval $0 < \Psi < 1$ and that

$$\frac{1}{\phi}H(\Psi) + \phi_4\left[\varepsilon x-(\varepsilon-1)\right] \geq \left[\min_{0<\Psi<1} H(\Psi)\right]\frac{1}{\phi} + \phi_4\left\{\min_{x\geq 1}\left[\varepsilon x-(\varepsilon-1)\right]\right\}$$

Since $\min\limits_{x\geq 1}\left[\varepsilon x-(\varepsilon-1)\right] = 1$, a sufficient condition for $b_1 < 1$ is that

$$\left[\min_{0<\Psi<1} H(\Psi)\right]\frac{1}{\phi} + \phi_4 > 1 \Leftrightarrow \min_{0<\Psi<1} H(\Psi) > \phi(1-\phi_4) \tag{A.26}$$

where $\phi(1-\phi_4) > 0$ because $(1-\phi_4) > 0$ for $x > 1$ and $\phi > 0$ for $\gamma > 0$. Note that $H'(\Psi) = \frac{(\rho+2\Psi)-2(\rho+\Psi)}{(\rho+2\Psi)^2}\frac{1}{(1-\Psi)} + \frac{(\rho+\Psi)}{(\rho+2\Psi)}\frac{1}{(1-\Psi)^2} = \frac{-\rho(1-\Psi)+(\rho+\Psi)(\rho+2\Psi)}{(\rho+2\Psi)^2(1-\Psi)^2} = \frac{2\Psi^2+4\rho\Psi-\rho(1-\rho)}{(\rho+2\Psi)^2(1-\Psi)^2} = \frac{\Psi^2+2\rho\Psi-\rho(1-\rho)/2}{2(\rho+2\Psi)^2(1-\Psi)^2} = 0$ has roots $\Psi = -\rho\pm\sqrt{\rho^2+\rho(1-\rho)/2} = -\rho \pm \sqrt{\frac{\rho}{2}(1+\rho)}$. Of these, only $\Psi^{\min} = -\rho + \sqrt{\frac{\rho}{2}(1+\rho)}$ is inside the $[0,1]$

interval, so H is minimized at $\Psi^{\min}$, where

$$\begin{aligned} H^{\min} &= H\left(\Psi^{\min}\right) = \frac{\sqrt{\frac{\rho}{2}(1+\rho)}}{\left(-\rho+2\sqrt{\frac{\rho}{2}(1+\rho)}\right)} \frac{1}{\left(1+\rho-\sqrt{\frac{\rho}{2}(1+\rho)}\right)} \\ &= \frac{\sqrt{\frac{\rho}{2}(1+\rho)}}{-\rho(1+\rho)+2(1+\rho)\sqrt{\frac{\rho}{2}(1+\rho)}+\rho\sqrt{\frac{\rho}{2}(1+\rho)}-2\frac{\rho}{2}(1+\rho)} \\ &= \frac{\sqrt{\frac{\rho}{2}(1+\rho)}}{-2\rho(1+\rho)+(2+3\rho)\sqrt{\frac{\rho}{2}(1+\rho)}} = \frac{1}{2+3\rho-2\rho(1+\rho)/\sqrt{\frac{\rho}{2}(1+\rho)}} \\ &= \frac{1}{2+3\rho-2\sqrt{2\rho(1+\rho)}} = \frac{1}{z(\rho)} \end{aligned}$$

Thus from (A.26), $1/z(\rho) > \phi\left(1-\phi_4\right) = \left(1-\frac{1}{\gamma}\right)\left(1-\alpha-\frac{\alpha^2}{\varepsilon-1}\right)$ is sufficient for $b_1 < 1$. *QED.*

Lemma A7: (a) The term $\left(1-\frac{1}{\gamma}\right)z(\rho)$ in (A.24) is increasing in γ and decreasing in α provided $\alpha < \gamma/2$; (b) the $1-\alpha-\frac{\alpha^2}{\varepsilon-1}$in (A.24) increasing in ε and decreasing in α. (c) If $1-\alpha-\frac{\alpha^2}{\varepsilon-1} \leq \frac{1}{2}$ and $\alpha \leq \gamma/2$, then condition (A.24) is satisfied for all $\gamma > 1$. (d) if $1-\alpha-\frac{\alpha^2}{\varepsilon-1} > \frac{1}{2}$ and $\alpha \leq 1/2$ there exists an upper bound $\bar{\gamma} > 1$ so that condition (A.24) is satisfied for all $\gamma \in (1, \bar{\gamma})$ and not satisfied for $\gamma > \bar{\gamma}$.

Proof: Note that $z'(\rho) = 3 - 2^{1.5}\frac{1+2\rho}{2\sqrt{\rho(1+\rho)}} = 0$ has roots $\rho_{1,2} = -\frac{1}{2} \pm \sqrt{\frac{1}{4}+2} = -2, +1$ and satisfies $z'(\rho) < 0$ for $\rho \in (-1, 1)$. Since $\alpha < \gamma/2$ implies $\rho < 1$ and since $\frac{\partial\rho}{\partial\gamma} < 0 < \frac{\partial\rho}{\partial\alpha}$ $\frac{\partial}{\partial\gamma}\left[\left(1-\frac{1}{\gamma}\right)z(\frac{\alpha}{\gamma-\alpha})\right] = \frac{1}{\gamma^2}z(\rho) + \left(1-\frac{1}{\gamma}\right)\frac{\partial\rho}{\partial\gamma}z'(\rho) > 0$ for all $\alpha \leq 1/2 < \gamma/2$, and $\frac{\partial\rho}{\partial\alpha} = \left(1-\frac{1}{\gamma}\right)\frac{\partial\rho}{\partial\alpha}z'(\rho) < 0$, proving (a). Part (b) holds by inspection. For (c), note that $\left(1-\frac{1}{\gamma}\right)z(\frac{\alpha}{\gamma-\alpha}) \to z(0) = 2$ as $\gamma \to \infty$, so $\left(1-\frac{1}{\gamma}\right)z(\frac{\alpha}{\gamma-\alpha}) < 2$ for any finite γ, which implies (c). For (d), note that $\left(1-\frac{1}{\gamma}\right)z(\frac{\alpha}{\gamma-\alpha}) \to 0$ as $\gamma \to 1$, so (A.24) holds in a neighborhood of $\gamma = 1$. For $1-\alpha-\frac{\alpha^2}{\varepsilon-1} > \frac{1}{2}$ and $\left(1-\frac{1}{\gamma}\right)z(\frac{\alpha}{\gamma-\alpha}) \to 2$ implies that (A.24) cannot hold as $\gamma- > \infty$. Existence of $\bar{\gamma}$ then follow from the mean value theorem and uniqueness of $\bar{\gamma}$ from monotonicity of $\left(1-\frac{1}{\gamma}\right)z(\frac{\alpha}{\gamma-\alpha})$. *QED.*

Corollary to A1-A7: If (A.25) holds for some $\alpha = \bar{\alpha}$ and $\gamma = \bar{\gamma}$, then (A.25) holds for all $\bar{\alpha} \leq \alpha \leq 1/2$ and all $1 < \gamma \leq \bar{\gamma}$. If in addition $x^1 > 1$, then $\mu_1 < -1$ and $0 < \mu_2 < 1$, so saddle-path stability holds.

Condition (A.25) can be evaluated numerically and is satisfied for plausible parameters. For example, for $\bar{\alpha} = 1/3$ and $\bar{\gamma} = 10$, one finds $\left(1-\frac{1}{\gamma}\right)z(\rho) = 1.4133 < \frac{\alpha}{1-\alpha} = 1.5$ and for $\bar{\alpha} = 0.2$ and $\bar{\gamma} = 5$, one finds $\left(1-\frac{1}{\gamma}\right)z(\rho) = 1.2285 < \frac{\alpha}{1-\alpha} = 1.25$. Thus the system is saddle-path stable for all $1/3 \leq \alpha \leq 0.5$ and $1 < \gamma \leq 10$ and for all $0.2 \leq \alpha \leq 0.5$ and $1 < \gamma \leq 5$. The sufficient condition tends to fail only if α is implausibly small and γ is large. (Note that the condition is not necessary; in some cases, one can use (A.24) to show that the system is stable even though (A.25) fails. For example, for $\alpha = 1/3$ and $\gamma = 15$, one can show that (A.24) applies for all $\varepsilon \leq 10$.)

7.6 Analysis in Section 5.2

7.6.1 Derivation of $\frac{dR_{t+1}}{d\theta_t}$ under Markov Strategies

Section 5.2 asserts (25), assuming no persistence. Proof: By the Markovian assumption on strategies we can write $\theta_{t+1} = g(\kappa_{t+1})$, where g is some unknown function. Differentiating $\ln R_{t+1}$, one obtains

$$\begin{aligned}\frac{d\ln R_{t+1}}{d\theta_t} &= (1-\alpha)\left[\frac{d\ln(1+\theta_{t+1})}{d\ln\kappa_{t+1}} - 1\right]\frac{d\ln\kappa_{t+1}}{d\theta_t} \\ &= (1-\alpha)\left[\lambda_{t+1} - 1\right]\frac{d\ln\kappa_{t+1}}{d\theta_t}.\end{aligned}$$

where $\lambda_{t+1} = \frac{d\ln(1+\theta_{t+1})}{d\ln\kappa_{t+1}} = \frac{\kappa_{t+1}g'(\kappa_{t+1})}{1+g(\kappa_{t+1})}$. Differentiating

$$\begin{aligned}\ln\kappa_{t+1} &= \left[\ln(1-\alpha) - \ln\eta\right] + (1-\alpha)\ln(1+\theta_t) \\ &\quad - \ln(1+\varepsilon\theta_t) - \ln\left(1+\beta^{-\frac{1}{\gamma}}R_{t+1}^{1-\frac{1}{\gamma}}\right) + \alpha\ln\kappa_t \qquad \text{(A.27)}\end{aligned}$$

with respect to θ_t, one obtains

$$\frac{d\ln\kappa_{t+1}}{d\theta_t} = \frac{1-\alpha}{(1+\theta_t)} - \frac{\varepsilon}{(1+\varepsilon\theta_t)} - \frac{\left(1-\frac{1}{\gamma}\right)}{1+B_t}\frac{d\ln R_{t+1}}{d\theta_t}$$

Substituting the above equation into $\frac{dR_{t+1}}{d\theta_t}$ above, we obtain

$$\frac{d\ln R_{t+1}}{d\theta_t} = (1-\alpha)\left[\lambda_{t+1}-1\right]\left\{\frac{1-\alpha}{(1+\theta_t)} - \frac{\varepsilon}{(1+\varepsilon\theta_t)} - \frac{\left(1-\frac{1}{\gamma}\right)}{1+B_t}\frac{d\ln R_{t+1}}{d\theta_t}\right\}.$$

Solving for $\frac{dR_{t+1}}{d\theta_t}$ and simplifying terms we obtain

$$\begin{aligned}\frac{d\ln R_{t+1}}{d\theta_t} &= \frac{(1-\alpha)\left[\lambda_{t+1}-1\right]\left[\frac{1-\alpha}{(1+\theta_t)} - \frac{\varepsilon}{(1+\varepsilon\theta_t)}\right](1+1/B_t)}{(1+1/B_t) + \left[\lambda_{t+1}-1\right](1-\alpha)\left(1-\frac{1}{\gamma}\right)/B_t} \\ &= \frac{(1-\alpha)(1-\lambda_{t+1})\left[\frac{\varepsilon}{(1+\varepsilon\theta_t)} - \frac{1-\alpha}{(1+\theta_t)}\right](1+1/B_t)}{1+(1-(1-\lambda_{t+1})\phi)/B_t} \qquad \text{(A.28)}\end{aligned}$$

As in the previous section, maximizing (6) with respect to θ_t again implies (A.15), where $\frac{dR_{t+1}}{d\theta_t} = R_{t+1}\frac{d\ln R_{t+1}}{d\theta_t}$ is now given by (A.28). An interior solution again requires $\frac{d\ln R_{t+1}}{d\theta_t} / (1 + 1/B_t) = \frac{\alpha}{(1+\theta_t)}$, which (after simplifying as in the previous section) reduces to (25). *QED.*

7.6.2 The Numerical Algorithm for the Equilibrium in Markov Strategies

We obtain a log-linearization of the model in Markov strategies and verify the quality of the solution by using numerical methods that use projection methods similar to den Haan and Marcet's (1990) parameterized expectations approach (PEA).

Specifically, we approximate period-t expectations about θ^*_{t+1} by using the function:

$$\left(1 + \theta^*_{t+1}\right) = \exp\left(\sum_{i=0}^{n} \mu_i \left(\ln\left(\kappa_{t+1}\right)\right)^i\right) \tag{A.29}$$

for some unknown coefficients $\boldsymbol{\mu} = \{\mu_0, \mu_1, \mu_2, ..\mu_n\}$, and where κ_{t+1} is known at time t. The general idea is to choose some particular coefficients $\widehat{\boldsymbol{\mu}}$ such that the distance between the forecast and the optimal choices dictated by the equations of the model are minimized. For our application, the coefficients are obtained after an iterative procedure where the model is solved for different points on the state grid[26], where the solution is conditional on a previous set of coefficients. The new set of equilibrium pairs $\{\kappa_t, \theta_t\}$ are used to generate a new set of coefficients that describe the optimal policy function (and the elasticity $\frac{d\ln\left(1+\theta^*_{t+1}\right)}{d\ln\kappa_{t+1}}$), and which also update the expectation function of the model. This is repeated until the distance of the forecast and actual choices consistent with that forecast is minimized.

Under equation (A.29) the term $\frac{d\ln\left(1+\theta^*_{t+1}\right)}{d\ln\kappa_{t+1}}$ needed in the first order condition of generation $[t]$ is given by

$$\frac{d\ln\left(1+\theta_{t+1}\right)}{d\ln\kappa_{t+1}} = \sum_{i=1}^{n} i\mu_i \left(\ln\left(\kappa_{t+1}\right)\right)^{i-1} \tag{A.30}$$

The particular steps of the algorithm are explained below.

Preliminaries. For the initial conditions that represent the grid of the state variable (κ) we use a neighborhood around the myopic steady state, given by $\left[.75\kappa^1, 1.5\kappa^1\right]$. This interval does not need to be symmetric, and in our

[26] See Christiano and Fisher (2000) for the use of this step in the PEA algorithm. Since simulating the model will naturally lead to points which have a high probability and not many points of states with low probability, they suggest several variations of the algorithm that amount to collocation in the grid, rather than increasing the number of periods of the simulation. Some of their best variations of the PEA algorithm don't require many collocation points (i.e. 5 collocation points) , nor require many terms in their expectation function.

particular case it is due to the fact that (with $\gamma > 1$) the Markov steady state level (κ^*) is always to the right of the myopic steady state (κ^1).

Step 1. Start with the solution of the rational expectations "myopic" equilibrium where immigration quotas θ_t take as given the future immigration quotas θ_{t+s}, for $s \geq 1$. The model is simulated for many periods $t = 0, 1, 2, ...T$ using perfect foresight under $j = 1, 2, ...J$ different initial conditions in the state grid. For each of the time series generated under each initial condition, a point (κ_0, θ_0^*) is obtained.[27] Hence there are J pairs $\left(\kappa_{0,j}, \theta_{0,j}^*\right)$ that show the optimal immigration quota consistent with the initial condition and with perfect foresight simulation of the model. Then those pairs are used to estimate equation (A.29) in order to obtain the first set of coefficients $(\widehat{\boldsymbol{\mu}}_1)$. More details about the specific regression are explained in step (4). Then form the forecasting function $\widehat{g}\left(\kappa_{t+1}; \widehat{\boldsymbol{\mu}}_1\right)$, which is parameterized by the initial set of coefficients $\widehat{\boldsymbol{\mu}}_1$. For economy of notation in what follows define the forecasting function parameterized by the coefficients of the s^{th} iteration by $\widehat{g}\left(\kappa_{t+1}; \widehat{\boldsymbol{\mu}}_s\right) \equiv \widehat{g}_s\left(\kappa_{t+1}\right)$.

Step 2. For each initial condition $\kappa_{0,j}$ $(j = 1, 2, ...J)$ and given the coefficients $\widehat{\boldsymbol{\mu}}_s$ that parameterize the $\widehat{g}_s\left(\kappa_{t+1}\right)$, a non-linear solver is used in order to (simultaneously) solve for $\theta_{0,j}^*$ and $\kappa_{1,j}^*$ for the following two equations that describe the evolution of the system (where for simplicity we ignore the subscript j that denotes a particular initial condition):

$$\kappa_1 \left(1 + \beta^{-\frac{1}{\gamma}} \alpha^{\left(1-\frac{1}{\gamma}\right)} \left(\frac{1 + \widehat{g}_s\left(\kappa_1\right)}{\kappa_1}\right)^{\phi}\right) = \frac{(1-\alpha)\left(1+\theta_0\right)^{1-\alpha} \kappa_0^{\alpha}}{\eta\left(1+\varepsilon\theta_0\right)}$$

$$\begin{aligned} & \beta^{-\frac{1}{\gamma}} \alpha^{\left(1-\frac{1}{\gamma}\right)} \left(\frac{1 + \widehat{g}_s\left(\kappa_1\right)}{\kappa_1}\right)^{\phi} \\ = & \frac{\left[\left(\frac{1-\alpha}{\alpha}\right)\frac{(\varepsilon-1)}{(1+\varepsilon\theta_0)} - \alpha\right] - \frac{d\ln(1+\widehat{g}_s(\kappa_1))}{d\ln\kappa_1}\left[(1-\alpha) + \left(\frac{1-\alpha}{\alpha}\right)\frac{(\varepsilon-1)}{(1+\varepsilon\theta_0)}\right]}{\left\{\frac{1}{\gamma} + \alpha - \frac{\alpha}{\gamma}\right\} + \phi\frac{d\ln(1+\widehat{g}_s(\kappa_1))}{d\ln\kappa_1}} \end{aligned}$$

where the term $\frac{d\ln(1+\widehat{g}(\kappa_1))}{d\ln\kappa_1}$ is given by equation (A.29). For each $\kappa_{1,j}$ obtained as the solution to the above equations, the non-linear solver can be used again in order to obtain $\left(\kappa_{2,j}, \theta_{1,j}^*\right)$ and repeat this step recursively up to some final period T.

Step 3. A measure of accuracy of the forecast $\widehat{g}_s\left(\kappa_t\right)$ and the optimal choice that solves the system of equations $\left(\theta_{t,j}^*\right)$ is constructed. We calculate the sum of squared residuals for simulation started by initial condition j (SSR_j) from the current optimally chosen levels (θ_t^*) and the levels that would be predicted directly by the forecasting function using the previous set of coefficients for all

[27] More than one point can be used (i.e $(\kappa_{0j}, \theta_{0j}^*)$ and $(\kappa_{1j}, \theta_{1j}^*)$ for the $j-th$ initial condition. We don't do this step because the next set of points are closer to each other (converging toward the steady state) and in practice didn't change the coefficients. When only the first point is used, we control exactly which points on the grid we want to approximate, which is a good thing when approximating numerically a particular function.

periods t. That is, we compute an error term (not the regression error term) obtained in (each of the J) current simulations of the equilibrium immigration quota $\left(\theta^*_{t,s}\right)$, which use perfect foresight and which assume that the future effects $\frac{d\ln(1+\theta_{t+1})}{d\ln\kappa_{t+1}}$ are given by (A.30). The error of prediction for each one of the J simulations (for each of the J initial conditions) when the expectation function uses coefficients of iteration s, for time t is given by $e_{t,j,s+1} = \theta^*_{t,j} - \widehat{g}\left(\kappa_t, \widehat{\mu}_s\right)$. Then $SSR_{j,s+1} = \sum_{t=0}^{T} \left(e_{t,j,s+1}\right)^2$. If this distance is at a minimum, then a solution has been found, but if not at a minimum, continue with the next step.[28]

Step 4. Given the new set of pairs $\left\{\theta^*_{0,j}, \kappa_{0,j}\right\}_{j=1}^{J}$, run the non-linear regression

$$\left(1+\theta^*_{0,j}\right) = \exp\left(\sum_{i=0}^{n} \mu_i \left(\ln\left(\kappa_{0,j}\right)\right)^i\right) + error$$

where the estimates for coefficients that parameterize this function when this is the $(s+1)$ time that this step is performed are given by $\widehat{\boldsymbol{\mu}}_{s+1}$. Given the new coefficients, go back to step 2 and repeat the steps until $\max\left[SSR_{1,s+1}, ..SSR_{J,s+1}\right]$ has been minimized.[29]

An example. Consider the case with the parameters as discussed in the text with $\alpha = \frac{1}{3}$, $\eta = \frac{1.25}{1.12}$, $\gamma = 4$, $\beta = .412$ and $\varepsilon = 1.9321$. We use $n = 3$ (a polynomial of the third degree) in order to minimize the distance. The steady state yields $\theta^* = 8.00\%$ and $\kappa^* = .0739$, with an elasticity $\frac{d\ln\left(1+\theta^*_{t+1}\right)}{d\ln\kappa_{t+1}}$ evaluated at steady state of 7.04%. In our experiments, adding more monomial terms to the expectation function doesn't result in more accuracy. The Markovian strategy of equilibrium is described in this case by

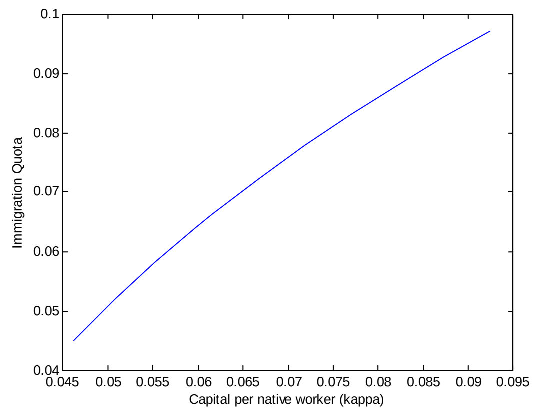
$$\left(1+\theta^*_t\right) = \exp\left(\sum_{i=0}^{3} \widetilde{\mu}_i \ln\left(\kappa_{t,j}\right)^i\right)$$

with coefficients $\{\widetilde{\mu}_0, \widetilde{\mu}_1, \widetilde{\mu}_2, \widetilde{\mu}_3\}$ given by {0.261032563347811, 0.0698516558436102, -0.000699103908625326, -0.000150142686729684}, and $max\left\{SSR_1, ...SSR_J\right\} = .5*10^{-15}$. The algorithm required 11 iterations in order to arrive at the solution where we used $T = 20$ and $N = 24$. The optimal policy function in Markov strategies for this particular example is shown below.

[28] Alternatively, we can define this algorithm in terms of finding the coefficients $\boldsymbol{\mu}^*$ such that, using the forecast $g\left(\kappa; \boldsymbol{\mu}^*\right)$ the model equations yields a set of pairs $\left\{\kappa_j, \theta_j\right\}_j^J$ such that $\boldsymbol{\mu}^*$ also solves the regression problem. That is, returns the same coefficients used in the simulation of the model.

[29] The coefficients used for the perfect foresight simulation of the model (vector $\widetilde{\boldsymbol{\mu}}$) can be those directly dictated by the regression $(\widehat{\boldsymbol{\mu}})$, or as den Haan and Marcet do, a linear combination of the estimated coefficients for the current iteration and the estimates used in the previous iteration can be used. That is, for the i^{th} iteration if the regression coefficients are $\widehat{\boldsymbol{\mu}}_i$ and the coefficients that are fed into the model in the previous iteration are $\widetilde{\boldsymbol{\mu}}_{i-1}$, then the next iteration uses coefficients given by $\widetilde{\boldsymbol{\mu}}_i = \rho\widehat{\boldsymbol{\mu}}_i + (1-\rho)\,\widetilde{\boldsymbol{\mu}}_{i-1}$, for a specific $0 < \rho < 1$. For our model we can update the new coefficients directly $(\rho = 1)$.

The optimal markovian policy function



Robustness. This algorithm is robust with respect to many variations in the procedure: to the number of periods T (Using $T = 10$ or $T = 200$ yields identical solutions in θ_t^* up to several decimal points), the number of initial conditions J (can use J of at least 4 and get very accurate results provided we use the same grid), as well as the number of pairs used for the regression procedure (can use pairs $\left\{\theta_{t,j}^*, \kappa_{t,j}\right\}_{j=1}^{J}$ for $t = \{0\}$ or $t = \{0, 1\}$, or $t = \{0, 1, 2\}$ in the regression and still obtain the same results). Variations in the size of the state grid don't seem to affect the steady state results even when we used a much smaller grid given by $\left[.99\kappa^1, 1.10\kappa^1\right]$.

Simplified Version. For $n = 1$, the projection (A.29) reduces to a log-linear approximation around the steady state. For purposes of computing steady states–our objects of main interest–the only relevant feature of g is the elasticity $\lambda^* = \frac{g'(\kappa^*)\kappa^*}{1+g(\kappa^*)} = \frac{d\ln(1+\theta^*)}{d\ln\kappa}$ at the steady state capital stock κ^*, because only λ_{t+1} appears in (25). Log-linearization yields an analytical solution for $g'(\kappa^*)$, which means $\{\kappa^*, \theta^*, \lambda^*\}$ can be written as a system of three non-linear equations that can be solved numerically (i.e., without having to approximate g away from κ^*). The log-linearizations turn out to provide values for θ^* that are very close to values obtained from solutions to the PEA algorithm. For example, using the parameters discussed above, the log-linearization yields $\theta^* = 7.98\%$, as compared to $\theta^* = 8.00\%$ with PEA.

7.7 Analysis in Section 5.3

We claim in Section 5.3. that the CRRA model with persistence has a first order condition that can be used to compute optimal solutions.

Proof: In the CRRA model with persistence, $R_{t+1,j}$ depends on the state $j = I, II$, as in Section 4.1. Now $\theta_{t+1}^I = g_p(\kappa_{t+1})$ and $\theta_{t+1}^{II} = \theta_t$. The individual problem (A.8) again implies (A.9), where

$$B_t = \left[(\kappa_{t+1})^{(\gamma-1)(1-\alpha)} \beta \alpha^{(1-\gamma)} \left(\sum_j p_j \cdot (1+\theta_{t+1,j})^{(1-\gamma)(1-\alpha)} \right) \right]^{1/\gamma} = (\kappa_{t+1})^{\phi} E_t$$

is decomposed multiplicatively into a function of κ_{t+1} and the expectational term

$$E_t = \beta^{\frac{1}{\gamma}} \alpha^{-(1-\frac{1}{\gamma})} \left(\sum_j p_j \cdot (1+\theta_{t+1,j})^{(1-\gamma)(1-\alpha)} \right)^{\frac{1}{\gamma}}.$$

Note that $\kappa_{t+1} = \psi_t w_t \frac{B_t}{1+B_t}$, where $\psi_t = \frac{(1+\theta_t)}{\eta(1+\varepsilon\theta_t)}$. Hence

$$B_t = (\kappa_{t+1})^{\phi} E_t = (\psi_t w_t \frac{B_t}{1+B_t})^{\phi} E_t = z_t (\frac{B_t}{1+B_t})^{\phi}$$

is (implicitly) a function $B_t = B(z_t)$, where $z_t = (\psi_t w_t)^{\phi} E_t$.

Voters maximize $U_t = \frac{1}{1-\gamma} w_t^{1-\gamma} [1+B_t]^{\gamma}$ by choice of θ_t. This implies

$$\begin{aligned} \frac{dU_t}{d\theta_t} &= \frac{(1-\gamma)U_t}{w_t}\frac{dw_t}{d\theta_t} + \frac{\gamma U_t}{1+B(z_t)} B'(z_t) \frac{dz_t}{d\theta_t}, \text{ or} \\ \frac{d\ln U_t}{(1-\gamma)d\theta_t} &= \frac{d\ln w_t}{d\theta_t} - \frac{\epsilon_B(z_t)}{1-1/\gamma}\frac{d\ln z_t}{d\theta_t} \end{aligned}$$

where $\epsilon_B(z_t) = \frac{B'(z_t)z_t}{1+B(z_t)} = \frac{B_t}{1-\phi+B_t}$. Hence interior solutions require

$$\frac{1}{\epsilon_B(z_t)} \frac{d\ln w_t}{d\theta_t} = \frac{1}{1-1/\gamma}\frac{d\ln z_t}{d\theta_t}. \tag{A.31}$$

Taking derivatives:

$$\frac{d\ln z_t}{d\theta_t} = \frac{d\ln E_t}{d\theta_{t+1,II}} + \frac{d\ln E_t}{d\theta_{t+1,I}} \frac{\partial \theta_{t+1,I}}{\partial \ln \kappa_{t+1}} \frac{\partial \ln \kappa_{t+1}}{\partial \theta_t} + \phi \left(\frac{d\ln w_t}{d\theta_t} + \frac{d\ln \psi_t}{d\theta_t} \right);$$

$$\begin{aligned} \frac{d\ln E_t}{d\theta_{t+1,II}} &= -(1-\frac{1}{\gamma})\frac{1-\alpha}{1+\theta_t} P_{II}, \text{ where} \\ P_{II} &= P_{II}(\theta_{t+1}^{II}, E_t) = p\frac{\beta\alpha^{(1-\gamma)}(1+\theta_{t+1}^{II})^{(1-\gamma)(1-\alpha)}}{E_t^{\gamma}}; \\ \frac{d\ln E_t}{d\theta_{t+1,I}} \frac{\partial \theta_{t+1,I}}{\partial \ln \kappa_{t+1}} &= \frac{1}{\gamma}(1-P_{II})(1-\gamma)(1-\alpha) \cdot \lambda_{t+1}; \end{aligned}$$

$$\begin{aligned} \ln \kappa_{t+1} &= \ln \psi_t + \ln w_t + \ln B_t - \ln(1+B_t) \text{ implies} \\ \frac{\partial \ln \kappa_{t+1}}{\partial \theta_t} &= \left(\frac{d\ln w_t}{d\theta_t} + \frac{d\ln \psi_t}{d\theta_t} \right) + \frac{1}{B_t}\frac{\partial B_t}{\partial \theta_t} - \frac{1}{1+B_t}\frac{\partial B_t}{\partial \theta_t} \\ &= \left(\frac{d\ln w_t}{d\theta_t} + \frac{d\ln \psi_t}{d\theta_t} \right) + \frac{\epsilon_B(z_t)}{B_t}\frac{d\ln z_t}{d\theta_t}; \text{ and} \end{aligned}$$

$$\frac{d\ln w_t}{d\theta_t}+\frac{d\ln \psi_t}{d\theta_t}=\frac{\alpha}{(1+\theta_t)}+\frac{1}{(1+\theta_t)}\frac{\varepsilon-1}{(1+\varepsilon\theta_t)},$$

one obtains,

$$\begin{aligned}\frac{d\ln z_t}{d\theta_t} &= -(1-\frac{1}{\gamma})\frac{1-\alpha}{1+\theta_t}P_{II}+\phi\left(\frac{d\ln w_t}{d\theta_t}+\frac{d\ln\psi_t}{d\theta_t}\right)\\ &\quad -(1-P_{II})\phi\lambda_{t+1}\left[\left(\frac{d\ln w_t}{d\theta_t}+\frac{d\ln\psi_t}{d\theta_t}\right)+\frac{\epsilon_B(z_t)}{B_t}\frac{d\ln z_t}{d\theta_t}\right],\end{aligned}$$

$$\frac{d\ln z_t}{d\theta_t}=\frac{-(1-\frac{1}{\gamma})\frac{1-\alpha}{1+\theta_t}P_{II}+\phi\left(\frac{\alpha}{(1+\theta_t)}+\frac{1}{(1+\theta_t)}\frac{\varepsilon-1}{(1+\varepsilon\theta_t)}\right)[1-(1-P_{II})\lambda_{t+1}]}{1+\frac{\phi}{1-\phi+B_t}(1-P_{II})\lambda_{t+1}\phi}.$$

provided $1+\frac{\phi}{1-\phi+B_t}(1-P_{II})\lambda_{t+1}\phi>0$. Inserting into (A.31) and combining terms, the first order condition is

$$\frac{1}{\epsilon_B(z_t)}\frac{d\ln w_t}{d\theta_t}=\frac{\frac{1}{1-1/\gamma}\frac{\partial\ln E_t}{\partial\theta_t}+(1-a)\left(\frac{d\ln w_t}{d\theta_t}+\frac{d\ln\psi_t}{d\theta_t}\right)[1-(1-P_{II})\lambda_{t+1}]}{1+(1-P_{II})\lambda_{t+1}\phi\frac{\epsilon_B(z_t)}{B_t}}$$

Using $\frac{1}{\epsilon_B(z_t)}=\frac{1-\phi+B_t}{B_t}=1+\frac{1-\phi}{B_t}$, this can be written as

$$\frac{1}{B_t}=\frac{\left(1-a+\left(\frac{1-\alpha}{\alpha}\right)\frac{\varepsilon-1}{(1+\varepsilon\theta_t)}\right)[1-(1-P_{II})\lambda_{t+1}]+\left(\frac{1-\alpha}{\alpha}\right)P_{II}-1}{1-\phi+\phi(1-P_{II})\lambda_{t+1}}. \tag{A.32}$$

This first order condition generalizes (25). *QED.*

Note that in any steady state, $\theta^I_{t+1}=\theta^{II}_{t+1}=\theta^*$ implies $P_{II}=p$. Constant $B_t=B^*$ and $\theta_t=\theta^*$ imply

$$\begin{aligned}\frac{1}{B^*} &= \frac{p\left(\frac{1-\alpha}{\alpha}\right)+\left(1-a+\left(\frac{1-\alpha}{\alpha}\right)\frac{\varepsilon-1}{(1+\varepsilon\theta^*)}\right)[1-(1-p)\lambda^*]-1}{1-\phi+(1-p)\lambda^*\phi}, \text{ and}\\ B^* &= \beta^{\frac{1}{\gamma}}\left(\frac{1-\alpha}{\alpha\eta(1+\varepsilon\theta^*)}\frac{B^*}{1+B^*}\right)^{1-\frac{1}{\gamma}}\end{aligned}$$

where $\lambda^*=\frac{g'(\kappa^*)\kappa^*}{1+g(\kappa^*)}$ depends on the policy function at κ^*. Hence given a numerical approximation for g, a steady state $\{B^*,\theta^*,\lambda^*\}$ can be characterized as solution to this system of non-linear equations. We obtain steady states numerically in two ways: by approximating g using PEA (as discussed above); and by deriving an analytical solution for $g'(\kappa^*)$ from log-linearizing (A.32) and (A.18), and then solving the steady state equations numerically. In our applications, both approaches yield virtually identical answers.