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Authors

Breslavsky, Dmytro
Brovarnyk, Oleksiy
Breslavsky, Ivan

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Mathematical Modelling of the Variation of Population and Towns Growth in the Territory of Kyiv Princes' State in 9th-11th Centuries

Dmytro Breslavsky¹, Oleksii Brovarnyk¹, Ivan Breslavsky²

¹*National Technical University 'Kharkiv Polytechnic Institute'*

²*Westlake University*

An approach and method for mathematical modelling of the process of population variation in the territory of a given state in a given historical period are proposed. The modelling is based on the use of historical and geographical data from individual parts of the analysed territory. The problem is formulated as a boundary-initial value problem, in which the main variable, normalized population, depends not only on time, but also on the spatial coordinates in a given area. The formulated problem is solved using the Finite Element Method in combination with the finite difference method of integration over time. Dedicated software was developed to perform numerical modelling. As an example, the dynamics of population variation in the territory of the state of the Kyiv princes in the period 700-1015 AD is considered. According to the results of the calculations, the locations of town origins are determined. A comparison of the coordinates of such areas with the real coordinates of the location of towns in a given historical period showed satisfactory quality of modelling.

Introduction

The problems of studying historical heritage by modern science have recently ceased to be the prerogative of historians. Thanks to the scientific works of specialists in economic history, who moved from the traditional approaches inherent in economics and history to qualitative analysis using mathematical apparatus, a new one was added to the list of scientific disciplines, which is called Cliometrics (North 1997, North & Nye 2003). Approaches to modelling in this direction are adjacent to general methods of qualitative analysis of social processes (Bartholomew 1982, Humphreys 2003).

Thanks to the works of Peter Turchin (2008, 2011, 2014), another scientific direction arose, which was called Cliodynamics. Historical processes can be

Corresponding author's e-mail: dmytro.breslavsky@khpi.edu.ua

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described as dynamic because they change over time: the population changes, states arise, grow, and collapse, etc. (Note that the name "dynamic", which characterizes the non-stationarity of the process, is inherent in this direction in some sciences and, unlike the physical sciences, does not imply the influence of inertial forces. Further, thanks to the practice in Cliodynamics and the very name of this science, this term will be used). Thus, Cliodynamics is intended to explain and study historical processes using mathematical models (Seabright 2004, Spinney 2012). In addition to Cliometry, Cliodynamics should be based on approaches and methods not only of mathematical statistics and related sections, but also on other methods of mathematical modelling and prediction.

Traditionally, at the first stages of the development of Cliodynamics, modelling methods based on the use of ordinary differential equations (Korotayev 2006) or systems of them (Weidlich 2006) were used to quantitatively describe historical processes. The most significant, from the authors' point of view, parameters of the processes were taken into account and the rate of their change in time was modelled. That is, from a mathematical point of view, the problem was formulated as an initial-value problem.

Such an approach, although it may lead to obtaining certain quantitative and qualitative results in the analysis of historical processes, nevertheless does not provide the opportunity to take into account important natural factors, for example, geographical, climatic, etc. (Maini 2020). Some steps in this direction were made in Spatial Econometrics (Paelinck et al. 1979, Anselin et al. 2004), in which conventional spatial econometric models represent spatial interaction through a discrete set of geographical units linked by a weight matrix. Recently, studies have been appearing aimed at taking into account spatial distributions in the study of historical processes (see, for example (Turchin et al. 2021)). Such consideration, from the point of view of mathematical physics approaches (Gupta 2021), leads to a reformulation of the problem as a boundary-initial value problem. This means that the main variables describing the historical process will be functions not only of time, but also of spatial coordinates. In this case, the problem will be described by partial differential equations.

In this paper, the authors attempt to formulate such a mathematical statement of the problem as a boundary-initial value problem and apply a well-known numerical method to its solution. As an example, the process of population dynamics in the territories belonging to the area of residence of mainly Slavic tribes, in which a state was created in the 9th-10th centuries under the rule of the Kyivan princes (other names are the Ancient Ukrainian State, Kyivan Rus, etc.), is considered.

Paul Humphreys (2003) notes that although traditionally the construction of models in the natural sciences assumed that there are laws behind the models, in reality there is no need for such a law-like generality, since the essence of the

construction of models is to adapt the model to the specific system being modelled. However, in the opinion of the authors of this paper, the use of models built on the laws and regularities of historical and natural processes is more justified. Further, such laws and regularities will be formalized and incorporated into the algorithms of numerical modelling.

The goal of the study and the description of the problem

The goal of this study is to present an approach to the formulation of the problem of population variation and distribution in a given country in a given historical period and the implementation of a mathematical method for its solution. Based on the analysis of population concentration in the territory, a conclusion is made about the most suitable regions for the emergence of towns. As a prerequisite for modelling, the historical and geographical features of the territories of the state under consideration are laid down.

As an example, the problem of modelling population variation and the emergence of urban-type settlements in the territory of the Kyiv princes in the 9th - early 11th centuries is considered.

The main hypotheses underlying the modelling are the following:

- The best living conditions of the population are found mainly along the rivers and in the southwestern part of the state, which leads to an increase in its number (Arkas 1912, Hrushevsky 1997);
- A fairly moderate impact of population movement on the territory of the state from other countries or territories is observed in this period (Arkas 1912, Hrushevsky 1997);
- The emergence of urban-type settlements is observed in areas of the greatest population concentration.

As a method adopted in the natural sciences for assessing the reliability of the results obtained in numerical modelling, a comparison of the places (geographic coordinates of the borders of regions - parts of territories) of the formation of towns on dates known from historical documents presented in databases was used.

Mathematical formulation of the population dynamics problem

As mentioned in the Introduction, mathematically the problem will be formulated as a boundary-initial value problem (Gupta 2021). The main variable is chosen to be the normalized population. This value will be a function not only of time, but also of spatial coordinates. As a simplification, it is assumed that the territory under consideration is a two-dimensional plane object, i.e. the curvature of the Earth's surface is neglected. Then the relative population η is represented as a function of two coordinates on the plane and time.

The country is described by a two-dimensional object Ω in the Cartesian coordinate system $x_i, i = 1, 2$ with a boundary in the form of a curve Γ . The relative distribution of the population over the territory and in time is denoted by the function $\psi(x_i, t)$ of coordinates and time t . The function $\Pi(x_i, t)$ describes the characteristic of population change within the country due to fertility.

On the other hand, there is an increase (in the algebraic sense) of the population due to its movement across the borders as a result of migration processes. The proposed model incorporates population dynamics arising from both natural change (births and deaths) and migration. The density of the population flow across the border of the object Γ is denoted by the function $\phi(x_i, t)$, and at any time the following relation is valid:

$$\phi_n = k_{x_i}(x_i, t) \frac{\partial \psi(x_i, t)}{\partial n}, i = 1, 2, \quad (1)$$

where $k_{x_i}(x_i, t)$ are functions that determine the intensity of the population flow across the border in a given direction x_i for a given time t , $\mathbf{n}$ is the normal vector to the boundary Γ .

As is standard in the analysis of flows in mathematical physics (Gupta 2021), we consider the difference function between population flows, which consists of the inflow and outflow of population, its change within the country, and the rate of population change over time.

Let us denote the components of the population flow vector across the boundaries per unit length per unit time as $\phi(\phi_x, \phi_y)$. Then the difference between the incoming and outgoing population flows moving through an infinitely small area element $dx_1 dx_2$ will be:

$$D = dx_2 \left(\phi_{x_1} + \frac{\partial \phi_{x_1}}{\partial x_1} dx_1 - \phi_{x_1} \right) + dx_1 \left(\phi_{x_2} + \frac{\partial \phi_{x_2}}{\partial x_2} dx_2 - \phi_{x_2} \right) \quad (2)$$

The total amount of population at a given time must be equal to the rate of population change within the country Π and the number of people who disappear for objective reasons (mortality) per unit of time. This requirement leads to the following relationship:

$$\frac{\partial \phi_{x_1}(x_i, t)}{\partial x_1} + \frac{\partial \phi_{x_2}(x_i, t)}{\partial x_2} = \Pi(x_i, t) - d(x_i, t) \frac{\partial \psi(x_i, t)}{\partial t}, i = 1, 2 \quad (3)$$

Here $d(x_i, t)$ is denoted as the population mortality function.

In the general case, the functions $\Pi(x_i, t)$, $k(x_i, t)$ and $d(x_i, t)$ are dependent on the directions of varying (coordinates) and there may be heterogeneity of the dependencies or their anisotropy.

Using (1), as in (Zienkiewicz & Morgan 2006), we obtain for the value of the population flow density in two directions:

$$\frac{\partial \phi(x_i, t)}{\partial x_1} = -k_{x_1}(x_1, t) \frac{\partial \psi(x_i, t)}{\partial x_1}, \frac{\partial \phi(x_i, t)}{\partial x_2} = -k_{x_2}(x_2, t) \frac{\partial \psi(x_i, t)}{\partial x_2}, i = 1, 2 \quad (4)$$

The system of partial differential equations (3) and (4) completely determines the distribution of the population in time and over the territory. As in other applications (Gupta 2021), this system can be transformed into a single second-order differential equation with one variable $\psi(x_i, t)$. Using (4) and substituting into (3), we obtain:

$$d(x_i, t) \frac{\partial \psi(x_i, t)}{\partial t} = \frac{\partial}{\partial x_1} \left(k_{x_1}(x_1, t) \frac{\partial \psi(x_i, t)}{\partial x_1} \right) + \frac{\partial}{\partial x_2} \left(k_{x_2}(x_2, t) \frac{\partial \psi(x_i, t)}{\partial x_2} \right) + \Pi(x_i, t), i = 1, 2 \quad (5)$$

Solving the differential equation (5) requires the determination of the initial and boundary conditions.

As an initial condition, we will use the population distribution function for the territory under consideration at the initial moment of distribution analysis, at $t=0$:

$$\psi(x_i, 0) = \bar{\psi}(x_i), i = 1, 2 \quad (6)$$

In further analysis, it is possible to use boundary conditions of the first (Dirichlet) and second (Neumann) kind. The condition of the first kind – on a part of the boundary of the region Γ_1 the population distribution $\psi_1(x_i, t)$ is given:

$$\psi(x_i, t)|_{\Gamma_1} = \psi_1(x_i, t), i = 1, 2 \quad (7)$$

The second-order condition is that on the part of the boundary of the region Γ_2 the value of the population flow through the boundary is given:

$$-k_{x_i}(x_i, t) \frac{\partial \psi}{\partial x_i}(x_i, t) \Big|_{\Gamma_2} = \phi_2(x_i, t), i = 1, 2 \quad (8)$$

We note that in the current mathematical formulation the problem is similar to the problem of unsteady heat conduction known in mathematical physics (Gupta 2021).

Method of Solution

The formulated boundary-initial value problem in the general case of inhomogeneity and anisotropy in space and time of the unknown functions included in it cannot be solved in analytical form for the non-canonical geometry of the domain. Therefore, it is proposed to apply the Finite Element Method (FEM) as the solution method (Zienkiewicz & Morgan 2006).

First, we consider the case of solving the problem of stationary population distribution at a certain point in time, i.e., we will assume that $\psi(x_i, t) = \psi(x_i)$:

$$\frac{\partial}{\partial x_1} \left(k_{x_1}(x_1) \frac{\partial \psi(x_i)}{\partial x_1} \right) + \frac{\partial}{\partial x_2} \left(k_{x_2}(x_2) \frac{\partial \psi(x_i)}{\partial x_2} \right) + \Pi(x_i) = 0, i = 1, 2 \quad (9)$$

This is necessary in connection with the subsequent use of the stepwise method of integrating the initial problem. We will also use only the boundary conditions (7) and (8).

Using the weighted residuals method (Zienkiewicz & Morgan 2006), we transform the differential equation (9) into a system of linear algebraic equations. The weighted residuals method is based on the approximation of the unknown function ψ using its series expansion in terms of some previously known basis functions $N_m(x_i)$ with unknown coefficients a_m :

$$\psi(x_i) \approx \hat{\psi}(x_i) = \sum_{m=1}^M a_m N_m(x_i) \quad (10)$$

Approximation (10) must operate with basis functions that must have the properties of continuity, differentiability, and completeness, i.e. if the number of functions m approaches infinity ($m \rightarrow \infty$), the approximating function will approach the analytic function ($\hat{\psi} \rightarrow \psi$) (Zienkiewicz & Morgan 2006).

Let us introduce the value of residual, which reflects the degree of approximation accuracy and is calculated by substituting the approximation $\hat{\psi}$ into the differential equation (9):

$$R_\Omega = \frac{\partial}{\partial x_1} \left(k_{x_1}(x_1) \frac{\partial \hat{\psi}(x_i)}{\partial x_1} \right) + \frac{\partial}{\partial x_2} \left(k_{x_2}(x_2, t) \frac{\partial \hat{\psi}(x_i)}{\partial x_2} \right) + \Pi(x_i) \quad (11)$$

It is clear that for a fixed M the value of the residual R_Ω will be different from zero. Further, due to the fact that the basis functions of the series N_m in the vast majority of cases (Zienkiewicz & Morgan 2006) do not automatically satisfy the boundary requirements, then, similarly to the residual introduced above for the domain, the residual for the element boundary R_Γ is introduced:

$$R_\Gamma = \psi(x_i)|_{\Gamma_1} - \psi_1(x_i) - k_{x_i}(x_i) \frac{\partial \psi}{\partial x_i}(x_i) \Big|_{\Gamma_2} - \phi_2(x_i) \quad (12)$$

The weighted residuals method provides allows for the simultaneous minimization of residuals in the solution of the differential equation (9) R_Ω and when fulfilling the boundary conditions (11) R_Γ . For this, the equation of the following form is considered (Zienkiewicz & Morgan 2006):

$$\int_{\Omega} W_l(x_i) R_V d\Omega + \int_{\Gamma} \bar{W}_l(x_i) R_S d\Gamma = 0 \quad (13)$$

where $W_l, \bar{W}_l$ are the weight functions, $l = 1, \dots, M$. These functions can be chosen as independent or related (Zienkiewicz & Morgan 2006). In the well-known Galerkin method (Zienkiewicz & Morgan 2006), which will be used further, the weight functions are taken to be the same as the basic ones $W_l = N_l$.

Let us note that minimization of residuals is a requirement for finding the most accurate solution for the given conditions.

Substitution of the approximation (10) into the basic relation (13) leads to obtaining the system of linear algebraic equations with respect to the unknown approximation coefficients a_m :

$$[K] \cdot \{a\} = \{f\} \quad (14)$$

where $\{a\}^T = \{a_1, a_2, \dots, a_M\}$ is the vector of unknown approximation coefficients, the coefficients of the matrix K and the vector of the right-hand sides f are calculated as follows:

$$K_{lm} = \iint_{\Omega} W_l(x_i) \left(\frac{\partial}{\partial x_1} \left(k_{x_1}(x_1) \frac{\partial N_m(x_i)}{\partial x_1} \right) + \frac{\partial}{\partial x_2} \left(k_{x_2}(x_2) \frac{\partial N_m(x_i)}{\partial x_2} \right) \right) d\Omega + \int_{\Gamma} \bar{W}_l(x_i) \left(N_m(x_i) - k_{x_i}(x_i) \frac{\partial N_m(x_i)}{\partial x_i} \right) d\Gamma \quad (15)$$

$$\hat{f}_l = - \int_{\Omega} W_l(x_i) \Pi(x_i) d\Omega - \int_{\Gamma} \bar{W}_l(x_i) (\psi_1(x_i) - \phi_2(x_i)) d\Gamma$$

Next, we will use the Finite Element Method (FEM) (Zienkiewicz & Morgan 2006), which is an extension of the weighted residuals method to the case of discontinuous basis functions. In simplified terms, the method can be explained as finding a solution to a differential equation of type (9) in a domain of complex form, which is divided into subdomains. These subdomains have mostly simple geometric shapes and are called finite elements (Zienkiewicz & Morgan 2006). The solution to the problem is sought at the points at which these elements are connected to each other and then approximated to the entire domain of definition. These points are called nodes.

For using the FEM with triangular three-node finite elements with linear basis functions (shape functions) N^β with approximation of the population value in the finite element β through the nodal values ψ^β :

$$\hat{\psi}^\beta(x_i) = \sum_{m=1}^3 \psi_m^\beta N_m^\beta(x_i). \quad (16)$$

Let us proceed to the weak formulation of the problem (Zienkiewicz & Morgan 2006). We use the Galerkin method for each finite element β , then on it $W_l^\beta = N_l^\beta$ and $\bar{W}_l^\beta = N_l^\beta$.

Transformation to the weak form using Green's formulas (Gupta 2021) leads to the following expressions for the components of the matrix $[K]$ and the vector $\{\hat{f}\}$ (Zienkiewicz & Morgan 2006):

$$K_{lm} = \iint_{\Omega} \left(\frac{\partial N_l(x_i)}{\partial x_1} k_{x_1}(x_1) \frac{\partial N_m(x_i)}{\partial x_1} + \frac{\partial N_l(x_i)}{\partial x_2} k_{x_2}(x_2) \frac{\partial N_m(x_i)}{\partial x_2} \right) d\Omega$$

$$\hat{f}_l = \int_{\Omega} N_l(x_i) \Pi(x_i) d\Omega - \int_{\Gamma_2} N_l(x_i) \phi_2(x_i) d\Gamma_2$$
(17)

As always in the FEM theory (Zienkiewicz & Morgan 2006), the residual on the part of the boundary Γ_1 is omitted, it is taken into account at the stage of assembling the matrix of the system by adding terms with nodal values to the corresponding nodes of the vector $\{\hat{f}\}$.

After obtaining the matrices for each finite element, the ensemble procedure of the entire system is performed and a system of linear algebraic equations is obtained with respect to the nodal values of the population values:

$$[\bar{K}] \cdot \{\psi\} = \{F\}$$
(18)

where $[\bar{K}]$ is the matrix of the system of linear algebraic equations, $\{F\}$ is the vector of nodal values, $\{\psi\}$ is the vector of nodal values of population values.

Further, the same transformations are carried out for the case of modelling a non-stationary process of the variation of the population values over time. As can be seen from equation (5), it is necessary to additionally take into account the term containing the first derivative of the population size. Such transformations are carried out using the finite difference numerical method (Hamming 2012).

As a result, we obtain a system of ordinary linear differential equations of the first order with respect to the unknown nodal values of the components of the vector $\{\psi\}$:

$$[\bar{D}]\{\psi\} = ([\bar{K}])\{\psi\} + \{F\}$$
(19)

where $[\bar{D}]$ is the matrix whose components arise when transforming equation (5) and reflect the impact of population mortality on its overall level.

The solution of the system of differential equations (19) presents the distribution of the population in time and over the territory, which is the domain of definition Ω . To solve it, we use numerical integration methods, for example, Euler or multi-step predictor-corrector types. The values are distributed discretely, in nodes, at a given time integration step. Further, using the approximation (16) at each step of integration over time, it is possible to determine

the population distribution over each finite element, and, therefore, over the entire domain of definition.

The described finite element algorithm is implemented in the form of software created at the National Technical University "Kharkiv Polytechnic Institute", Ukraine. A description of the software implementation of similar software solutions, individual blocks of which were used in the development of this software, can be found in (Breslavsky et al. 2017).

Results of numerical modelling

As an example of the use of the method presented above, we will consider the results of numerical modelling of the population growth process in the territory of the Kyivan princes (Fig. 1a, (Tolochko 2007)). As the domain of the problem definition, we will choose a curvilinear object, the boundaries of which correspond to the boundaries of the specified state in 1015 AD. The starting point was set at 700 AD, when the main migration processes associated with the so-called Migration Period had ended. Due to the lack of data on quantitative values of the population at that time, the modelling is carried out using the relative value. That is, the population varied between 1 (maximum) and 0 (minimum) population in a given location. It is assumed that the territory is initially uniformly sparsely populated. The results of numerical modelling of such distributions are presented in the form of maps (Figs. 2-4). As boundary conditions in this formulation, a slight population movement on the northern borders is considered, associated with the migration of representatives of the Balto-Scandinavian peoples (Varangians).

Due to the FEM capabilities to consider domains of definition with heterogeneous properties, the population source function Π was given piecewise for groups of finite elements. In this case, the first hypothesis of the investigations was used regarding the predominant distribution of the population along the banks of rivers and in the southwestern part of the territory of the original settlement of Slavic tribes. So, if the river passes through a given finite element, the element's source function value increases. To ensure the relativity of the population distribution, the functions $d(x_i, t)$ and $k_{x_i}(x_i, t)$ were scaled.

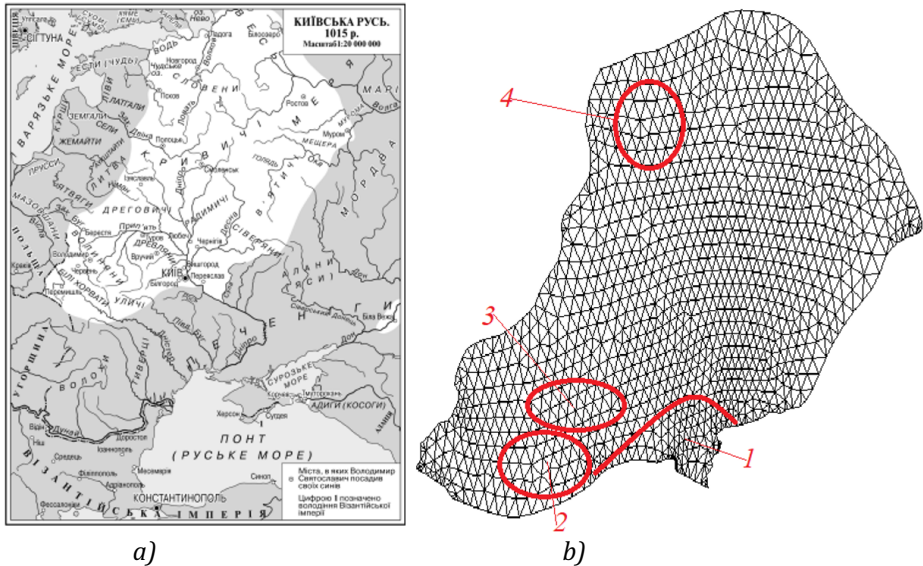


Figure 1. a) Map of the state of the Kyiv princes in 1015 AD (Tolochko 2007) and b) finite element model of the region. Locations of interest are numbered.

To assess the reliability of numerical solutions, simulations were carried out for different values of the time step when integrating the initial value problem and different finite element models when solving the boundary value one. After the convergence studies, a model with 2131 finite elements was selected. Such a model with certain geometric transformations for better display is given in Fig. 1b.

The results of the numerical modelling of the relative distribution of the population across the territory are presented in Fig. 2 (at the time of 940 AD) and Fig. 3 (at the time of 1015 AD). The obtained distributions show the concentration of the population in the south-western regions, and we also note the regions with a clear concentration in the north-east, which corresponds to the areas of further Slavic colonization.

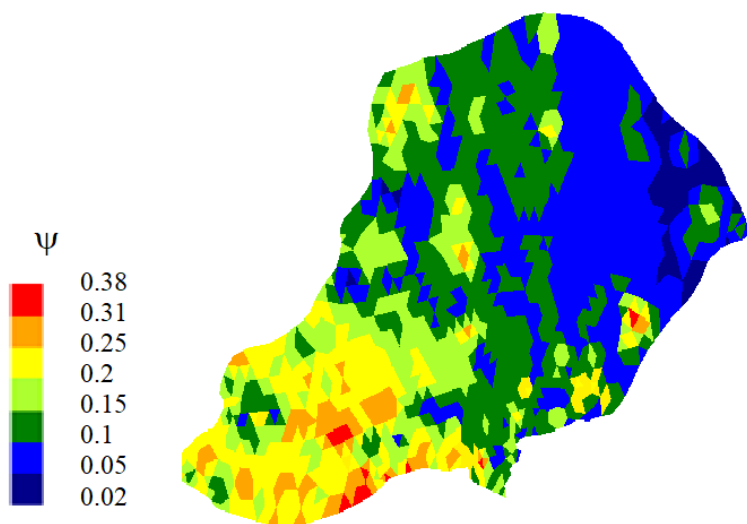


Figure 2. Relative distribution of population, 940 AD.

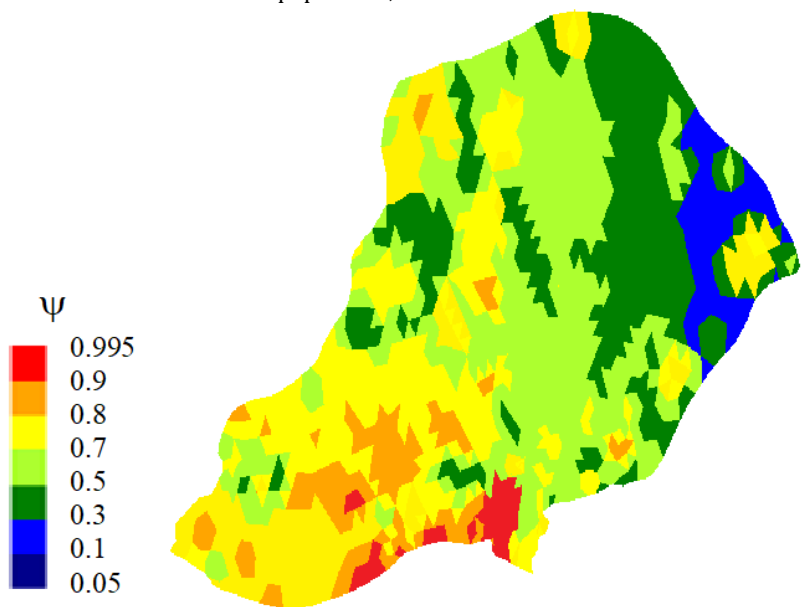


Figure 3. Relative distribution of population, 1015 AD.

Figure 4 shows the zones (marked in light blue, number 1 in the legend), in which, according to the calculations, the greatest concentration of population and the emergence of towns are expected. Yellow (number 2) shows the regions of actual town emergence, using coordinates and foundation years obtained from the Ukrainian Wikipedia (n.d.), sourced from historical references (Tolochko 2007, *Chronicle about...* 2013, *Tale of Bygone Years* 1950, *Complete Collection...* 1856, "Russian Chronicle: Behind the Patian List" 1989, *Słownik...* 1880, 1882; Tolochko et al. 2007, *Encyclopedia...* 2016–2017, Abramovich n.d., Andriyashev 1887, Terskyi 2006, Yanin & Aleshkovsky 1971, Shtykhov & Levko 2012). For this purpose, specialized software was developed. It is described in (Breslavsky & Brovarnyk 2024). Finally, green (number 3) shows the regions in which the coincidence of the coordinates of towns from historical data and the coordinates of the regions with the maximum population was obtained.

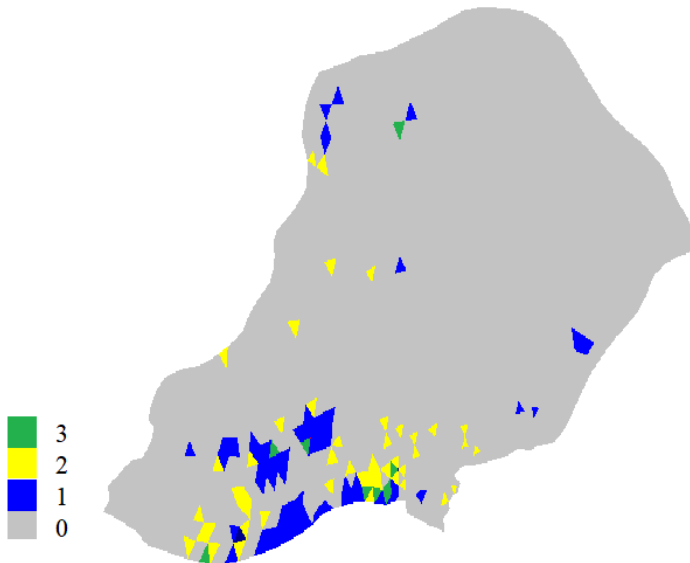


Figure 4. Distribution of the territory by zones of urban emergence: 1 – according to numerical modelling data; 2 – determined from historical sources; 3 – coincidence between calculated and historical data.

To analyse the results obtained, we will number some of the regions schematically shown in Fig. 1. In total, complete coincidence of locations was achieved for 8

towns out of 56 existing at that time. These are, first of all, the Kyiv region (zone 1), south-western Rus (zones 2, 3) and north-western territories (zone 4). It should also be noted that many other towns are located close to the zones of maximum population concentration. We should also emphasize that the exact location of Veliky Novgorod (a separate zone in the north) is obtained. This is explained by the influence of the location of the Volkhov River. It is worth noting the population concentration zone in the Oka River area (eastern region), in the region of which its growth continued in the future. Practically only for four towns of the western central region such proximity of zones was not obtained. This may be explained by factors not accounted for, such as trade and international relations, the distribution of natural resources, and other geographical features in addition to rivers. In the opinion of the authors of the paper, such results can be assessed as satisfactory and the numerical modelling method can be assessed as promising. The impact of considered migration from other territories or countries turned out to be insignificant.

Conclusions

An approach and method for mathematical modelling of the population variation in the territory of a certain state in a given historical period are presented. The main idea underlying the method is the analysis of the influence on the process of not only social but also physical factors, primarily geographical ones. That is, from a mathematical point of view, a boundary-initial value problem is posed, in which the main variable, which is the normalized population value, depends not only on time, but on the spatial coordinates as well. The formulated problem is solved using the widely used Finite Element Method (FEM) in combination with the Finite Difference Method (FDM) of integration over time (Zienkiewicz & Morgan 2006). Dedicated software was developed to perform numerical modelling.

As an example of application, the simulation of population dynamics in the territory of the Kyivan princes in the period 700-1015 AD was performed. Using the obtained population distributions in time and by territory, the places of origin of towns were determined. Comparison of the coordinates of such areas with the real coordinates of the location of towns at that time showed a satisfactory quality of modelling using the proposed method. Such coordinates were automatically obtained from known historical sources, for which a special software application was developed.

The developed modelling method can be useful not only for understanding and analysing qualitative and quantitative historical processes in specific territories, taking into account a large number of features, but also for testing hypotheses regarding the influence of certain real events or natural characteristics on the subsequent history of the objects under consideration.

The proposed approach and method require further testing on other examples, for example, population dynamics in other countries and historical periods. It is also possible to take into account other factors, such as climatic factors, historical development trends, etc. These are prospective topics for future research and may be advanced through collaboration with other scholars of historical processes.

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