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How Students Interpret Literal Symbols in Algebra: A Conceptual Change Approach

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Abstract

In this paper we present the results of an empirical study which examined students' difficulties in interpreting the use of literal symbols in algebra. Fifty seven students, 38 8th graders (approximate age 14 years) and 21 9th graders (approximate age 15 years) completed two questionnaires with algebraic objects that contained literal symbols. The results showed that students tended to interpret literal symbols as standing for natural numbers only. We concluded that students' prior knowledge of natural number interferes in their interpretation of literal symbols in mathematics.

Introduction

In the transition from arithmetic to algebra the concept of variable appears. Mathematically speaking a variable is usually represented by a literal symbol, a letter from the alphabet, which could stand for numerical values or other algebraic objects. When the variable stands for numerical values it can act as an unknown, as a changing quantity, or as a generalized number which refers to any numerical value. Literal symbols are often part of algebraic objects; for example, the algebraic objects '-b' and '4b' contain the literal symbol 'b' as well as other symbols such as a minus sign or the number 4.

The way students interpret the use of literal symbols as mathematical objects has been the subject of a great deal of research worldwide. Kucheman (1978) approached students' misunderstandings and difficulties with variables within a developmental framework. He classified students' responses to questions involving the use of literal symbols in algebra into general categories and he ordered them into Piagetian hierarchical levels. According to him the "letter as generalized number" is described as the upper level of understanding and only students who attain the formal operational stage are ready to understand this notion.

Stacey and MacGregor (1997), as long as other researchers as such Weinberg, Stephens, McNeil, Krill, Knuth and Alibali (2004), criticized this strict Piagetian hierarchy on the grounds that it does not take into account the role of experience. Stacey and MacGregor (1997) argued that pre-algebra students tend to assign to the literal symbols different meanings depending on their prior knowledge and experience. For instance, they assign the numerical value 8 to the literal symbol h, when this is used to represent a boy's height, because h is the eighth letter of the alphabet. Some students tend to interpret literal symbols as standing for abbreviated words such as D for David, h for height, etc. The researchers tried to trace the origins of these misunderstandings in students' intuitive assumptions, prior analogs with other symbol systems, interference with new mathematical learning, and problematic instruction.

In the present research, the conceptual change framework is adopted as an approach to students' difficulties in understanding the use of literal symbols in algebra. The value of the conceptual change approach to mathematics learning is only recently explored. Vosniadou and Verschaffel (2004) and Vosniadou, Vamvakoussi and Christou (2005) argue that the conceptual change framework can be used as a guide to identify concepts in mathematics that are going to cause students great difficulty, to predict and explain students' systematic errors and misconceptions, and to provide student-centered explanations of counter-intuitive mathematical concepts. According to the conceptual change theoretical framework developed by Vosniadou (1999), children's initial, naive knowledge of the physical world is organized in a 'framework theory' which consists of certain ontological and epistemological beliefs that constrain the way children understand scientific explanations of physical phenomena. For example, in the case of physics, young children believe that physical objects are stable, solid, do not move by themselves and fall down when not supported (Spelke, 1991). Likewise in mathematics it appears that, from a young age, children develop a concept of number as natural number only. This can be a source of difficulties when numbers other than natural numbers are introduced in the mathematical curriculum. For example, many errors and misunderstandings are caused by students' tendency to apply properties of natural numbers to fractions (Gelman, 2000, Stafylidou and Vosniadou 2004) to rational numbers (Resnick, Nesher, Leonard, Magone, Omanson and Peled, 1989; Vamvakoussi and Vosniadou, 2004) to negative numbers (Gallardo 2002, Vlassis 2004) and to algebraic notation and rules (Kieran, 1990).

A basic property of natural numbers is that they share a univocal form. This means that there is a one to one correspondence between the symbolic representation of the natural number and the numerical value that it represents. In the natural number set, every number represents only one value and different symbols represent different values. This does not occur in other number sets, such as the real number set. In the real numbers set a numerical value can be represented in multiple ways; for example, the numbers 2,0, $\frac{4}{2}$, $\frac{32}{16}$, and $\sqrt{4}$, can be different symbolic representations of the same numerical value 2. Another property of real numbers which does not apply in the case of the natural number set is that the opposite of a natural number is not a natural number.

When introduced to literal symbols in algebraic objects, students tend to interpret the literal symbols as standing only for natural numbers and not for fractions, decimals, or other non-natural numbers in general. The working hypothesis of the present study is that students' interpretation of numbers as natural numbers constrains their understanding of the use of literal symbols in algebra. Two questionnaires were given to students asking them to write down numerical values that they thought *can* (in Questionnaire A) or *cannot* (in Questionnaire B) be assigned to given algebraic objects that contained literal symbols with different superficial characteristics. We expected that there would be a strong bias on the part of the students to substitute the literal symbols only with natural numbers.

Method

Participants

Fifty seven students from two public high schools in Athens, Greece participated in this study: 36 8^{th} graders (approximate age 14 years) and 21 9^{th} graders (approximate age 15 years); 31 of them were girls and 26 boys. Twenty nine students completed Questionnaire A (QR/A) and twenty eight completed Questionnaire B (QR/B).

Materials

In both questionnaires students were given the following instructions: "In algebra, we use letters (like a, b, x, y, etc) to stand for numbers and relations between numbers. In this questionnaire we use such letters. Read the following seven questions carefully and write down as many examples of values as you can". In QR/A we continued with the following instructions: "Write down numerical values that you think can be assigned to Q1: a, Q2: -b, Q3: 4d, Q4: 1/d, Q5 a/b, Q6: a+a+a and Q7: k+3". In QR/B, the instructions were: "Write down numerical values that you think *cannot* be assigned to Q1: a, Q2: -b, Q3: 4d, Q4: 1/d, Q5 a/b, Q6: a+a+a and Q7: k+3".

Procedure

The participants completed the questionnaires in the presence of the experimenter and their mathematical teacher in their class. The questionnaires A and B were distributed randomly. Each student had only one questionnaire to complete. The experimenter had the opportunity to discuss with the students any clarification problem they had in answering the questions.

Results

Questionnaire A: Q1 & Q2

In QR/A, Q1 and Q2, the students were asked to write down the numerical values which they thought could be assigned to 'a' and '-b'. Table 1 shows the way we categorized students' responses.

Table 1: Percentage of Students' Responses in the Different Categories of Responses in QR/A for Q1 and Q2

Questionnaire A (Write down numerical values that you think can be assigned to the algebraic objects)		
Categories of responses	Q1: a	Q2: -b
No Answer	--	--
Positive Whole Numbers (natural numbers) (1, 2, 3, etc)	66% (+F/+N)	8% (-F/+N)
Negative Whole Numbers (e.g. -1, -2, -3, etc)	4% (-F/+N)	72% (+F/+N)
Positive Numbers (non-natural, e.g. 2.33, etc)	--	4% (-F/-N)
Negative Numbers (non-wholes, eg.-2.33, etc)	--	--
Scientific (all values can be assigned to it)	30%	16%

The scientific response was to write down that all kind of values can be assigned to each algebraic object, or to give some numerical values of any type of numbers as examples. For example, in 'a' or '-b' we considered the scientific responses to be that "all kind of values can be assigned to each algebraic object", or examples of non-natural values as 1, 2, -1, -2, $\frac{2}{3}$, -$\frac{2}{3}$, 0.266, -2.66, etc. About less than one third of the students (30% in Q1 and 16% in Q2, QR/A) gave the scientific responses in QR/A. It could be argued that in QR/A there cannot be any mathematically wrong responses because all values can be assigned to each algebraic object, (negative numbers, fractions, etc). Nevertheless, while the responses in all the other categories in Table 1 can be considered as correct, they are different from the responses expected from a mathematically sophisticated participant and provide useful information about the way students interpret the use of such symbols.

An examination of students' non-scientific responses revealed a clear tendency to interpret 'a' as a literal symbol which stands for natural numbers (positive wholes 66% in Q1, QR/A) and '-b' as a symbol which stands for the opposite of natural numbers (negative wholes, 72% in Q2, QR/A). A chi square analysis of association between students' positive and negative responses to 'a' and '-b' (QR/A) showed that the difference between students' responses was statistically significant [$x^2(2)=29.7$, $p<.001$)]. The students who responded with positive values to 'a' responded with negative values to '-b'. This means that when the algebraic object changed its superficial characteristics, such as its sign, students were affected by it and made similar changes to the values they assigned to the literal symbol. We consider these changes to be changes in the *form* of the algebraic object, and not in the *number* itself. At the same time the students continued to substitute the literal symbol itself (the 'b' in the algebraic object '-b') only with natural numbers.

In short, it appears that students are influenced by two things when they assign values to the algebraic objects that contain literal symbols: a) the form of the algebraic object (i.e., the superficial characteristics of the algebraic object such as the presence of a negative sign or the presence of another number as 4 in '4b'), and b) their interpretation of the literal symbols as always standing for natural numbers. We marked students' responses that retained the form of the algebraic object as '+F' and when they did not as '–F'; similarly, when they substituted literal symbols only with natural numbers it was marked as '+N' and when they did not, as '-N'. As shown in Table 1, in QR/A the majority of the students tended to assign numerical values that retained both the form of the algebraic object and the natural number (+F/+N), both in Q1 and in Q2.

Questionnaire B: Q1 and Q2

Table 2 shows students' responses to Q1 and Q2 in QR/B, where students were asked to write down numerical values that they thought *cannot* be assigned to the algebraic objects 'a' (Q1) and '-b'(Q2). Less than one third of the students gave scientific responses to Q1 (29%) and Q2 (18%) in QR/B. Unlike QR/A, in QR/B all responses other than the scientific ones are incorrect because there are no values that cannot be assigned to the algebraic objects. Students in these grades have already been taught that literal symbols in algebra stand for all numbers.

An examination of students' incorrect responses showed that the majority of students (46% in Q1, QR/B) said that negative whole numbers cannot be assigned to 'a' and that positive whole (natural) numbers cannot be assigned to '-b' (50% in Q2, QR/B). An additional 14% of the students responded that positive (non-natural) values cannot be assigned to '–b' (Q2, QR/B) and 10% that negative (non-whole) values cannot be assigned to 'a' (Q1, QR/B). Students' tendency to respond with negative values to 'a' and with positive values to '-b', when asked to write down numerical values that *cannot* be assigned to 'a' and '-b', was statistically significant [$x^2(2)=27.022$, $p<.001$)]. This result shows that when students are asked to write down numerical values that *cannot* be assigned to a given algebraic object, they tend to change only its form, by putting a minus sign in front of it. As in the previous case, however, the students substituted the literal symbol itself only with natural numbers.

Table 2: Percentage of Students' Responses in Different Categories of Responses in QR/B for Q1 and Q2

Questionnaire B (Write down numerical values that you think *cannot* be assigned to the following algebraic objects)		
Categories of responses	Q1: a	Q2: -b
No Answer	4%	6%
Positive Whole Numbers (natural numbers) (1, 2, 3, etc)	3% (+F/+N)	50% (-F/+N)
Negative Whole Numbers (e.g. -1, -2, -3, etc)	46% (-F/+N)	4% (+F/+N)
Positive Numbers (non-natural, e.g. 2.33, etc)	8% (-F/-N)	14% (-F/-N)
Negative Numbers (non-wholes, eg.-2.33, etc)	10% (-F/-N)	8% (-F/-N)
Scientific (all values can be assigned to it)	29%	18%

In order to further compare QR/A and QR/B, a chi square analysis of the categories of students' non scientific responses was conducted. The results showed statistically significant differences between students' responses in the two questionnaires [$x^2(2)=57.344$, $p<.001$] that were due to the fact that students tended to respond with values that were placed in the category '+F/+N' in QR/A and in category '-F/+N' in QR/B. This result suggests that students tended to change the form of the given algebraic object, when they were asked to write down numerical values that they thought *cannot* be assigned to the given algebraic object, by putting a minus sign in front of it, but they systematically substituted the literal symbol itself only with natural numbers.

Questionnaires A and B: Q3 to Q7

Table 3 shows the way we categorized students' responses in the remaining questions of QR/A. In this categorization we distinguished responses that paid attention to the form of the algebraic object ($\pm$F) versus the nature of the number itself ($\pm$N). For example, students' responses to '4d' as 4·1, or 4·2, were scored "+F/+N" (+Form/+Natural). Table 4 shows the way we categorized students' responses to the remaining questions of QR/B. The categorization is the same as in QR/A, presented in Table 3. For example, when students were asked to write down numerical values that they thought *cannot* be assigned to '4b', responses such as 1, 2, 3, or 4·(-1), 4·(-2), were scored as "-F/+N" (-Form /+Natural). We must make clear that we interpret the use of the negative sign to represent for the students not a change in the number itself but a change in the superficial

Table 3: Categorization of Students' Responses in QR/A for Q3 to Q7

Questionnaire A (Write down numerical values that you think can be assigned to the following algebraic objects)					
	Categories of Responses				
Questions	+F/+N	-F/+N	+F/-N	-F/-N	scientific
Q3: 4g	4·1, 4·2	1, 2, 3	[4·2.3, 4·$\frac{1}{2}$]	2.3, -2.3	all values
Q4: $\frac{1}{g}$	$\frac{1}{2}$, $\frac{1}{3}$	1, 2, 3	[$\frac{\frac{1}{2}}{3}$]	2.3, -2.3	all values
Q5: $\frac{a}{b}$	$\frac{3}{4}$, $\frac{2}{3}$	1, 2, 3	[$\frac{\frac{1}{2}}{25}$]	-1, -2, -3, 2.3, $\frac{2}{3}$	all values
Q6: d+d+d	1+1+1 2+2+2	1, 2, 3	[$\frac{2}{3}+\frac{2}{3}+\frac{2}{3}$, 2.3+2.3+2.3]	$\frac{2}{3}$, -$\frac{2}{3}$	all values
Q7: κ+3	2+3, 3+3 larger than 3	1, 2, 3	[2.3+3, $\frac{2}{3}$+3]	2.3, 2.77	all values

characteristics of the algebraic object (see Gallardo, 2002). Example of responses that were expected from students in a given response category but did not actually appear in our sample were put in square brackets both in Table 3 and in Table 4. For example, none of the students gave values such as 4·2,3 to '4d' (category +F/-N). Table 5 shows the percentages of students' responses in QR/A and QR/B for questions Q3 to Q7 combined. Less than one third of the students gave the scientific response both in QR/A and QR/B. It appears that there is a greater number of scientific responses in QR/B compared to QR/A. However a chi square analysis showed that these differences were not statistically significant.

The majority of students who gave non scientific responses tended to assign values to the algebraic objects that retained both the form of the algebraic object and the interpretation of the literal symbols as always standing for natural numbers. In QR/A, 49% of the students gave these kinds of responses. As expected, in QR/B, where students were asked to write down numerical values that *cannot* be assigned to the algebraic objects, many students changed the form of the algebraic object but they substituted the literal symbol itself only with natural numbers (-F/+N). An important finding that comes out from Table 5 is that the category "+F/-N" in both QR/A and QR/B is empty. There was not even a single case were a student changed the natural number without also changing its form, although it could be argued that some of these numbers (for example the double fractions) are rare and not commonly used in every day mathematics activities.

Table 4: Categorization of Students' Responses in QR/B for Q3 to Q7

Questionnaire B (Write down numerical values that you think *cannot* be assigned to the following algebraic objects)					
	Categories of Responses				
Questions	+F/+N	-F/+N	+F/-N	-F/-N	scientific
Q3: 4g	[4·1, 4·2]	1, 2, 3 4·(-1), 4·(-2)	[4·2.3, 4·$\frac{1}{2}$]	-1, -2, -3 -2.3, -$\frac{2}{3}$	all values
Q4: $\frac{1}{g}$	[$\frac{1}{2}$ $\frac{1}{3}$]	1, 2, 3· $\frac{1}{-2}$, $\frac{1}{-3}$	[$\frac{\frac{1}{2}}{3}$]	2.3, -2.3	all values
Q5: $\frac{a}{b}$	[$\frac{2}{3}$, $\frac{3}{4}$]	$\frac{-2}{-3}$, $\frac{-3}{-4}$	[$\frac{\frac{1}{2}}{25}$]	2.3, -2.3	all values
Q6: d+d+d	[1+1+1, 2+2+2]	1, 2, 3 (-1)+(-1)+(-1), (-2)+(-2)+(-2)	[$\frac{2}{3}+\frac{2}{3}+\frac{2}{3}$ 2.3+2.3+2.3]	-1, -2, -3, -2.3, -$\frac{1}{2}$	all values
Q7: κ+3	[2+3, 3+3]	(-2)+3, (-3)+3	[2.3+3, -2.3]	-1, -2, -3 -2.3, -$\frac{2}{3}$	all values

Chi square analysis of students' non scientific responses showed statistically significant differences between the category '+F/+N' in QR/A and the category '-F/+N' in QR/B, [$x^2(2)=85.39144$, $p<.001$)] confirming students' tendency to change the superficial characteristic of the algebraic object by putting a minus sign in front of a natural number when asked to write down numerical values that *cannot* be assigned to the given algebraic objects. Most of the students who responded by changing both the form and the number (category -F/-N as shown in Table 4, 16%), responded with negative whole numbers to all the algebraic objects except the ones that had the form of a fraction.

Table 5: Percentages of Students' Responses in QR/A and QR/B for Q3 to Q7

	Questionnaires	
Categories of Responses	QR/A	QR/B
No Answer	12%	26%
+F/+N	49%	0%
-F/+N	10%	22%
+F/-N	0%	0%
-F /-N	4%	16%
Scientific (all values)	25%	36%

In order to further examine changes in form, students' responses were subjected in one way ANOVA, where responses that retained the form were marked as 1, and responses that did not retain the form were marked as 2. Scientific responses were marked as 3. The results of the ANOVA showed a main effect for Questionnaire Type ($F(1, 281)= 6.126$, $p<0.05$) due to the fact that students retained the form of the algebraic objects in QR/A but not in QR/B. Note that in QR/B students were asked to write down numerical values that *cannot* be assigned to the algebraic objects. In another ANOVA, we examined the hypothesis that students interpret the literal symbol as always standing for natural numbers. In this ANOVA, responses that contained only natural numbers were marked as 1 while responses that were not were marked as 2. Scientific responses were again marked as 3. The results showed no statistically significant differences between the two questionnaires. In both questionnaires, students appeared reluctant to assign numerical values other than natural to the given literal symbols, systematically replacing the literal symbols with natural numbers. This finding is consistent with the theoretical hypothesis of this research that students tend to interpret literal symbol as only standing for natural numbers.

Discussion

The present study tested the hypothesis that students will tend to interpret literal symbols as standing only for natural numbers using two questionnaires. In QR/A, students were asked to write down numerical values that *could* be assigned to a set of algebraic objects. In this questionnaire, about a quarter of students gave the scientific response that all values can be assigned to the literal symbols and about half of the students substituted the literal symbols with natural numbers (+F/+N). The remaining responses were either "no answer" or responses where only the form of the algebraic objects was changed. There were only 4% of the responses where the literal symbol was substituted by a non-natural number of a different form and *no responses* where the form of the literal symbol was retained and the number was a non-natural number.

In QR/B a different sample of students were asked to write down numerical values that *could not* be assigned to the same set of algebraic objects. In this questionnaire there was an increase, which was not statistically significant, in the number of scientific responses (36%) but also in "no answer" responses (26%). About a quarter of students responded by substituting the literal symbols with negative whole numbers. Another 16% substituted the literal symbols with numbers of a different form that were also non-natural. As was the case in QR/A, there were no response where the students substituted the literal symbols with non-natural number without changing also the form of the algebraic object.

In short, the results of the study supported the hypothesis that there would be a strong tendency to interpret literal symbols as standing only for natural numbers. In addition, the results indicated that students pay attention to the form of the literal symbol. When they were asked to write down numerical values that cannot be assigned to the same set of algebraic object, they exhibited a strong tendency to change the form of the algebraic object by adding a negative sign. The analysis of variance showed significant differences between QR/A and QR/B only when comparing changes in the form of the literal symbol but not when comparing changes in the number itself (i.e., from natural to non-natural numbers). It was noticeable that no students substituted the literal symbols with non-natural numbers without changing their form as well.

One could argue that the substitution of the literal symbols with negative numbers in QR/B should be interpreted not only as a change in the form of the algebraic object but also as a change in the number itself (given that negative numbers are non-natural). However, we believe that students do not think of negative numbers as non-natural, but rather, as mentioned also by Gallardo (2002), they think that negative numbers are 'signed' numbers, where the minus sign is added without changing the number itself. It appears that students' strategy to change the superficial characteristic of the literal symbols by putting a minus sign in front of it belongs to an intermediate level of interpreting the use of letters as mathematical objects and adds further support to the hypothesis that even advanced high school students think of numbers as natural numbers only, a tendency that can hinder further development of algebraic thinking.

It could be argued that the response "all values can be assigned to each algebraic object" may not be a scientific response because the students may not think of all non-natural numbers when giving the response. It could also be argued that our participants substituted literal symbols with natural numbers because natural numbers are much more common than non-natural numbers and not because they do not know that literal symbols can stand for non-natural numbers. A deeper investigation of students' beliefs regarding exactly which kind of numbers could substitute literal symbols could be tested by using forced-choice questionnaires that contain non-natural numbers. We are in the process of conducting such a study and preliminary results support our hypothesis.

In general, the results of the present study support the conceptual change approach in mathematics (see Vosniadou and Verschaffel, 2004) and show that there is a strong tendency in students to interpret all numbers as natural numbers. This tendency may hinder the development of advanced mathematical thinking. The results of the present research are consistent with other findings in the development of the number concept (Gelman, 2000; Resnick, Nesher, Leonard, Magone, Omanson and Peled, 1989; Stafylidou and Vosniadou, 2004; Vamvakoussi and Vosniadou, 2004a). We believe that the conceptual change framework can help to systematize results from previous research and give a better explanation of some of students' difficulties in interpreting the use of literal symbols in algebra. It is of extreme importance for students to understand the generalized nature of literal symbols in mathematics. During mathematics instruction students will deal with new concepts such as the function, the absolute value of a number, the limit of a function and many others, in which literal symbols are used to express relationships between numbers. To understand these concepts, it should be clear to them that literal symbols in mathematics stand for numerical values of all kind and not only for natural numbers.

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