UC Merced
Proceedings of the Annual Meeting of the Cognitive Science Society

Title
A computational model of counting along a mental line

Permalink
https://escholarship.org/uc/item/7np6762w

Journal
Proceedings of the Annual Meeting of the Cognitive Science Society, 43(43)

ISSN
1069-7977

Authors
Chouteau, Stephanie
Mazens, Karine
Thevenot, Catherine
et al.

Publication Date
2021

Peer reviewed

eScholarship.org Powered by the California Digital Library
University of California

A Computational Model of Counting along a Mental Line

Stéphanie Chouteau (stephanie.chouteau@univ-grenoble-alpes.fr)
Univ. Grenoble Alpes, CNRS, LPNC, 38000 Grenoble, France

Karine Mazens (karine.mazens@univ-grenoble-alpes.fr)
Univ. Grenoble Alpes, CNRS, LPNC, 38000 Grenoble, France

Catherine Thevenot (catherine.thevenot@unil.ch)
Institute of Psychology, University of Lausanne, Géopolis
1015 Lausanne, Switzerland

Jasinta Dewi (jasinta.dewifreitag@unil.ch)
Institute of Psychology, University of Lausanne, Géopolis
1015 Lausanne, Switzerland

Benoit Lemaire (benoit.lemaire@univ-grenoble-alpes.fr)
Univ. Grenoble Alpes, CNRS, LPNC, 38000 Grenoble, France

Abstract

Are mental additions of single-digit numbers solved through direct retrieval from long-term memory or through persistent use of an automatized counting procedure along a mental line? In this paper, we present an experiment based on small additions along an artificial mental line, which tends to show that for very small addends (+2 to +4), counting may still be used at the end of a 3-week training. To investigate this issue, we developed the AutoCoP computational model, which describes how small additions could be solved, based on attention, working memory and experience. The simulations of AutoCoP based on this experiment showed that the effects detected at the behavioral level are reproduced and consistent with the theory, which assumes the use of a counting procedure in experts.

Keywords: Numerical cognition; mental addition; computational modeling; working memory; mental line

Introduction

Mental arithmetic and more specifically solving of basic additions have been the focus of many studies in the field of numerical cognition for several decades. Understanding the mechanisms underlying basic arithmetic is of great interest, as the resulting theories may be applied in the field of learning or education to improve teaching practices or to remedy disorders such as dyscalculia. Should kids learn by heart their addition tables or rather learn how to count faster and faster? There is a relative consensus in the scientific community, that retrieval of arithmetic facts from long-term memory is the dominant strategy used to solve simple addition problems in adults. According to associationist models and especially the Instance Theory of Automatization (Logan, 1988), the initial sequential multi-step algorithm used in basic arithmetic such as in additions, creates episodic traces associating each element involved in the operation. When the number of traces associated to a problem is high enough, the result is directly retrieved from long-term memory. Using chronometric data, almost all studies show that the duration for solving basic additions increases with the size of the smallest operand and decreases from 400 ms/increment in 6-year-old children to 20 ms/increment in adults (Groen & Parkman, 1972). According to associationist theories, this remaining slope detected in experts could correspond to the sporadic use of a counting procedure in case of retrieval failures or could be explained by the structural or functional characteristics of the retrieval process used at the memory network level storing the associations (Uittenhove et al., 2016, p. 291). Nevertheless, several recent studies cast doubt on this consensus and suggest that long-term memory retrieval may not be the dominant strategy used by experts to solve basic additions (Fayol & Thevenot, 2012; Barrouillet & Thevenot, 2013; Uittenhove et al., 2016; Mathieu et al., 2016). These studies suggest, on the contrary, that experts may use automatized and unconscious counting procedures corresponding to the quick scrolling of ordered numerical representations such as a mental number line (Figure 1).

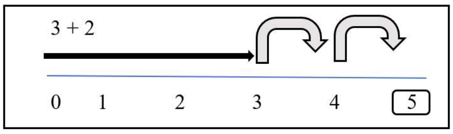

Figure 1. Representation of the counting procedure activated when solving a simple additive problem

Using an alphabet-arithmetic verification task (e.g. A+2=C?), Thevenot et al. (2020) confirmed these hypotheses and pointed out that even after intensive training, participants still use a counting strategy and increase the speed of execution of the counting steps for the small operands (+2 to +4), whereas for a specific category of problems corresponding to the largest addends, a minority of participants seems to favor memory retrieval. This conclusion calls into question the consensus according to which memory retrieval is primarily used for the smallest additions, which have been encountered first in preschool and have been more accurately and more frequently solved during training.

These hypotheses could fit the Race Model described by Logan and Cowan (1984) in which memory retrieval competes with the counting algorithm. For larger problems, the algorithm may be slower, which explains why memory traces have a better chance of winning than when the algorithm is faster, as with small problems. According to these studies, it seems currently impossible to exclude the possibility that individuals are still counting after extensive practice of small additions.

A method to investigate this issue is to develop a computational model to precisely describe how additions involving very small operands could be solved based on what is known about human cognition and especially attention and working memory. The goal is first to get a proof of concept able to reproduce the known effects, then a tool that would help researchers to investigate the theoretical models of solving basic additions.

AutoCoP: a new counting model based on an automatized procedure relying on working memory

The AutoCoP model of mental additions proposed in this study is based on the theoretical model presented by Uittenhove et al. (2016), which describes mental additions involving very small operands as an automatized and unconscious procedure scrolling an ordered representation such as a number line sequence. This model does not consider the notion of strategic variability as described in the Race Model. In its first version, AutoCoP focuses on the speed up of the counting procedures and not on memory retrieval that may occur in some participants with the largest addends. According to this model, fluency in mental additions would not result from a transition from costly counting procedures to memory retrieval, but rather from slow to automated and ultra-fast procedures. At the beginning of the training, these algorithms are requiring control and awareness of each of their steps but at the end, they are performed out of the control of the participant who remains unaware of the process itself. According to Uittenhove et al. (2016), the high speed of this process may be due to the fact that it occurs during a single focus of attention for very small additions. Working memory plays a central role in the execution of this procedure and is continuously updated during this process. Therefore, intermediate inputs and outputs involved in the compiled procedure may not be accessible to the consciousness of the person who performs it. This model is also inspired by the Adaptive Control of Thought-Rational (ACT-R) (Anderson et al., 2004). In ACT-R, learning begins with a first stage during which individuals control their processing strategies. This step is followed by a stage in which knowledge is compiled to create fast procedures. Nevertheless, in contrast to Anderson's model, Autocop relies on automatization rather than compilation. Indeed, whereas in Logan and Anderson's models, "learning mechanisms reduce the number of things to think about" (Logan, 2018, p. 453) either through retrieval or chunking, we propose that "the number of things to think about" remains the same with practice but that their execution and mental succession is accelerated until automatization. Therefore, the experience acquired during the training makes it possible to speed up procedural strategies until they are automated and no longer under individuals' cognitive control. From a behavioral point of view, training results in a decrease in solution times until it reaches a plateau.

AutoCoP: description and implementation

AutoCoP simulates the training of adding numerical operands (e.g. +2) to items belonging to a sequence of ordered elements corresponding to any mental line, either existing (e.g. 1 2 3 4…, A B C D …) or artificial (e.g. X G R Q D F V K B…). For instance, based on the alphabetical line, the model estimates the duration to compute A+2 or to check whether A+2=C is correct. For this purpose, the model simulates the basic cognitive processes to go from one element to the next one until it reaches the correct result. The model also considers the individual's experience on counting along the sequence. The experience is here assumed to depend on the number of times each particular step on the mental line has been performed. Therefore, each element of the sequence has an experience value, e.g. a participant may have high experience moving from element i to i + 1 but a weaker experience moving from element k to k + 1. In Table 1, at the beginning of the training, there is no experience associated with the elements of the sequence X G R Q D F V K B P, i.e. each letter has an experience value equal to 0. When performing an operation like X+6, one has to move from X to G, G to R, R to Q, Q to D, D to F and F to V which increases the experience value of each element X, G, R, Q, D and F.

Table 1: Example illustrating how experience is updated during the training on the X G R Q D F V K B P line: each number represents the number of times each next element has been retrieved.

Operation	Experience of each element									
	X	G	R	Q	D	F	V	K	B	P
None	0	0	0	0	0	0	0	0	0	0
X+6=V	1	1	1	1	1	1	0	0	0	0
D+3=K	1	1	1	1	2	2	1	0	0	0
G+2=Q	1	2	2	1	2	2	1	0	0	0

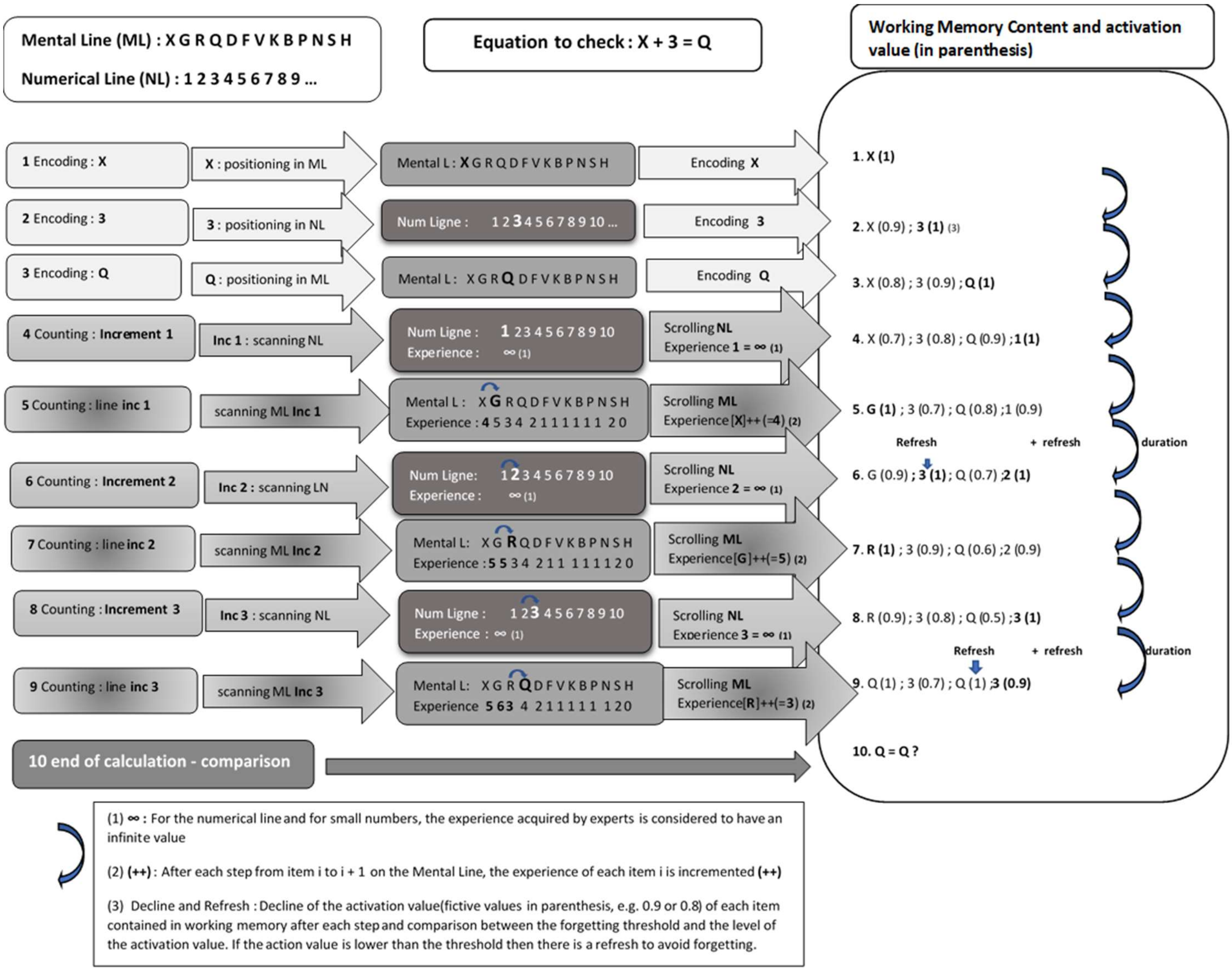

Figure 2. The different stages of an equation verification according to the AutoCoP model

Working memory occupies a central place in this model, both at processing and at data storage levels. Indeed, all data stored in working memory must be kept at a sufficient level of activation throughout the process. Thus, each element stored in working memory is associated with an activation value, which evolves during the processing episode. The way it evolves was inspired by the Time-Based Resource Sharing Model (TBRS, Barrouillet et al., 2011) and its implementation (Portrat & Lemaire, 2015). The TBRS model considers that information stored in working memory decays over time when attention is focused on another task. Attentional focus is seen as a mechanism that can only be dedicated to one process at a time and that may quickly switch from a processing stage to a refreshing stage within working memory.

The implementation of AutoCoP (Figure 2) aims to simulate a verification task of additions such as X+3=Q, which is correct using the artificial mental line of Table 1, or D+4=K, which is not. As described before, each elementary step of the algorithm gives rise to the update of the activation value of each element stored in working memory and its refreshing if necessary. Let us now detail the basic steps of the model with the equation X+3=Q to be verified.

Initialization phase (steps 1 - 3 in Figure 2): All the elements of the equation (e.g. X, 3, and Q corresponding to the augend, addend and result) are encoded in working memory. Their activation value (between 0 and 1) is initialized to 1.

Iteration phase (steps 4 - 9 in Figure 2): The iterative counter, initialized to 0, is incremented following the numerical line (1 2 3 4 …) up to the next value. The next element on the mental line is retrieved and replaces the augend in working memory. The duration of this step depends on the experience of the element being processed according to a mathematical function that will be presented later. Then, the experience of the newly replaced element in working memory is increased by 1. If the iterative counter is equal to the addend, then the process is finished at the end of the step. Otherwise, the iteration continues until it reaches the addend. In line with the TBRS model, the activation value of each working memory element at a given moment (see Figure 2: the value in parenthesis next to the elements stored in the working memory part) is updated by considering the duration of each step of the process. If the forgetting threshold is reached for any stored element, then a refreshing step is carried out which adds an extra delay to the duration of the process. In the previous example, the working memory therefore contains successively the following elements (augend, addend, result, increment):

[X 3 Q 0] [G 3 Q 1] [R 3 Q 2] [Q 3 Q 3]

End of processing (step 10 in Figure 2): At the end of the iteration phase, the result obtained is compared to the result of the equation and the answer (yes/no) is provided.

Mathematical modeling

As previously described, the duration of the calculation considers the possible extra delay for refreshing each element stored in the working memory if the activation value is lower than the forgetting threshold as well as the duration to move from one element to another on the mental lines. The implementation of the TBRS model relies on an exponential function (Oberauer and Lewandowsky, 2011) to represent the decay of the elements stored in working memory. The activation value a_i after a duration t, of an element i of the working memory follows the following decay function:

$$a_i(t) = a_i(0).\, e^{(-D\ )} \quad (1)$$

t in second, D decay between 0 and 1

If the activation value ai of element i becomes lower than the forgetting threshold, then the element is refreshed and its activation value a_i is updated to 1.

For an element i of the mental line, the duration to retrieve the next one, that is to move from i to i+1, consists of the sum of two durations: an incompressible duration of approximately 20 ms which corresponds to the duration observed in experts and a duration depending on a decreasing function considering the experience of the element i and based on Equation 2 (Figure 3).

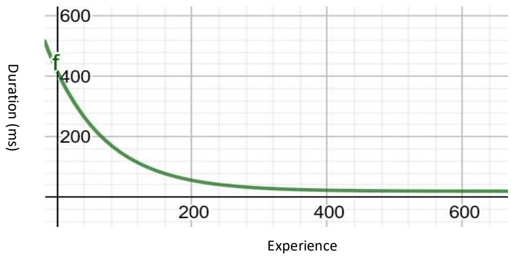

Figure 3: Relationship between experience on the mental line and reaction time

$$Duration(i \to i+1) = RDB + RDF.\, e^{\left(\frac{-(experience(i \to i+1))}{RDD}\right)} \quad (2)$$

RDB: Retrieval Duration Baseline (20 ms)
RDF: Retrieval Duration Factor
RDD: Retrieval Duration Divisor

In Equation 2, the duration is maximal (i.e. equal to RDB + RDF) when the experience is zero and minimal (i.e. equal to RDB) when the experience increases towards infinity.

This computational model was implemented in a C program, which is available at our Open Science Foundation web page. We now present the human data that are used to assess the model.

A new experiment based on a new mental line

Numerous studies on small basic additions show a monotonous linear effect of the smaller operand on solution times (Barrouillet & Thevenot, 2013). On alphanumeric addition training tasks (based on the alphabetical sequence), Thevenot et al. (2020) showed that resolution times decrease as the training progresses but eventually reach a plateau. At the end of the training, the effect of the operand on solution times continues (except for the largest addend). In this analysis, the authors found a positive slope of 217 ms/addend and of 235 ms/addend (for experiment 1 with problems involving addends 2 to 4 and for experiment 2 with problems involving addends 2 to 5) and a clear linear pattern of solution times as a function of addends for those operands.

The problem with the existing numerical and alphabetical mental lines is that participants already learned them since kindergarten and therefore know them perfectly well, which precludes the description of what happens at the beginning of learning. Our solution was to rely on an artificial mental line that no participant would know before the experiment. In addition, it would allow us to check whether the effects observed with the standard alphabetical line also occur with other types of mental line. A learning experiment with 19 students aged between 18 and 29 years was conducted over 3 weeks. Participants learned to add digits (addends +2 to +5) to the first eight letters of a new sequence (e.g. X G R Q D F V K B P N S H J), which they had to learn the week-end preceding the beginning of the experiment. In each of the 15 daily training sessions, participants had to check equations (8 letters × 4 addends from +2 to +5) based on this new artificial line and which were presented in random order. Half of the presented equations were associated with the correct answer (e.g. X+3=Q – "true") whereas the other half was associated with an incorrect answer (e.g. G+4=V – "false"). Every combination of letters, addends (+2 to +5), and response validity ("true" or "false") was presented six times per session. Thus, every session involved 384 trials (i.e., 8 letters x 4 addends x 2 possible answers x 6 repetitions), which were divided into three blocks separated by a break.

Our experimental results are very similar to those of Thevenot et al. (2020) on the alphabetical mental line. The overall correct answer rate was 97.33% (*SD* = 0.82%). We conducted a 15 (Session: 1 to 15) x 4 (Addend: 2 to 5) repeated-measures ANOVA on correct answers and "true" equations, which showed that the mean solution times significantly decreased over sessions ($F(14,252)=114.4$, $SD=0.86$, $p<.001$) from 5 177 ms (SD=3 013 ms) in session 1 to 1 412 ms (SD=729 ms) in session 15. We compared the mean solution times of each session n to session n+1 to determine whether a plateau has been reached at the end of the training. From session 9, the differences between successive sessions begin to disappear: between session 9 and session 10, the difference is not significant ($t(18)=2.608$, $p=0.154$), same as between session 11 and 12 ($t(18)=-1.364$, $p=1$) and between session 14 and 15 ($t(18)=1.55$, $p=1$). These results suggest a stagnation of mean solution times from session 9.

A main effect of addend was also found ($F(3,54)=39.01$, $\eta_p^2=0.684$, $p<.001$) showing that solution times generally increased as a function of addend, with 1 807 ms (SD=1 283), 2 254 ms (SD=1 649), 2 450 ms (SD=1 827) and 2 582 ms (SD=2 070) respectively for addends +2, +3, +4, +5. More important in these results is the interaction between session and addend ($F(42,756)=21.57$, $\eta_p^2=0.545$, $p<.001$) showing that the effect of addend decreased across sessions (Figure 4).

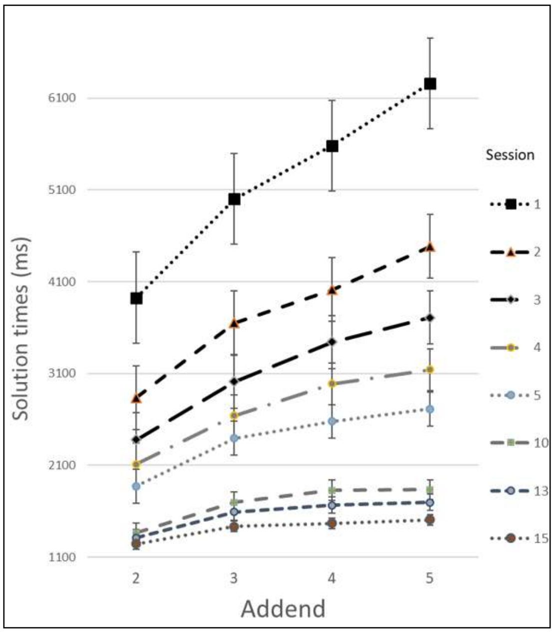

Figure 4. Mean solution times as a function of addend for Sessions 1, 2, 3, 4, 5, 10, 13 and 15. Error bars represent standard errors

We calculated the slope of solution times as a function of addend and it remained significant from the beginning to the end of training (Session 1: M=760 ms/addend (SE=44, $p<.001$) - Session 5: M=271 ms/addend (SE=20, $p<.001$) Session 10: M=155 ms/addend (SE=14, $p<.001$) and Session 15: M=82 ms/addend (SE=10, $p<.001$). Nevertheless, the difference in slope was no longer significant between sessions 10 and 15 ($t(35.67)=1.268$, $p=0.1065$).

The results also showed that the effect of the addend on solution times persists over all sessions, indicating that participants are somehow still counting even after 3 weeks of learning. However, from session 5, there is no longer significant difference between the solution times of addends +4 and +5. This last result seems to indicate that +5 problems are not processed as the other problems during the course of the training as in Thevenot et al.'s (2020) study.

Learning an artificial mental line would therefore produce the same effects as mental lines such as the number line or the alphabetical line. The plateau reached at the end of the training coupled with a persistent effect of the addend tends to support an acceleration of the counting procedure for very small addends (+2 to +4). These experimental effects are the benchmarks of our model simulations.

Simulations

The goal of our simulation is firstly to estimate the two parameters RDD and RDF described in Equation 2 which control the speed of scrolling the mental line as a function of learning, then to check whether the effects produced by the model are similar to those detected in the experiment based on the artificial line and finally, to verify whether the mean solution times estimated by the model with those parameters are close to those observed at the behavioral level. The model simulates the verification task of 32 "true" equations presented to 19 virtual participants in random order during 15 sessions as described in the experiment. The model produces one solution time per addend and per session based on the mean solution times obtained from the 19 participants during the simulation.

Initializing model parameters

The model parameters were initialized with the following values:
- Encoding duration of augend, addend and result: **80 ms** per element (Widaman et al., 1989);
- Decay parameters and forgetting threshold (from the TBRS model): **0.5** (Equation 1) (Oberauer and Lewandowsky, 2011);
- Duration of the motor command performed at the end of the process (to press the key on the keyboard): **300 ms** (Rosenbaum, 1980);
- Duration of the comparison between the result obtained and the result presented: **200 ms** (Kang and Ratcliff, 2018);
- RDB parameter (Equation 2) which corresponds to the minimum duration of a step: **20 ms** (Groen and Parkman, 1972);
- Refreshing time elements in working memory: **80 ms** (Oberauer and Lewandowsky, 2011).

Parameter estimation

The data collected from the experiment based on the artificial line were used to estimate the parameters RDD and RDF described in the Equation 2. Session 1 was excluded because it includes a large extra delay, probably due to the fact that participants were discovering the experiment for the first time. Therefore, the simulation was performed from sessions 2 to 15 for addends +2 to +5. The estimation consists of searching the RDD and RDF parameter values that minimize the MSE (Mean Squared Error) between the mean data calculated by the model and the mean experimental data per addend and per session. The overall shape of the MSE as a function of RDD and RDF values is illustrated in Figure 5. The mean MSE obtained is 274 ms (for RDD=72 and RDF=740), which represents 12% of the mean solution times (2 271 ms, *SD*=1 754 ms).

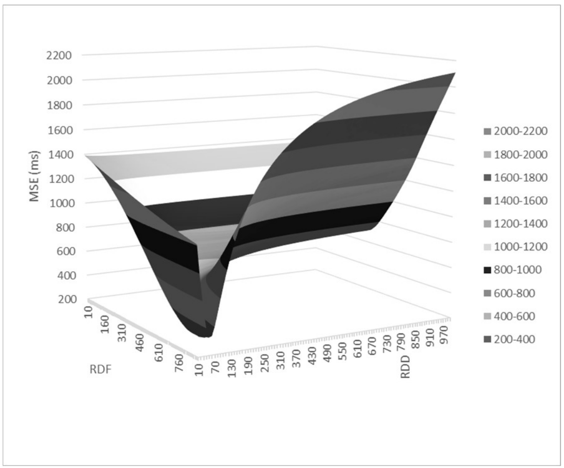

Figure 5. MSE (ms) as a function of RDD and RDF

Comparison of simulated and experimental data

Exactly like theoretical models in cognitive psychology are first assessed on their ability to predict behavioral effects, exploratory computational models such as ours need also to be first tested based on their ability to reproduce effects with simulated data. If simulations show the basic effects, then deeper investigation is conducted to check whether the data magnitude is correctly reproduced.

Figure 6 represents the mean simulated solution times with RDD=72 and RDF=740, by addend for sessions 2, 3, 4, 5, 7, 8, 10, 13 and 15. The model produces globally the same effects as the experiment on the artificial line for very small addends:

- An effect of the addend, which results in an increase of solution times as a function of the addend;
- An effect of the sessions, which results in a decrease in solution times as the training progresses;
- An interaction effect between sessions and addends, the effect of which decreases as the sessions progress.

Nevertheless, the model underestimates the solution times for addends +2 to +4 and overestimates solution times for addend +5. In fact, it does not consider the bend of the curve for the last addend detected by Thevenot et al. (2020). This drop could correspond to the beginning of memory retrieval which is not considered at the moment. Thus, the simulated data almost follow a linear trend for all addends until session 7 but from session 8, the model predicts a slight acceleration between addend +4 and addend +5. Nevertheless, for addends+2 to +4, the estimated slopes in sessions 10 and 15 are respectively 233 and 111 ms/addend for experimental data and 255 and 152 ms/addend for simulated data, which is in the same order of magnitude.

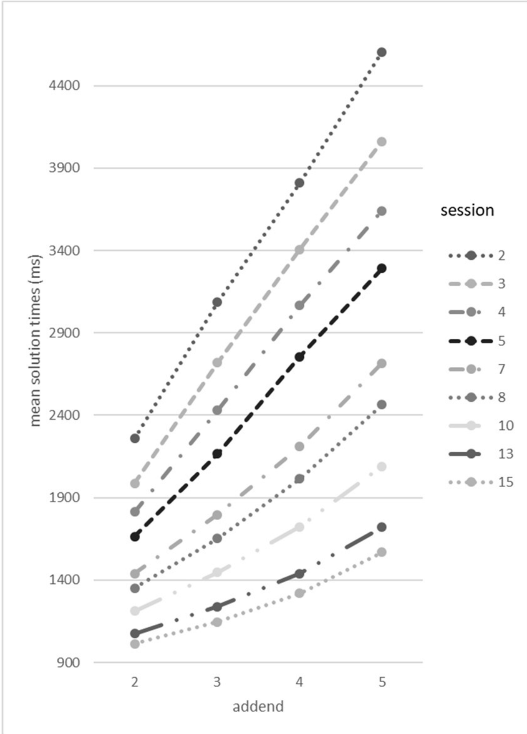

Figure 6. Mean solution times as a function of addend for RDD=72 RDF=740 - simulated data

Discussion

In the end, the model manages to reproduce human behavior on the artificial line for the smallest addends (from +2 to +4), namely a decrease in solution times as the sessions progress.

To explore different theoretical hypotheses, it is fruitful to carry out simulations in order to predict the behavior under other conditions. The AutoCoP model hypothesizes the automation of the counting procedure along the mental line, resulting in a residual slope of the solution times as a function of the addend. To make a prediction about what could happen after a very long training on our artificial material, similar to what occurred with "fossilized" mental lines (i.e. numerical mental line), we performed a simulation on a mental line for addends +2 to +5 over 100 sessions. The residual slope of 48 ms/addend obtained in session 100 is of the same order of magnitude as that measured by Uittenhove et al. (2016) (45 ms/addend).

At the end of the training, simulations always show a significant slope, in accordance with an ultra-fast counting procedure as proposed by AutoCoP model. However, the notion of competition between the counting algorithm and direct memory retrieval which could explain the inflection of solution times for the largest addend is part of our future work.

References

Anderson, J. R., Bothell, D., Byrne, M. D., Douglass, S., Lebiere, C., & Qin, Y. (2004). An integrated theory of the mind. *Psychological Review 111(4),* 1036-1060.

Barrouillet, P., Portrat, S., Camos, V. (2011). On the law relating processing to storage in working memory. *Psychological Review, 118(2)*, 175-192.

Barrouillet, P., & Thevenot, C. (2013). On the problem size effect in small addition: Can we really discard any counting-based account? *Cognition, 128*, 35-44.

Fayol, M., & Thevenot, C. (2012). The use of procedural knowledge in simple addition and subtraction problems. *Cognition, 123*, 392-403.

Groen, G. J., & Parkman, J. M. (1972). A chronometric analysis of simple addition*. Psychological Review, 79,* 329-343.

Kang, I. & Ratcliff, R. (2020). Modeling the interaction of numerosity and perceptual variables with the diffusion model. *Cognitive Psychology, 120,* 101288.

Logan, G.D. (1988). Toward an instance theory of automatization. *Psychological Review*, *95,* 492-527.

Logan, G. D. (2018). Automatic control: How experts act without thinking. *Psychological Review, 125(4)*, 453-485.

Logan, G. D., Cowan, W. B. (1984). On the ability to inhibit thought and action: A theory of an act of control. *Psychological Review*, *91,* 295-327.

Mathieu, R., Gourjon, A., Couderc, A., Thevenot, C., & Prado, J. (2016). Running the number line: Rapid shifts of attention in single-digit arithmetic. *Cognition, 146,* 229-239.

Oberauer, K., & Lewandowsky, S. (2011). Modeling working memory: A computational implementation of the time-based resource-sharing theory. *Psychonomic Bulletin & Review, 18,* 10–45.

Portrat, S., Lemaire, B. (2015). Is attentional refreshing in working memory sequential? A computational modeling approach. *Cognitive Computation*, *7(3),* 333-345.

Rosenbaum, D. (1980). Human Movement Initiation: Specification of Arm, Direction, and Extent. *Journal of Experimental Psychology: General, 109 (4),* 444-474.

Thevenot, C., Dewi, J. , Bagnoud, J., Uittenhove, K., & Castel, C. (2020). Scrutinizing patterns of solution times in alphabet-arithmetic tasks favors counting over retrieval models. *Cognition, 200*, 104272.

Uittenhove, K., Thevenot, C., & Barrouillet, P. (2016). Fast automated counting procedures in addition problem solving: When are they used and why are they mistaken for retrieval? *Cognition, 146,* 289-303.

Widaman, K., Geary, D., Cormier, 0., & Little, T. (1989). A componential model of mental addition. *Journal of Experimental Psychology: Learning, Memory, & Cognition, 15,* 898-919.