Thought Experiments in Scientific Discovery:
What Emergent Mental Capabilities Underlie their Efficacy?

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Abstract

Examples from Archimedes, Galileo, Newton, Einstein and others suggest that fundamental laws of physics were—or, at least, could have been—discovered by experiments performed not in the physical world but only in the mind. Though problematic for a strict empiricist, this mode of discovery may have become possible, in part, through the evolutionary emergence in humans of deeply internalized implicit knowledge of abstract principles of transformation and symmetry.

Keywords: Thought Experiments; Physical Laws; Imagined Transformations; Mental Rotation; Symmetry; Rationality; Moral Laws; Determinism; Free Will

The Problem

Thought experiments are widely reported to have played a prominent role in the discoveries of physical laws. I shall describe some specific thought experiments, similar to those that were (or may have been) carried out by Archimedes, Galileo, Newton, and Einstein and that appear to be sufficient to establish fundamental laws of physics, without having to carry out any of these experiments physically. How is this possible? Where does such knowledge originate if not, as supposed by strict empiricists, from an individual's direct interactions with the physical world?

The answer I propose grew out of my evolutionary perspective together with my cognitive psychological researches on mental transformations (Shepard & Cooper, 1982, Shepard & Metzler, 1971) and on generalization (Shepard, 1987; and the related far-reaching developments subsequently achieved by Tenenbaum and his coworkers—e.g., Tenenbaum & Griffiths, 2001a, 2001b). Here, however, I focus primarily on the role of mental transformations and an associated symmetry principle of invariance under transformation. (Some of the ideas I shall present here I have previously sketched, e.g., in Shepard, 1994, 2001, 2003, and in my 1994 William James Lectures at Harvard.)

For readers engaged in the computational modeling of cognitive processes, I invite consideration of how the kinds of transformational processes and symmetry principles I consider here might be implemented in the modeling of processes of scientific discovery.

Empirical Science and Mathematics

Traditionally, a sharp distinction is maintained between the empirical sciences including physics, on one hand, and mathematics and logic, on the other. Observations, measurements, and experiments on physical objects and phenomena are generally considered to be essential for the advancement of the empirical sciences. But mathematics is supposed to be concerned with what propositions are logically entailed by other propositions, without regard for whether any of these propositions are strictly true of anything in the physical world.

Also, in physics, what are taken to be the elementary or primitive objects at any given stage of development of the field are always provisional and subject to later re-conceptualization—often in terms of still more elementary entities. Thus, water, experienced as a continuous fluid, is reconceived as composed of increasingly more elementary entities such as: discrete molecules, electrons and protons, quarks, vibrations in a convoluted high-dimensional manifold, and so on—with no clear end in sight.

But in mathematics, what are taken to be the elementary or primitive objects are dictated by the mathematician rather than by nature. Such objects as the integers of arithmetic or number theory or the points and lines of geometry are completely transparent from the outset. We do not suppose that an integer or a point is ever going to be found to be composed of some previously unsuspected more elementary parts. Instead, the advances yet to be made in number theory or in geometry are expected to concern what relations among numbers or among points and lines will be found to be entailed by whatever axioms we have formulated for number theory or geometry.

Nevertheless, mathematics and physics are alike in that, in both cases, we aspire to a consistent theoretical system of basic assumptions and derivable implications. But whereas in physics a system is valued primarily to the extent that the derivable implications succeed in explaining or predicting what we observe or measure in the physical world, in pure mathematics the system may be valued for the beauty and symmetry of the system and its results–independently of the extent to which they agree with anything we observe or measure in the physical world. Nevertheless, results of pure mathematics that were originally valued for their own intrinsic beauty were often later found to be useful or even crucial for the development of physical theory. Notable examples include the non-Euclidean geometries of Minkowski and of Riemann for special and general relativity, respectively, and matrix algebra, complex numbers, quaternions, and infinite-dimensional Hilbert spaces for quantum theory.

Moreover, physics and mathematics are, alike, creations of the human mind. Absent a comprehending mind, the equations of either mathematics or of theoretical physics would not exist and, even if they did, would exist only as

uninterpreted squiggly patterns in the physical world. How do we humans come to formulate and to comprehend these mathematical equations and, in the case of physics, these laws of nature? When we try to do this for ourselves, we may find that the cognitive processes in the cases of physics and mathematics are often more similar than the traditional distinction between mathematical and the empirical sciences would suggest. To illustrate, I present, first, an example from mathematics (specifically, geometry) and, then, several simple examples from physics (primarily, mechanics).

Pythagoras's Theorem for Right Triangles

The theorem attributed to Pythagoras that relates the length of the hypotenuse of a right triangle to the lengths of its two other sides, is surely a theorem of mathematics. It may not hold exactly in the physical world. Even before Einstein, it clearly did not hold for large triangles on the spherical surface of the earth. Then, according to general relativity it does not exactly hold for triangles in three-dimensional physical space, more or less warped as it is in the vicinity of any massive bodies. Still, according to pure geometry, it does hold precisely for triangles in a flat plane. How can we convince ourselves of this latter, mathematical fact?

There are various intuitive demonstrations of the validity of the Pythagorean theorem. Figure 1 illustrates one such demonstration. (I don't know who originally devised this demonstration. I first learned of it from Douglas Hofstadter, personal communication.) The theorem states that for any right triangle with shorter sides of lengths A and B and hypotenuse of length C, $A^2 + B^2 = C^2$. As indicated in the figure, this is equivalent to the statement that the sum of the areas of the squares constructed on the two shorter sides of the triangle equals the area of the larger square constructed on the hypotenuse.

The truth of the theorem can be confirmed by imagining a few simple transformations. First, imagine a straight line constructed orthogonal to the hypotenuse and passing through the opposite vertex (as indicated in the figure by the dotted segment that thus divides the original triangle into the two areas labeled "a" and "b"). Second, imagine each constructed square (of area A^2, B^2, or C^2) and its adjacent triangle (of area a, b, or a+b) rotated (and also flipped in the case of the largest square-plus-triangle) as a rigid unit to yield the three house-shaped objects lined up across the bottom of the figure. Third, from a consideration of the complementary angles of the original triangle, confirm that the three triangles at the tops of the three squares (lined up below), though differing in size, are necessarily identical in shape. Finally, from this identity of shape and the obvious fact that a + b = c, reach the conclusion for the proportional areas of the squares that $A^2 + B^2 = C^2$. Q E D.

I suggest that the operations imagined here are of essentially the kind that my students and I investigated in our studies of "mental rotation" (Shepard & Cooper, 1982; Shepard & Metzler, 1971). Also crucial, here, is the realization that shape and size are invariant under the rigid transformations of translation, rotation, and horizontal flipping (which is a rotation in depth). Such invariance

under transformation corresponds to a symmetry principle, which will figure more prominently in ensuing examples.
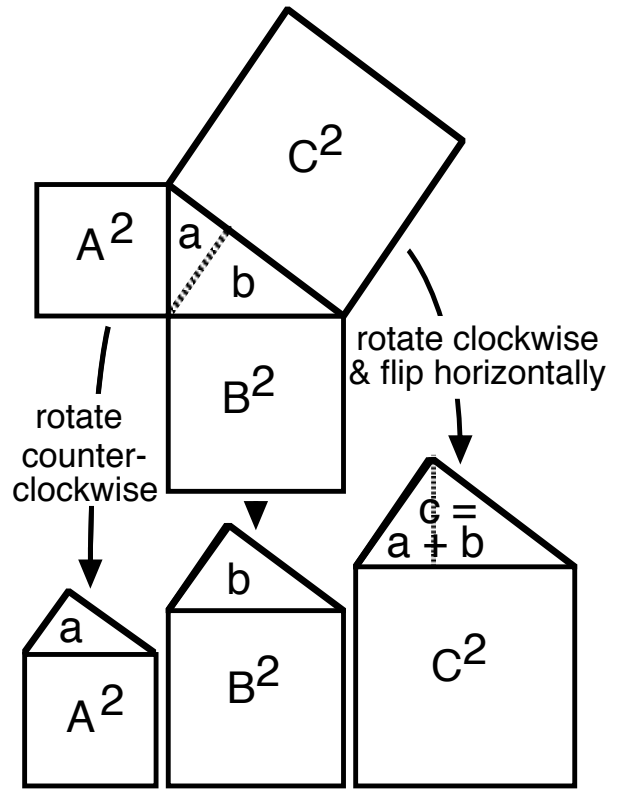

Figure 1: Proof of the Pythagorean Theorem

Archimedes's Law of the Lever

Unlike the theorem of Pythagoras, which belongs to pure mathematics, Archimedes law of the lever appears to be a law of physics. It can be stated as follows: Physical objects placed along a beam resting on a central fulcrum will balance if and only if the sum of the products of the weights and their distances from the fulcrum is equal for the objects on the left and for the objects on the right of the fulcrum. Archimedes may have verified this law by placing actual physical objects on actual physical balance beams. But he need not have done so. He may very well have seen that this law must hold by thought experiments and a simple principle of symmetry, as I illustrate in A-E of Figure 2.

If he had imagined the situation shown in A, Archimedes would immediately have seen by symmetry that the four weights must balance. Because the weights are identical by hypothesis, any permutation of them must leave the situation unchanged, with no reason for one side or the other to tilt down. Now imagine that without altering the weight of any part of the beam itself, it is modified to have a secondary fulcrum as shown in B. If the four identical weights are now placed as illustrated in C, everything will still balance as it did in A. Moreover, this remains true for any placement of the two weights on the secondary Beam 2 that is symmetrical around that secondary fulcrum. For any such symmetrical placement, the combined weight of those

two objects is communicated to the primary beam, as before, at the location of that secondary fulcrum. Each of the cases exhibited (C, D, & E) satisfies Archimedes's condition. As shown by the equation over each case, the negative sum of the products on the left of the fulcrum cancels the positive sum of the products on the right.
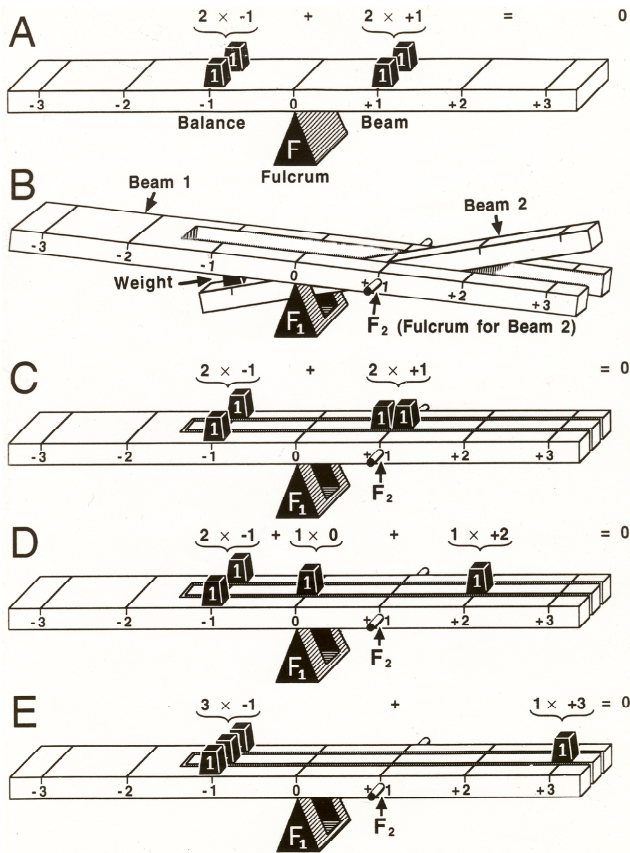

Figure 2: Demonstration of Archimedes's Law of the Lever

More generally, beginning with any distribution of any number of weights along a beam, we can use the same symmetry principle to confirm the following: First, we note that whether Archimedes's condition holds or not (i.e., whether the beam balances or not) the fact of its balance or imbalance is preserved under any transformation of moving any two weights together at the midpoint between them. Second, iteration of such transformations converges toward the situation in which all the weights are located at Archimedes's centroid of the original distribution. (At any stage of this process, the centroid will be at the fulcrum if and only if Archimedes's condition holds at that stage.) Thus are we able to verify Archimedes law, without ever placing any actual weights on any actual beam. So, what was taken to be an empirical fact about the physical world turns out to be entailed by an abstract, mathematical principle of invariance under transformation–or, equivalently, of symmetry, as we can verify by thought alone, much as we did in the case of Pythagoras's mathematical law concerning right triangles.

After choosing the mathematical example of Pythagoras and the physical example of Archimedes, I noticed another connection between them: Imagine that you wish to verify that thee long rods of the same diameter and material would form a right triangle. But (a) the long narrow room in which you must do this is not wide enough to accommodate the triangle itself and (b) no ruler or measuring tape is available. I leave it as an exercise for the reader to prove that if a long (unmarked) beam and fulcrum are available, a triangle formed by the three bars would form a right triangle if and only if a balance is achieved when the two shorter bars are laid side-by-side on one side extending from the fulcrum and the longest bar is laid on the other side extending from the fulcrum (as illustrated in Figure 3).
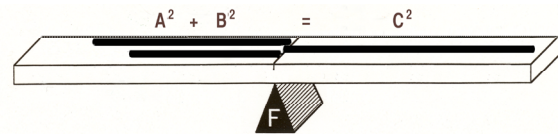

Figure 3: Beam balance equivalence of right triangle

Galileo's Law of Falling Bodies

Perhaps the most extensively analyzed thought experiment of all time is the one Galileo used to refute Aristotle's claim that falling bodies drop with speeds proportional to their weights (see Gendler, 1998). Presumably, Aristotle did not reach his erroneous conclusion by performing an actual experiment. If he reached it from a thought experiment, it evidently was one to which he did not devote enough thought. He may have imagined hefting a light object in one hand and a heavy one in the other and leaped to the hasty conclusion that if the objects were released, the greater downward force of the heavier object on his hand would manifest itself as a faster descent. But such a conclusion ignores inertia–something that Artistotle might have realized before Galileo if he had also thought about the greater effort needed to accelerate a more massive object to the same speed as a lighter object.

Galileo's thought experiment was conclusive. It yielded the correct conclusion: The speed of descent is independent of the weight of the object (to the extent that air resistance is negligible). This implies that the greater downward force that Aristotle imagined he would feel from the heavier object is in fact exactly the force needed to accelerate the more massive object to the very same speed. I shall describe a thought experiment that is slightly different from the one actually offered by Galileo but that is, I believe, equally conclusive and more revealing of the relevance of the principle of symmetry.

I imagine Galileo imagining himself at the top of the leaning tower at Pisa with three identical bricks. By the symmetry principle of invariance under permutation of identical objects, if the three bricks were dropped together, they should reach the ground at the same time. As in the case of Archimedes's identical weights equally distant on each side of the fulcrum, there is no reason for one to go down more than the other. Now suppose two of these three

bricks are glued together to form a single, twice-as-heavy brick. Surely the virtually weightless film of glue used to make the two bricks into one would not cause that now heavier brick to fall twice as rapidly as the separate, third brick. Once again, the correct conclusion is reached without having to perform any actual experiment.

Newton's Law of Action and Reaction

Newton's inspiration for his universal law of gravitation is sometimes attributed to a thought experiment in which he imagined throwing an apple (according to legend, one that dropped on him as he sat under a tree). Newton would have realized that if he threw the apple with greater and greater force, the apple would fall to ground at a greater and greater distance from him. Knowing that the earth is a sphere, he would have concluded that if he were able to throw the apple with sufficient force, it would fall not to earth but around the earth. He might then conjecture that the moon was hurling with such speed as to be similarly ever falling around the earth.

I now consider, instead, a different thought experiment from which Newton might have arrived at his Third Law of Motion—the law that every action has an equal and opposite reaction. I choose the thought experiment for this law for two reasons: First, of his three laws of motion, this is the one that was most original with Newton. Second, this thought experiment derives in a particularly simple and transparent way from the principle of symmetry.

I imagine Newton imagining himself arched over the water with his feet on the gunwale of his boat and his hands on the gunwale of another boat of the same size that he is endeavoring to push away from his boat. From the obvious symmetry of the situation, Newton would realize that there is no way that he can push the other boat away from his without equally pushing his boat away from the other. He might also go on to realize that if he were stranded in the middle of a lake with an oar-less boat load of apples, he could propel himself back toward shore by hurling the apples, one by one, to the rear with great force (just as space vehicles now accelerate through empty space by ejecting molecules of gas at very high velocity).

Einstein's Theory of Relativity

It is widely recognized both that Einstein was, like Galileo, a master of the thought experiment and that Einstein was attracted to symmetries. Examples of such symmetries are those between observers in relative motion, between electric and magnetic fields, and between gravitational force and acceleration. He first developed special relativity through many thought experiments about how the same events would be differently perceived by observers uniformly moving relative to each other at appreciable fractions of the velocity of light. Continuing my theme of the physics of falling bodies, however, I confine myself, here, to the famous thought experiment that he used to support the claim of general relativity that light is deflected in passing near a massive body. (It was the empirical confirmation of this claim—obtained during Eddington's expedition to measure such a deflection during a solar eclipse—that first brought Einstein instant and lasting celebrity around the world.)

In this thought experiment Einstein imagined himself inside an elevator that is in free fall (perhaps as a result of its supporting cable having been severed). Floating weightlessly in the falling elevator, like an astronaut in a vehicle free-falling around the earth, Einstein would experience no gravitational force. Because no force is acting within the elevator, a beam of light passing through the elevator would surely traverse a straight line. In the absence of gravitational force, the symmetry between up and down removes any reason for the light to be curved either upward or downward.

Then Einstein imagines that he is, instead, a stationary observer outside the falling elevator. He would now feel the force of gravity holding him down on the fixed platform from which he watches the elevator accelerating downward in its free fall. But clearly, a light beam that traverses a straight line relative to the observer in the elevator must traverse a downward curving (in fact, parabolic) path to the stationary external observer. That observer will thus attribute both his own experience of a downward pressure of his feet on the platform and the downward bend of the light to the gravitational attraction of the massive earth.

One of the beauties of Einstein's special and general theories of relativity is that they provide elegant mathematical formulations of the symmetries of invariance under transformations between observers that are moving relative to each other–either uniformly as in the special theory or in acceleration as in the general theory. Yet Einstein developed these revolutionary theories neither by performing physical experiments himself nor by studying empirical data collected by others. He developed them by performing thought experiments. The very first of these–in which he imagined moving along with a light wave at light speed–led to the counter-intuitive result in special relativity that the velocity of light is invariant for all observers regardless of how fast each may be moving.

Empiricist Versus Rationalist Views of the Efficacy of Thought Experiments

I have had space, here, to consider just a few simple examples illustrating the striking effectiveness of thought experiments and symmetry principles in the discovery of physical laws. Elsewhere, I have discussed other examples —concerning the discovery of laws of electromagnetism and the propagation of light (by Maxwell); laws of thermo-dynamics, statistical mechanics, and entropy by (Carnot, Clausius, and Boltzmann); laws of quantum mechanics by (Palanck, Einstein, Bohr, Schrödinger, Heisenberg, Feynman, and others); and even (going beyond physics), principles of evolutionary biology (by Darwin). (See Shepard, 2001, 2003, for a few of the examples I presented in my 1994 William James Lectures at Harvard.)

Still, many scientists find it difficult to believe that laws governing the natural world are discoverable through thought alone. They prefer explanations that are more

compatible with their deeply entrenched empiricist presuppositions. Some may embrace the idea that most so-called thought experiments are merely mental re-imaginings of the scientist's own previous interactions with the world. Galileo might be supposed already to have noted (perhaps unconsciously) that the rate of fall of compact objects is in fact largely independent of their weights. Possibly the thought experiment he described was mainly an attempt to render explicit and vivid, for didactic purposes, what he had previously learned from his own experiences. Others may suggest that to the extent that a thought experiment is effective, it must really be just a disguised version of a deductive argument based on already accepted premises.

Such efforts to explain away the effectiveness of thought experiments strike me as incomplete or unconvincing, and perhaps a little desperate. Moreover, the claim by some philosophers of science that thought experiments must be reducible to deductive arguments has been effectively countered by Gendler (1998). I suggest that the dismissive stances taken toward thought experiments have been motivated by an out-dated "blank-slate" conception of the human mind. They largely ignore the empirical and logical evidence that has accumulated from many quarters, including evolutionary, developmental, linguistic, cognitive and brain sciences, as well as from machine learning theory. Clearly, individuals do not come into the world devoid of any advance preparation for the kind of world in which they must learn to make their ways (Shepard, 2001, p. 712).

Perhaps the resistance to such evidence arises from a mistaken notion that proponents of thought experiments assume that the knowledge with which we are innately endowed is explicit knowledge of quite specific facts and laws of nature. My claim is, rather, that the "knowledge" with which we have been endowed by natural selection is only implicit and, moreover, that it primarily pertains to the most abstract, enduring, and invariant features of our world. Some of these features may be physical in nature—such as that physical space is three-dimensional and locally Euclidean, hence accommodating just six degrees of freedom of rigid motion. Others features may be even more abstract, more logical or mathematical in character—such as the principle of symmetry (Shepard, 1994, 2001).

A noteworthy fact about the currently most successful theories of the physical world, namely, general relativity and quantum mechanics, also seems to be insufficiently appreciated by strict empiricists. Each of these theories possesses such elegance and tight internal constraints of consistency and symmetry that it has proved extremely difficult to find any minor changes that can be made to either theory without disrupting its whole structure. In retrospect, it almost seems that if we had only been smart enough, we could have seen that each theory could not be otherwise. Yet, despite their enormous successes, these two theories are known to be inconsistent with each other. Hence, the discovery of a still more general theory that consistently subsumes both of them as limiting cases would be a towering achievement comparable to the discovery of each of the two component theories themselves—without having collected a single additional empirical datum.

When we discover such a general theory, we may wonder why its laws take the particular form that they do. Only three possible answers occur to me. The first is that the form of these laws is simply an arbitrary brute fact; there is no reason why they are this way rather than some other. The second is that these laws form the only mathematically possible self-consistent set. And the third is that the universe in which we reside is only one of infinitely many, and that the entire ensemble includes universes with laws of every mathematically possible form. The first of these alternatives seems to be implicitly presupposed by strict empiricists and the second by pure rationalists. The third possibility falls somewhere between. For, in accordance with the anthropic cosmological principle, the laws governing our universe are then constrained—perhaps tightly—by the requirement that they make possible the evolution of intelligent life.

The Evolution of the Rational Capabilities Necessary for Successful Thought Experiments

It falls to us, as cognitive scientists, to elucidate the mental capabilities that make possible the discovery of the laws of nature through thought and, as evolutionary cognitive scientists, to suggest how these capabilities may have arisen through natural selection. At the outset I noted that the ability to imagine identity-preserving transformations, such as mental rotations, may have an important role to play. I have proposed that neuronal mechanisms that originally evolved in the service of the perceptual representation of the external world may have evolved additional capabilities. These may have included, successively, capabilities (a) for perceptual completion when the sensory input is brief, degraded, or incomplete, (b) for anticipation of probable ensuing or accompanying events and, finally, (c) for the autonomous imagining of possible transformations in the sensory absence of such transformations or their objects. Possession of this final capability would presumably confer significant benefit in planning and problem solving.

The emergence of the ability to construct explicitly articulated consistent theories about this world, however, presumably required some additional developments. I suggest that the following three may have been critical for the successful employment of thought experiments and abstract principles such as that of symmetry for this purpose: (a) a motivation, arising from our evolutionary branching into what has been termed the "cognitive niche," to understand and comprehend the world; (b) the concomitant emergence of a capacity for holding immediate self interest in abeyance and for contemplating an explicit set of alternatives objectively with respect to explicitly chosen abstract criteria; and, of course, (c) the emergence of language, thus enabling the sharing, preservation, and further analysis of results. The first two of these capabilities may not be uniformly well developed or exercised in the human population. Thinkers such as Archimedes, Galileo, Newton, and Einstein are still notably rare.

Biological evolution is, of course, a gradual process, taking countless generations to manifest appreciable change, as the continuities of bodily structure and behavior along

evolutionary lines attests. Yet, the unprecedented and uniquely human "step to rationality," as I have termed it (Shepard, 2001), has occurred virtually discontinuously on the evolutionary time scale. The uncertain but certainly enormous potential consequences of this development for the future of our species and our world motivates me to end this article with a brief sketch of how the ideas I have been proposing about the cognitive grounds of science might be extended to illuminate the cognitive grounds of ethics.

The Cognitive Grounds of Moral Principles and of Deterministic Free Will

Philosophers have long made extensive use of thought experiments in attempting to evaluate alternative proposed principles of moral action. One example of direct and obvious relevance here is the type of thought experiment on which Rawls (1971) based his highly influential theory of justice. Rawls asked us to imagine a hypothetical situation in which those endeavoring to frame rules for the just governance of society are prevented by a "veil of ignorance" from knowing what role each will be assigned in that society. Such framers would thus be prevented from favoring rules that would benefit the particular personal situations to which they would otherwise be heir through no virtue or effort of their own—for example, situations varying in health, strength, wealth, power, beauty, youth, race, or gender. In terms of the more mathematical concepts I have been using, what is being sought here is just the symmetry of invariance under permutation of individuals. Clearly, such a symmetry principle entails the "Golden Rule," which may well be the candidate moral principle that comes closest to eliciting universal verbal assent (if not behavioral compliance).

I do recognize and understand the "is"-"ought" distinction and the pervasive skepticism of most scientists and many philosophers about the possibility of a universal, objective basis for ethics. Yet I confess to finding repugnant the alternative of a complete biological/cultural relativism in which we, as scientists, may well investigate how what an individual in fact does or says is determined by the accidents of that individual's genetic inheritance and cultural milieu, but in which we have no justification for a conviction that some intentions, actions, or outcomes are ultimately and inherently better than others.

What I now regard as the best and perhaps only hope for avoiding nihilistic ethical relativism is to strive for a sufficiently deep appreciation and grasp of what may at first seem an extremely abstract and airy principle—namely, the symmetry principle of invariance under permutation (Shepard, 2001, pp. 744-748). Then, and perhaps only then, one who is neither a solipsist nor a fatalist and who values his or her own life, freedom, and well being, may come to know—as well as anyone knows anything—the binding force of the Golden Rule.

Of course, an individual who is not in an appropriate sense free to make his or her own moral choices cannot be held responsible for those choices. This consideration has led me to seek a reconciliation between the need I thus fervently feel for and account of such a "freedom of the will," on the one hand, and my unshakeable rational conclusion that every event is either deterministically caused or else occurs purely by chance.

The reconciliation that I propose depends on the unprecedented capability that has emerged with our "step to rationality." This is the capacity that I have already noted for holding immediate self interest in abeyance and for contemplating an explicit set of alternatives objectively with respect to explicitly chosen abstract criteria (Shepard, 2001, pp. 744-748). If the alternatives are weighed in terms of the products of their computed subjective values and their computed subjective probabilities, as in utility theory, the process can be imagined as a literal "weighing" on an Archimedian balance beam, with locations along the beam representing the computed values and the masses of the weights representing the computed probabilities.

My proposal is compatible with the possibility that one's decisions are strictly determined by preceding events, as might be specified in a computational model. But some of those preceding causes are also what we call reasons. What is required for one's decision to be free, is that one can cognitively represent the entire set of selected alternatives, can compute (or re-compute) the value of any one, and can adopt the one with the highest computed value, without any extraneous constraint or compulsion. A free decision is thus made for one's own reasons. One could have—and would have—decided differently if one's careful weighing of the alternatives had tipped the balance in the other direction.

References

Gendler, T. S. (1998). Galileo and the indispensability of scientific thought experiment. *The British Journal for the Philosophy of Science, 49*, 397-424.

Rawls, J. (1971). *A theory of justice*. Cambridge, MA: Harvard University Press.

Shepard, R. N. (1987). Toward a universal law of generalization for psychological science. *Science, 237*, 1317-1323.

Shepard, R. N. (1994). Perceptual-cognitive universals as reflections of the world. *Psychonomic Bulletin and Review, 1*, 2-28.

Shepard, R. N. (2001). On the possibility of universal mental laws: A reply to my critics. *Behavioral and Brain Sciences, 24*, 712-748.

Shepard, R. N. (2003). How a cognitive psychologist came to seek universal laws. *Psychonomic Bulletin and Review, 11*, 1-23.

Shepard, R. N., & Cooper, L. A. (1982). *Mental images and their transformations*. Cambridge, MA: MIT Press.

Shepard, R. N., & Metzler, J. A. (1971). *Mental rotation of three-dimensional objects. Science, 171*, 701-703.

Tenenbaum, J. B., & Griffiths, T. L. (2001a). Generalization, similarity, and Bayesian inference. *Behavioral and Brain Sciences, 24*, 629-640.

Tenenbaum, J. B., & Griffiths, T. L. (2001b). Some specifics about generalization. *Behavioral and Brain Sciences, 24*, 762-778.