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The (Neuro)cognitive Roots of Fraction Knowledge

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Abstract

The current study investigated whether cognitive architectures tuned to the magnitudes of nonsymbolic ratios support the acquisition of symbolic fraction concepts and subsequent achievement in algebra. Participants' performance on a novel battery of nonsymbolic ratio comparison tasks predicted their symbolic fraction knowledge and algebra achievement even after controlling for performance on control tasks. These results provide initial behavioral evidence that recently discovered brain systems which represent nonsymbolic rational magnitude may be important for math learning, especially for developing a strong understanding of fraction concepts.

Keywords: Mathematical Cognition, Approximate Number System, Fractions, Math Achievement.

Introduction

Recent research has highlighted the critical importance of fraction concepts for promoting math achievement (Siegler et al., 2012). Fraction knowledge serves as a “gateway” to algebra and higher mathematics (Booth & Newton, 2012), and cultivating this knowledge has rightfully become a major focus of math education reform initiatives (National Mathematics Advisory Panel, 2008). However, cultivating fraction knowledge is far from easy; students often struggle to understand fractions, and several theorists have even proposed that fraction concepts are fundamentally incompatible with the core neurocognitive architectures that support mathematical cognition (Dehaene, 1997; Feigenson, Dehaene, & Spelke, 2004; Geary, 2007). In this paper, we investigate a competing hypothesis that may help to illuminate how individual differences in ratio-specific neurocognitive architectures influence fraction learning and eventual achievement in higher mathematics.

According to this hypothesis, successful fraction learning builds upon a set of potentially primitive neurocognitive architectures – the Rational Brain System (RBS) – which are tuned to the magnitudes of nonsymbolic ratios (e.g. the relative length of two lines or the relative numerosities of two dot arrays) (See Figure 1). In the current study, we investigate a critical behavioral implication of this proposal – that individual differences in RBS acuity, as measured by

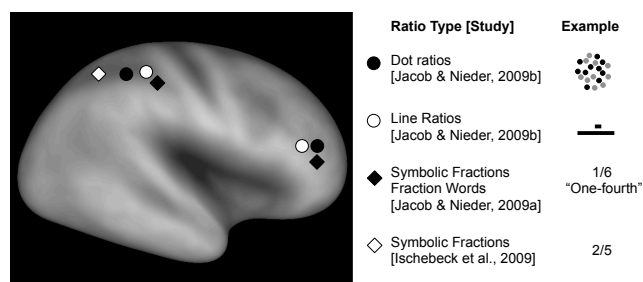


Figure 1: Approximate locations of brain regions showing distance-dependent activation to the holistic magnitudes of nonsymbolic ratios and symbolic fractions.

the ability to compare nonsymbolic ratios, predict individual differences in the learning of fraction concepts and subsequent algebra achievement.

Core Architectures for Math Cognition

In recent decades, advances in the cognitive and neural sciences have helped researchers to chart the neurocognitive foundations of mathematical cognition. This research has begun to identify candidate “neurocognitive startup tools” upon which number knowledge is founded. These startup tools include the exact small number system that enables subitizing (the Object Tacking System, or OTS) and the Approximate Number System (ANS) that enables humans and many nonhuman animals to represent the approximate number of items in a set (Piazza, 2010). These evolutionarily primitive architectures are recycled by formal learning and promote the acquisition of key early math concepts that parallel their nonsymbolic representational affordances (Dehaene & Cohen, 2007). For example, the OTS-based ability to represent exact small numerosities and the ANS-based ability to represent approximate numerosity may promote the learning of whole number concepts and whole number symbols.

The role of the ANS in supporting early math learning has been implicated by positive correlations between individual differences in ANS acuity, measured by the ability to compare the numerosities of two sets, and individual differences in measures of symbolic whole number

knowledge and arithmetic achievement (Halberda, Mazocco, & Feigenson, 2008). However, these architectures become less useful in promoting math learning as the structure of mathematical constructs diverges from their representational affordances.

A major divergence from the affordances of these core architectures comes relatively early with the introduction of fraction concepts in grades 3 and 4. For the first time, the magnitude of the symbols students must learn no longer maps easily onto their intuitive OTS or ANS representations. Whereas the magnitude of the symbol “7” may have mapped onto an ANS representation built by evolution and/or experience to represent approximately seven discrete items, the magnitude of the symbol “1/2” emerges from the relative magnitude of the two symbolic components and can in fact refer to an infinite number of magnitude pairs (1/2, 2/4, 4/8...).

Students struggle mightily to understand the holistic magnitudes of fractions. For example, when a nationally representative sample of 17 year olds was asked whether $12/13 + 7/8$ was closest to 1, 2, 19, or 21, more chose either 19 or 21 than 2 (Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1980). Misconceptions like these illustrate a clear failure to recognize that the magnitude of a fraction emerges from the relation between its two components rather than from the magnitudes of the individual components themselves.

The pervasive problems that students experience have led some to claim that fractions are fundamentally incompatible with human cognitive architecture (Dehaene, 1997; Feigenson et al., 2004; Geary, 2007). However, a growing body of evidence suggests that humans may actually possess neural circuitry, the Rational Brain System (RBS) which is ideally suited to representing fraction concepts.

The Rational Brain System

Although fractions are not usually formally introduced until grades 3 and 4, children far younger than this demonstrate some understanding of nonsymbolic ratios. For example, McCrink & Wynn (McCrink & Wynn, 2007) demonstrated that 6-month old infants habituated to specific nonsymbolic ratios (e.g., a 2:1 ratio composed of yellow Pacmen intermixed with blue pellets) and subsequently looked longer at novel ratio stimuli that differed by a factor of two (e.g., a 4:1 ratio). By 4 years of age, children can order pictures of part-whole ratios based on holistic magnitude (Goswami, 1995) and perform above chance on tasks that require the addition and subtraction of nonsymbolic part-whole fractions (Mix, Levine, & Huttenlocher, 1999) even though much older children struggle to perform analogous tasks with symbolic fractions. This competence with nonsymbolic ratios seems to be flexible and abstract, allowing children to draw proportional analogies even when materials are visually dissimilar (e.g., drawing analogies between pizzas and chocolates and glasses of lemonade). Even monkeys can be trained to identify matching

nonsymbolic ratios in delayed match-to-sample tasks, and this training transfers to novel untrained ratios.

Functional imaging with humans (Jacob & Nieder, 2009b) and single-cell recordings from monkeys (Vallentin & Nieder, 2008) have begun to reveal that key components of the RBS span a fronto-parietal network (see Figure 1). Within this network, subpopulations of neurons show classical tuning curves to specific ratios. For example, some populations of neurons respond most strongly to all pairs of lines in which one of the lines is $\frac{1}{4}$ as long as the other, but respond more weakly to other relative lengths. Importantly, RBS tuning is not sensitive to the absolute sizes of components used to instantiate a ratio but tracks the holistic magnitude corresponding to the relationship between two given components.

The Rational Brain and Fraction Learning

Lewis, Matthews, & Hubbard (under review) have recently articulated a theory explicating how these “rational brain” networks influence fraction learning and, in turn, subsequent algebra achievement. This model suggests that both formal and informal learning experiences help to generate links between perceptually based representations of nonsymbolic ratio magnitudes and fraction symbols – both verbal fraction labels like “one half” and “one third” and orthographic symbols like “2/3” and “3/4”. Notably, these links are formed regardless of whether instruction explicitly leverages perceptual sensitivity to nonsymbolic ratios. Visuospatial representations (e.g. pies, numberlines, and counters) are used extensively to support learners’ acquisition of fraction knowledge. Even when these representations are used to complement instruction that emphasizes counting and the application of whole number schemas, RBS networks – which antedate formal educational experiences – respond to the nonsymbolic rational magnitudes encoded by these representations. Thus, there is at least a modicum of percept-to-symbol linking that occurs even when instruction does not explicitly encourage it. These links become stronger with more years of experience using graphical representations of fractions.

A key prediction of the RBS argument is that these symbolic-to-nonsymbolic links serve as an important basis for learners’ understanding of the holistic magnitude of fractions, a key understanding that promotes the acquisition of procedural and conceptual fraction knowledge. In turn, improved understanding of fractions supports subsequent algebra achievement. Because the precision of these symbolic-to-nonsymbolic links is dependent on the acuity of a learner’s RBS, Lewis et al. predicted that individual differences in RBS acuity would predict individual differences in fraction understanding and algebra achievement.

The Current Study

Although there is compelling evidence demonstrating human sensitivity to nonsymbolic rational magnitudes, the hypothesized connections between RBS acuity, fraction

learning, and algebra achievement have yet to be investigated. The current study was designed to do just that. Specifically, we tested whether individual differences in nonsymbolic rational acuity would predict individual differences in symbolic fraction knowledge and algebra achievement.

Therefore, we constructed a battery of nonsymbolic comparison tasks that included a standard measure of ANS acuity, a measure of acuity for line length, and four novel measures of RBS acuity (see Figure 2). In addition to the battery of comparison tasks, participants completed three measures of symbolic fraction knowledge (symbolic fraction comparison, fraction number line estimation, and a paper and pencil fraction knowledge assessment), and an additional task to control for differences in cognitive control/conflict resolution (flanker task).

The key research question was whether RBS acuity would predict fraction knowledge and algebra achievement above and beyond ANS acuity and other control measures. If so, these results would be the first to demonstrate the importance of the RBS to math learning.

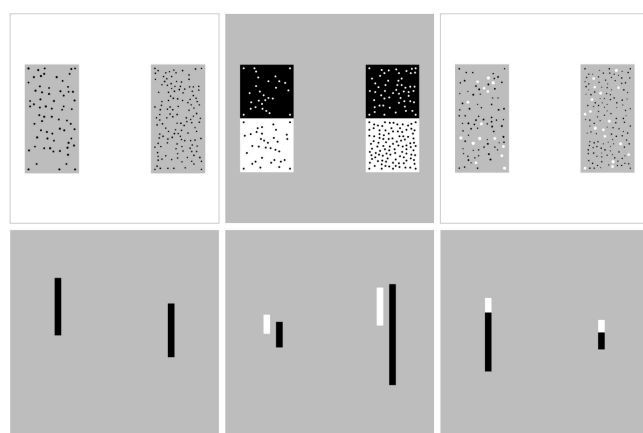


Figure 2: Sample stimuli from the nonsymbolic comparison battery. Top row from left to right: dot numerosities, separate dot ratios, and integrated dot ratios. Bottom row from left to right: line lengths, separate line ratios, and integrated line ratios.

Methods

Participants

101 undergraduate students at a large American university participated for partial course credit. Two participants did not understand the computerized tasks and were excused from that portion of the experiment.

Procedure

Participants first completed the computerized nonsymbolic comparison tasks. Next, they completed two computerized measures of symbolic fraction knowledge (fraction comparison and numberline estimation) followed by the flanker task. Finally, participants completed a paper-and-

pencil assessment to assess their fraction knowledge and granted us access to their university algebra entrance exam scores.

Nonsymbolic Comparison Tasks

Nonsymbolic comparison tasks assessed acuity for discriminating numerosity, line length, and nonsymbolic rational magnitude. Each trial began with a fixation cross for 1000 ms. This was followed by the presentation of two stimuli. Stimulus pairs were visible for 750 ms for the dot and line comparison tasks and 1500 ms for the nonsymbolic ratio comparison tasks. Participants were instructed to select the more numerous dot array, the longer line, or the larger nonsymbolic ratio by pressing the “j” key on a computer keyboard for the right stimulus and the “f” key for the left stimulus. There was no time limit imposed on trials, but responses were fast (e.g., an average reaction time of approximately 1200 ms for the separate line ratio task). Each comparison task began with 10 practice trials and contained 45 experimental trials (40 for the dot and line comparison tasks) of varying difficulty determined by the ratio of stimuli magnitudes in a trial. For example, a ratio of 2:1 might be realized by the comparison of 100 dots to 50 dots in the ANS task or by the comparison of a line ratio of $\frac{1}{2}$ to a line ratio of $\frac{1}{4}$ in the integrated line ratio comparison task. Ratios for each task were chosen based on pilot experiments and were intended to span from a ratio easy enough to elicit near perfect performance to a ratio difficult enough to elicit chance performance at the group level. All participants judged the same stimuli, but the order of these stimuli was randomized for each participant. In each task, the correct selection was presented approximately half of the time on the right. Performance was defined as the proportion correct on the experimental trials.

To reduce reliance on the lengths/numerosities of individual components in the ratio comparison tasks, in approximately half of the trials the components of the larger ratio were shorter/less numerous than the corresponding components of the smaller ratio, and in the remaining trials the components of the smaller ratio were shorter/less numerous than the corresponding components of the larger ratio.

To reduce reliance on cumulative area or dot size in the three tasks that employed dot stimuli, in approximately half of the trials, sets had the same cumulative area, but variable dot sizes, and in the other half, dot size was constant, but cumulative area differed.

Dot Numerosity (ANS Acuity) Participants were asked to “determine whether there are more dots on the left or right hand side of the screen” and were explicitly instructed to “base your decisions on the number of dots, not the sizes of the dots”. The range of item difficulties were chosen based on Halberda et al. (2008). The number of dots in each set ranged from 50 to 175.

Line Length Participants were asked to “decide which line is longer”. Lines were randomly offset from the midline by up to half of the line length.

Separate Line Ratios Participants were asked to “determine which set of lines has the higher ratio of white to black line lengths”.

Integrated Line Ratios Participants were asked to “decide which line has a higher proportion of white relative to total line length”.

Separate Dot Ratios Participants were asked to “determine which set of dots has a higher ratio of white to black dots” and were instructed to “choose the higher ratio based on the number of white to black dots, not size”. Component dot numerosities ranged from 11 to 119.

Integrated Dot Ratios Participants were asked to “determine which set of dots has a higher proportion of white dots relative to the total number of dots in that set” and were instructed to make their decisions “based on the number of white dots relative to the total number of dots, not the sizes of the dots”. Component dot numerosities ranged from 10 to 143.

Symbolic Fraction Comparison

Participants selected the larger of two symbolic fractions in each of 30 single digit fraction pairs. To reduce reliance on componential strategies, fraction pairs shared no common components (all numerators and denominators in a pair were unique). Pairs were sampled at random from all possible combinations of single digit irreducible fractions except $\frac{1}{2}$ and presentation side with the restriction that in half of the trials, the larger fraction was presented on the left. Each trial began with a fixation cross for 750 ms and then the fraction pair until the participant responded or 5000 ms had elapsed at which point the trial timed out and participants had to press the space bar to continue. Participants pressed the “f” key to indicate that the fraction on the left was larger and the “j” key to indicate that the fraction on the right was larger. Prior to the start of the experiment, participants made 5 practice judgments. All participants judged the same fraction pairs, but the order of the fraction pairs was randomized for each participant. Performance was defined as proportion correct on the experimental trials.

Numberline Estimation

The numberline estimation task was patterned after Siegler, Thompson and Schneider (2011). Participants used a computer mouse to indicate the position of the fractions ($\frac{1}{19}$, $\frac{1}{7}$, $\frac{1}{4}$, $\frac{3}{8}$, $\frac{1}{2}$, $\frac{4}{7}$, $\frac{2}{3}$, $\frac{7}{9}$, $\frac{5}{6}$, or $\frac{12}{13}$) on a numberline with end points 0 and 1. Performance was defined as each participants’ mean percentage absolute error (PAE), where $PAE = (|Answer - Correct Answer| / Numerical Range)$. A smaller PAE indicates a more accurate response.

Flanker Task

Each trial began with the presentation of a fixation cross for 500ms. This was followed by an array of five equally-sized and equally-spaced white arrows on a black background. This array remained on the screen for 800ms or until participants made a response. Participants were instructed to “choose which direction the center arrow is pointing” by pressing “f” for left and “j” for right. Participants first completed 12 practice trials and then 80 experimental trials. Forty of these trials were “congruent”; the four flanking arrows pointed in the same direction as the center arrow (e.g., “> > > >”). The other 40 trials were “incongruent”; the four flanking arrows pointed in the opposite direction (e.g., “< < < <”). In half of the trials, the center arrow pointed left; and in the other half, it pointed right. The order of trials was randomized for each participant. Performance was determined by the proportion of correct choices, and trials in which no response was provided in the allotted 800ms were scored as incorrect.

Fraction Knowledge Assessment (FKA)

The FKA is a 38-item test of procedural and conceptual fraction knowledge. We constructed this assessment using items from key national and international assessments including the NAEP and TIMSS and from instruments developed by psychology and math education researchers (e.g., Hallet et al., 2012). The FKA had strong reliability as measured by classical test theory (Cronbach’s $\alpha = .88$)

Algebra Entrance Exam

The algebra entrance exam is a 35 item subtest of the University of Wisconsin System Mathematics Placement Test taken by all freshmen in the University of Wisconsin system. The algebra subtest has strong reliability as measured by classical test theory (Cronbach’s $\alpha = .90$). Exam scores were not available for all participants.

Outlier Removal and Recoded Cases

For the nonsymbolic comparison tasks and the fraction comparison task, we removed data from trials that were more than 3 standard deviations faster or slower than a participant’s mean response time for that task. At the task level, we removed data from any participants who scored more than 3 standard deviation above or below the group mean for a given task. This resulted in the removal of data from 5 participants.

In a few cases, participants appeared to give reversed responses. For example, two participants in the flanker task had performance of less than 20 percent correct. These participants’ responses were recoded. All reported tests for nonrational acuity remain statistically significant if these cases are not recoded.

Table 1: Hierarchical Regression Analyses

		Fraction Comparison				Numberline Estimation - Log(PAE)			
		Slope	SE	p	Base R^2	Slope	SE	p	Base R^2
Base Model	Line Length	0.02	0.18	0.92	0.08	-0.80	0.28	0.28	0.04
	ANS Acuity	0.52	0.21	0.02 *		-0.58	0.52	0.52	
	Flanker	0.35	0.34	0.31	ΔR^2	-1.65	0.25	0.25	ΔR^2
Base + Ratio	Separate line ratios	0.37	0.15	0.01 *	0.06	-1.69	0.01	0.01 *	0.08
	Integrated line ratios	0.28	0.12	0.03 *	0.05	-0.40	0.44	0.44	0.01
	Separate dot ratios	0.33	0.10	0.00 ***	0.10	-1.03	0.02	0.02 *	0.06
	Integrated dot ratios	0.06	0.14	0.67	0.00	-0.43	0.46	0.46	0.01
		Fraction Knowledge Assessment				Algebra Entrance Exam			
		Slope	SE	p	Base R^2	Slope	SE	p	Base R^2
Base Model	Line Length	10.47	7.79	0.18	0.08	-1.938	169.564	0.991	0.04
	ANS Acuity	19.05	9.36	0.05		366.12	211.46	0.087	
	Flanker	13.75	15.06	0.36	ΔR^2	26.791	321.942	0.934	ΔR^2
Base + Ratio	Separate line ratios	18.57	6.36	0.00 ***	0.08	384.277	148.357	0.011 *	0.08
	Integrated line ratios	12.59	5.38	0.02 *	0.05	280.526	118.157	0.02 *	0.06
	Separate dot ratios	14.43	4.51	0.00 ***	0.09	193.6	105.825	0.071	0.04
	Integrated dot ratios	8.81	6.16	0.16	0.02	143.573	133.155	0.284	0.01

Results

To test whether individual differences in RBS acuity predict fraction knowledge and algebra achievement above-and-beyond individual differences in ANS acuity, line length acuity, and flanker performance, we conducted a series of hierarchical regression analyses (see Table 1). In the first stage of each analysis, we entered the control variables. In the second stage, we added the RBS variables (using a separate model for each comparison task) to test whether individual differences in RBS acuity would uniquely predict performance on the fraction knowledge and algebra measures.

Even after controlling for ANS acuity, line length acuity, and flanker performance, acuity for separate line ratios significantly predicted performance on all three fraction knowledge measures and the algebra entrance exam; acuity for integrated line ratios predicted performance on all of these measures except the numberline task; and acuity for separate dot ratios significantly predicted performance on all outcomes except algebra performance. Acuity for integrated dot ratios did not significantly predict any of the outcomes.

Discussion

These results represent the first evidence that individual differences in acuity for nonsymbolic rational magnitudes correlate with fraction knowledge and algebra achievement. Although this evidence is behavioral, it supports hypotheses

derived from emerging neuroscientific evidence for a Rational Brain System. In the long run, understanding how to exploit the RBS may help educators to design more effective fraction instruction. In the short run, however, a number of open questions remain.

Open Questions & Educational Implications

Perhaps the most pressing question is to determine whether learners' representations of symbolic fractions do indeed build upon the neural circuits of the RBS. We are currently conducting an fMRI experiment to investigate this question. Another open question is to determine how the RBS relates to systems like the ANS that are tuned to the magnitudes of the components that make up a nonsymbolic ratio. This question is particularly interesting given that key components of both the RBS and ANS appear to be located near the intraparietal sulcus.

A broader set of questions concerns how the RBS supports fraction learning, and whether fraction learning can be improved by more directly leveraging this system. Figure 3 presents a conceptual model of how the RBS may influence students' understanding of fractions. According to this model, instruction on fractions generates links between symbols and RBS representations of rational magnitude. These links form an important basis of a learners' understanding of the holistic magnitude of fractions, a key understanding that promotes the acquisition

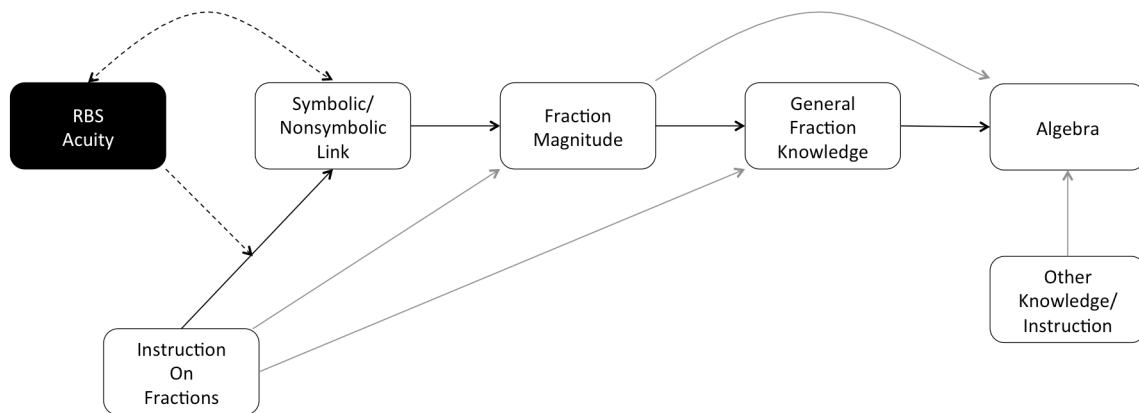


Figure 3: A conceptual model of how RBS acuity supports fraction learning and Algebra achievement.

of procedural and conceptual fraction knowledge. In turn, general fraction knowledge supports algebra learning.

This model suggests that students with higher RBS acuity will be more successful in fraction and algebra learning – a prediction that is supported by the current research. It also suggests that fraction instruction can be improved by generating stronger links between fraction symbols and RBS representations. One way to do this may be to reduce the classroom emphasis on activities like partitioning that may activate incompatible ANS representations and whole-number schemas. However, more targeted research with school-aged children will be necessary to evaluate this model and its educational implications.

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